Spectral Inequality for Elliptic Operators and Applications to the Control Problems of Heat Equations

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Outline

1. *Lebeau-Robbiano type spectral inequality*

2. *The existence of time optimal controls for heat equation*

3. *Robust null controllability for heat equations*

4. *Further comments and open problems*

1 Lebeau-Robbiano type spectral inequality

- Let M be a d ($d \in \mathbb{N}$) dimensional connected compact C^∞-smooth Riemannian manifold with the boundary Γ, and ω a nonempty open subset of M. Denote by the Laplace-Beltrami operator (on M) given by $\mathcal{D}_M$.

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- We define an unbounded operator A on $L^2(M)$ by

$$\begin{cases} D(A) = H^2(M) \cap H_0^1(M), \\ Au = -\Delta_M u, \quad \forall\, u \in D(A). \end{cases} \tag{1}$$

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- Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues of A, and $\{e_i\}_{i=1}^\infty$ the corresponding eigenfunctions satisfying $|e_i|_{L^2(M)} = 1$. It is easy to show that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $\{e_i\}_{i=1}^\infty$ constitutes an orthonormal basis of $L^2(M)$.

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- **Theorem:** It holds that

$$\sum_{\lambda_i \leq r} |a_i|^2 \leq Ce^{C\sqrt{r}} \int_\omega \Big| \sum_{\lambda_i \leq r} a_i e_i(x) \Big|^2 dx, \qquad (2)$$

for every $r > 0$ and every choice of the coefficients $\{a_i\}_{\lambda_i \leq r}$ with $a_i \in \mathbb{C}$.

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for every $r > 0$ and every choice of the coefficients $\{a_i\}_{\lambda_i \leq r}$ with $a_i \in \mathbb{C}$.

- This result provides a delicate lower bound estimate for the local energy of partial sum of eigenfunctions for Laplace-Beltrami operators (in C^∞-smooth Riemannian manifolds). The power $\frac{1}{2}$ in the above $e^{C\sqrt{r}}$ is sharp.

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- Lebeau-Robbiano(1995), the null controllability of the heat equation.
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- Wang(2008), the Bang-Bang principle for time optimal control problem for the heat equation with a locally distributed controller.

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- 1. In [Lü,2013], the author proved that (2) holds when M is a d ($d \in \mathbb{N}$) dimensional connected compact C^1-smooth Riemannian manifold with an C^2-smooth boundary ∂M. The proof is based on global Carleman estimate.

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- 2. In [Apraiz-Escauriaza-Wang-Zhang, 2013], the authors proved that (2) holds when ∂M is Lipschitz and M is analytic and locally star shaped. The key point in their proof is the analyticity of the solution to an elliptic equation in the interior of the domain.

2 The existence of time optimal controls for heat equation

- Consider the following controlled heat equation

$$\begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } M \times (0, +\infty), \\ y = 0 & \text{on } \partial M \times (0, +\infty), \\ y(0) = y_0 & \text{in } M. \end{cases} \tag{3}$$

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- Here $y_0 \in L^2(M)$, $u(\cdot) \in L^\infty(0, +\infty; \mathcal{U})$ with

$$\mathcal{U} \triangleq \left\{ v \in L^2(M) \ : \ v = \sum_{i=1}^{\infty} v_i e_i \text{ with } |v_i| \leq a_i \right\} \tag{4}$$

for some $\{a_i\}_{i=1}^{\infty} \in \ell^2_+$.

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$$t^* \triangleq \inf\big\{t : u(\cdot)\in L^\infty(0,+\infty;\mathcal{U}),\ y(t;u)=0\big\}$$

is a positive number, we call it the optimal time; when $t^* < +\infty$, a control $u^*(\cdot)\in L^\infty(0,+\infty;\mathcal{U})$ and satisfies $y(t^*;u^*)=0$, is called a time optimal control (or simply, an optimal control); *a pair* $(\widetilde{t},\widetilde{u}(\cdot))$ *holding properties:* $\widetilde{u}(\cdot)\in L^\infty(0,+\infty;\mathcal{U})$, $\widetilde{t}<+\infty$ *and* $y(\widetilde{t};\widetilde{u})=0$, *is called an admissible pair.*

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- Once we can show that the set of admit controls is nonempty, then, by some standard arguments, one can obtain the existence of the optimal control.

- This existence of the admissible control depends on the controller χ_ω, the initial datum y_0 and the set $\mathcal{U}$. When ω and y_0 are fixed, it only depends on the set $\mathcal{U}$.

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- Even in the case that $\omega = M$, the problem **(P)**, with some $\{\overline{a}_i\}_{i=1}^{+\infty}$, has no admissible pairs.
- Let $\omega = M$ and $y_0 = \sum_{i=1}^{+\infty} \frac{1}{2^i} e_i$. Set

$$\widetilde{a}_i = \frac{\lambda_i}{2^i e^{i\lambda_i}}, \quad i = 1, 2, \cdots .$$

Clearly, the element $\{\widetilde{a}_i\}_{i=1}^{+\infty}$ belongs to ℓ_+^2. However, the problem **(P)** with $\{\widetilde{a}_i\}_{i=1}^{+\infty}$, has no any admissible pair.

- **Theorem 2**: Let $\mathbb{A}$ be the subset of ℓ^2_+, such that for each of its elements, $\{a_i\}_{i=1}^{+\infty}$, there is a polynomial p with $\{1/p(i)\}_{i=1}^{+\infty} \in \ell^2_+$, such that $a_i \geq 1/p(i)$ for all $i \in \mathbb{N}$. Then any element $\{a_i\}_{i=1}^{+\infty}$ in $\mathbb{A}$, the set of admissible controls is nonempty.

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- Here $y_0 \in L^2(M)$, $u(\cdot) \in L^\infty(0, +\infty; \mathcal{U})$ with

$$U \triangleq \left\{ v \in L^2(M) \ : \ v = \sum_{i=1}^{\infty} v_i e_i \text{ with } |v_i| \leq a_i \right\} \tag{5}$$

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for some $\{a_i\}_{i=1}^{\infty} \in \ell^2$.

- Theorem 2 means that once the speed of a_i tends to 0 is slower than the inverse of a given polynomial of i, then the admissible controls always exist.

- **Lemma 1:** Let B_m be the matrix $\left(\int_\omega e_i e_j dx\right)_{1\leq i,j\leq m}$, where $m \in \mathbb{N}$. Then B_m is positive definite. Furthermore, for any $\gamma \in \mathbb{R}^m$, it holds that $|B_m^{-1}\gamma|^2_{\mathbb{R}^m} \leq C_1^2 e^{2C_2\sqrt{\lambda_m}}|\gamma|^2_{\mathbb{R}^m}$.

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- *Proof.* For each $\beta=(\beta_1,\beta_2,\cdots,\beta_m)^T\in\mathbb{R}^m$, it holds that

$$|\beta|^2_{\mathbb{R}^m}=\sum_{i=1}^m\beta_i^2\le C_1e^{C_2\sqrt{\lambda_m}}\int_\omega\Big|\sum_{i=1}^m\beta_ie_i\Big|^2dx=C_1e^{C_2\sqrt{\lambda_m}}\beta^TB_m\beta.$$

This shows that B_m is a positive definite matrix and

$$|\beta|^2_{\mathbb{R}^m}\le C_1e^{C_2\sqrt{\lambda_m}}|\sqrt{B_m}\beta|^2_{\mathbb{R}^m},\ \text{for all } \beta=(\beta_1,\beta_2,\cdots,\beta_m)^T\in\mathbb{R}^m.$$

Let $\beta=B_m^{-1}\gamma$. Then, we obtain that

$$|B_m^{-1}\gamma|^2_{\mathbb{R}^m}\le C_1^2e^{2C_2\sqrt{\lambda_m}}|B_mB_m^{-1}\gamma|^2_{\mathbb{R}^m}=C_1^2e^{2C_2\sqrt{\lambda_m}}|\gamma|^2_{\mathbb{R}^m}.$$

Thus, we complete the proof.

- Let t_1 and t_2 be such that $0 \leq t_1 < t_2 < +\infty$. Consider the following controlled system of ordinary differential equations:

$$\begin{cases} z_t = A_m z + B_m f \quad \text{in } [t_1, t_2], \\ z(t_1) = z_0. \end{cases} \tag{6}$$

Here, B_m is the matrix given in Lemma 1, $f(\cdot)$ is a control taken from $L^\infty(t_1, t_2; \mathbb{R}^m)$, $z_0 \in \mathbb{R}^m$, and $A_m = \mathrm{diag}(-\lambda_1, \cdots, -\lambda_m)$.

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- **Lemma 2:** Let $m \in \mathbb{N}$. Then for each $z_0 \in \mathbb{R}^m$, the control $\widetilde{f}(\cdot)$ defined by $\widetilde{f}(t) \equiv -B_m^{-1}\left(\int_{t_1}^{t_2} e^{-A_m(s-t_1)} ds\right)^{-1} z_0$ drives the solution $z(\cdot; \widetilde{f})$ to the origin at time t_2. Furthermore,

$$|\widetilde{f}|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)} \le C_1^2 e^{2C_2\sqrt{\lambda_m}} \left| \left(\int_{t_1}^{t_2} e^{-A_m(s-t_1)} ds\right)^{-1} z_0 \right|^2_{\mathbb{R}^m}.$$

- For each $k \in \mathbb{N}$, *write* P_{m_k} *for the orthogonal projection from* $L^2(M)$ *to* $Span\{e_1, \cdots, e_{m_k}\}$*.* Let us choose two sets of suitable disjoint intervals $\{I_k\}_{k=1}^{\infty}$ and $\{J_k\}_{k=1}^{\infty}$ such that $I_j \cap J_k = \emptyset$ for $j, k \in \mathbb{N}$ in the following way.

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- On $I_1 = [T_1, T_2]$, we let the heat equation freely evolve. Let T_2 large enough such that $|y(T_2)|_{L^2(M)}$ small enough so that on each interval $J_1 = (T_2, T_3)$, by Lemma 2, we can take a control $f^{(1)} \in L^\infty(T_2, T_3; \mathcal{U})$ from the finite dimensional space $\text{Span}\{e_1, e_2, \cdots, e_{m_1}\}$ such that $P_{m_1}(y(T_3)) = 0$.

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- For the initial value on I_k, $k = 2, 3, \cdots$, we define it to be the ending value of the solution to the equation on J_{k-1}. The initial value of the equation on J_k, $k = 1, 2, \cdots$, is given by the ending value of the solution for the equation on I_k.

- Notice that for each $k \in \mathbb{N}$, the control $f^{(k)}$ is independent of time t. On one hand, Lemma 2 provides an estimate for the control $f^{(k)}$. On the other hand, we can have a $L^2(M)-$norm estimate for the ending value of the solution to the equation on I_k. These two estimates yield that $f^{(k)}(t) \in \mathcal{U}$, a.e.

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- Finally, we prove that

$$\widetilde{u}(t) = \begin{cases} 0, & \text{if } \ t \in (0, +\infty) \setminus \bigcup_{k=1}^{+\infty} J_k, \\ f^{(k)}, & \text{if } \ t \in J_k, \text{ with } \ k \in \mathbb{N} \end{cases} \tag{7}$$

is an admissible control by showing that $\widetilde{t} \triangleq \lim_{j \to \infty} T_j < \infty$.

3 Robust null controllability for heat equations with unknown switching control mode

- Consider the following controlled heat equation:

$$\begin{cases} y_t - \Delta y = \big[\gamma\chi_{\omega_1} + (1-\gamma)\chi_{\omega_2}\big]u & \text{in } M \times (0,T), \\ y = 0 & \text{on } \partial M \times (0,T), \\ y(0) = y_0 & \text{in } M. \end{cases} \tag{8}$$

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- ω_1 and ω_2 are two nonempty open subsets of M such that $\omega_1 \cap \omega_2 = \emptyset$, and $\gamma(\cdot) : \mathbb{R} \to \{0,1\}$ be an unknown measurable function.

- We consider the null controllability problem of system (8) which consists in driving the solution to rest, for initial state $y_0 \in L^2(M)$, by means of a suitable control u, independent of γ.

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- Control systems with switching controllers arise in many fields of applications. Most of the existing works focus on designing smart switching control laws(R. Shorten et al, 2007; E.Zuazua, 2011).

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- Control systems with switching controllers arise in many fields of applications. Most of the existing works focus on designing smart switching control laws(R. Shorten et al, 2007; E.Zuazua, 2011).
- We address a different issue: that of building possible strategies of robust control so that the control u at every time instant t is guaranteed to fulfill the control requirement at the final time $t = T$ and this regardless of the possible future evolution of the switching law γ in the future time interval $[t, T]$.

- **Theorem 3:** There is a sequence $\{t_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} t_i = T$ and $0 = t_1 < t_2 < \cdots$ so that for every $y_0 \in L^2(M)$, we can find a control $u(\cdot) \in L^\infty(0, T; L^2(\omega_1 \cup \omega_2))$, such that

$$u(t)=\begin{cases} \text{a function indepednent of } t, & \text{if } t \in (t_{2k-1}, t_{2k}),\ k \in \mathbb{N}, \\ 0, & \text{if } t \in (t_{2k}, t_{2k+1}),\ k \in \mathbb{N}, \end{cases}$$

which drives y to the rest at $t = T$. Furthermore, there exists a constant $L > 0$ such that

$$|u|^2_{L^\infty(0,T;L^2(\omega_1\cup\omega_2))} \leq L|y_0|^2_{L^2(M)} \tag{9}$$

for all measurable switching functions $\gamma(\cdot) : \mathbb{R} \to \{0, 1\}$ and all y_0 in $L^2(M)$.

- The control u depends on the history of the switching function γ. This is reasonable, since possible variations of γ modify the dynamics of the system. In fact, we can show that the control can not be completely independent of γ.

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- **Lemma 3**: If for some $y_0 \in L^2(M)$ and time $T > 0$, we can find a control $u(\cdot) \in L^2(0, T; L^2(\omega_1 \cup \omega_2))$ which is independent of $\gamma(\cdot)$, such that the corresponding solution $y(\cdot; y_0, \gamma, u)$ fulfills $y(T; y_0, \gamma, u) = 0$ for all γ, then $y_0 = 0$ in $L^2(M)$ and $u = 0$ in $L^2(0, T; L^2(\omega_1 \cup \omega_2))$.

Main idea of the proof of Theorem 3:

- Let $0 \leq t_1 < t_2 < +\infty$. Consider the following system of controlled ordinary differential equations:

$$\begin{cases} z_t = A_m z + \gamma B_m^{(1)} f_1 + (1-\gamma) B_m^{(2)} f_2 \quad \text{in } [t_1, t_2], \\ z(t_1) = z_0. \end{cases} \tag{10}$$

Here, $f_1(\cdot)$ and $f_2(\cdot)$ are controls taken from $L^\infty(t_1, t_2; \mathbb{R}^m)$, $z_0 \in \mathbb{R}^m$, and $A_m = \mathrm{diag}(-\lambda_1, \cdots, -\lambda_m)$ with $0 < \lambda_1 \leq \cdots \leq \lambda_m$,

$$B_m^{(1)} = \Big(\int_{\Omega_1} e_i e_j dx \Big)_{1 \leq i,j \leq m}, \ B_m^{(2)} = \Big(\int_{\Omega_2} e_i e_j dx \Big)_{1 \leq i,j \leq m}.$$

- **Lemma 4:** Let $m \in \mathbb{N}$. Then for each $z_0 \in \mathbb{R}^m$, the controls $f_1(\cdot)$ and $f_2(\cdot)$ defined by

$$\begin{cases} f_1(t) \equiv -(B_m^{(1)})^{-1}\Big(\displaystyle\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1} z_0, \quad t \in (t_1, t_2), \\ f_2(t) \equiv -(B_m^{(2)})^{-1}\Big(\displaystyle\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1} z_0, \quad t \in (t_1, t_2), \end{cases}$$

  drive the solution $z(\cdot; z_0, \gamma, f_1, f_2)$ to the origin at time t_2. Furthermore, these controls satisfy the estimate that

$$\begin{cases} |f_1|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)} \le C_1^2 e^{2C_1\sqrt{\lambda_m}} \Big|\Big(\displaystyle\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1} z_0\Big|^2_{\mathbb{R}^m}, \\ |f_2|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)} \le C_2^2 e^{2C_2\sqrt{\lambda_m}} \Big|\Big(\displaystyle\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1} z_0\Big|^2_{\mathbb{R}^m}. \end{cases}$$

- Let

$$T_k = \begin{cases} 0, & \text{if } k = 1, \\ T \sum_{i=1}^{k-1} 2^{-i}, & \text{if } k > 1, \end{cases} \tag{11}$$

and

$$\widetilde{T}_k = \begin{cases} \frac{T}{4}, & \text{if } k = 1, \\ T \Big(\sum_{i=1}^{k-1} 2^{-i} + 2^{-k-1} \Big), & \text{if } k > 1. \end{cases} \tag{12}$$

- We define the following sequences of time intervals:

$$I_k = [T_k, \widetilde{T}_k) \tag{13}$$

and

$$J_k = [\widetilde{T}_k, T_{k+1}). \tag{14}$$

We put

$$r_k = \frac{16C_1^2}{(T_{k+1} - \widetilde{T}_k)^4}, \quad \text{for } k = 1, 2, \cdots \tag{15}$$

Then we know that

$$r_k \to \infty \text{ as } k \to \infty. \tag{16}$$

- For each $k \in \mathbb{N}$, let us denote by P_k the orthogonal projection from $L^2(M)$ to $\mathrm{Span}_{\lambda_i \leq r_k}\{e_i\}$. On each interval I_k, we control the heat equation with a control switching from ω_1 to ω_2 in an unknown mode. By Proposition 2, we can find a control $u^{(k)}(\cdot) \in L^\infty(I_k; L^2(\omega_1 \cup \omega_2))$ such that the corresponding solution $y^{(k)}(\cdot)$ to the equation on I_k satisfies

$$P_k\big(y^{(k)}\big(\widetilde{T}_k\big)\big) = 0.$$

- On every interval J_k, we let the heat equation freely evolve. We start by having the initial datum for the equation on I_1 to be y_0. For the initial datum on I_k, $k = 2, 3, \cdots$, we define it to be the ending value of the solution to the equation on J_{k-1}. The initial datum of the equation on J_k, $k = 1, 2, \cdots$, is given by the ending value of the solution for the equation on I_k. If there is no eigenvalue of $-A$ in $(r_k, r_{k+1}]$, we simply set $u^{(k)}(\cdot) = 0$ on I_k.

- Notice that for each $k \in \mathbb{N}$, by Proposition 2, the control $u^{(k)}(\cdot)$ is independent of time t and the value of $\gamma(\cdot)$ in I_k. Further, Proposition 2 provides an estimate for the control $u^{(k)}(\cdot)$. On the other hand, thanks to the energy decay of the heat equation, we can get a suitable $L^2(M)$-norm estimate for the ending value of the solution to the equation on J_k. These two estimates yield that the control

$$u(\cdot) = \sum_{k=1}^{\infty} \chi_{I_k}(\cdot) u^{(k)} \in L^\infty(0, T; L^2(\omega_1 \cap \omega_2)),$$

drives the solution of system (8) to 0 at time T.

- In order to adopt the above strategy, we need to know the ending values of the solution on every $I_k (k \in \mathbb{N})$. These values cannot be obtained by the initial datum of the solution on every I_k, $k = 1, 2, \cdots$, if we do not know the value of $\gamma(s)$ for $s \in I_k$. Hence, we have to observe them. This is reasonable and necessary according to Proposition 1. Moreover, this is operable since we only need the previous and present state of the system rather than the future of $\gamma(\cdot)$.

4. Further comments and open problems

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- The Spectral inequality for the eigenfunctions of more general operators, i.e., Stokes operator, strongly coupled elliptic systems, etc.
- The existence of the time optimal control under rectangular constraints for general parabolic equations.

- **Observability inequalities.** Our results yield the observability inequality

$$|\varphi(0)|^2_{L^2(M)} \leq C\int_0^T \min\Big\{\int_{\omega_1} |\varphi(x,t)|^2 dx, \int_{\omega_2} |\varphi(x,t)|^2 dx\Big\} dt, \tag{17}$$

for every $\varphi(\cdot)$ solves

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } M\times(0,T), \\ \varphi = 0 & \text{on } \partial M\times(0,T), \\ \varphi(T) = \varphi_T & \text{in } M, \end{cases}$$

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- As far as we know this observability inequality is new and we do not know how to prove it directly.

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- For instance, to find the control from inequality (17), it is sufficient to minimize the functional

$$J(\varphi_T)=\int_0^T\int_{\omega_1}\gamma(t)|\varphi(x,t)|^2dxdt+\int_0^T\int_{\omega_2}(1-\gamma(t))|\varphi(x,t)|^2dxdt$$
$$+\int_M\varphi(x,0)y_0(x)dx$$

on the Hilbert space H, which is the completion of $C_0^\infty(M)$ with respect to the following norm:

$$|\varphi_T|_H^2\triangleq\int_0^T\int_{\omega_1}\gamma(t)|\varphi(x,t)|^2dxdt+\int_0^T\int_{\omega_2}(1-\gamma(t))|\varphi(x,t)|^2dxdt.$$

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- Unfortunately, the control obtained in this way will depends globally on γ.

- **Wave equations.** The results of this paper do not hold for the wave equation even in $1-d$.

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- Assuming that ω_1 and ω_2 are two open non-empty subintervals of the interval M where the wave equation is posed, the exact controllability property of the wave equation is ensured when the time of control is sufficiently large. But this does not suffice to guarantee the exact controllability for all possible switching functions γ. Indeed, it is easy to build a switching function γ such that there exists a broken characteristic line reflected on the boundary but that never meets the control sets ω_1 and ω_2 when they are active. In this situation the wave equation is not controllable.

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$$\begin{cases} \dfrac{dx}{dt} = ax + \left[\gamma b_1 + (1-\gamma) b_2\right] u \quad \text{in } [0, T], \\ x(0) = x_0. \end{cases} \tag{18}$$

Here $a > 0$, and $b_1 > 0$ and $b_2 > 0$. This is the simplest possible situation from a control theoretical point of view. The initial state $x_0 \in \mathbb{R}$ and the switching function γ belongs to the set of all measurable functions from $[0, T]$ to $\{0, 1\}$. The controls $u, u_1, u_2 \in L^2(0, T; \mathbb{R})$.

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- One can easily show that an analogous result of Theorem 3 is not true for the above system.

Thank you!