Global exact simultaneous controllability of an arbitrary number of 1D bilinear Schrödinger equations

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Partial differential equations, optimal design and numerics
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Joint work with **Vahagn Nersesyan** (UVSQ).

Model studied : N identical and independent $1D$ particles in a potential

$$\begin{cases} i\partial_t \psi^j = \left(-\partial^2_{xx} + V(x)\right)\psi^j - u(t)\mu(x)\psi^j, & x \in (0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \end{cases} \qquad (\mathbf{S}_N)$$

where

- **State :** $\left(\psi^1,\dots,\psi^N\right) \in \mathcal{S}^N$, **control :** $u : (0,T) \to \mathbb{R}$,
- $V : (0,1) \to \mathbb{R}$ potential,
- $\mu : (0,1) \to \mathbb{R}$ dipole moment.

Goal : Simultaneous control of $(\psi^1,\dots,\psi^N)$ with a single control u.

Outline

Notations

- $\mathcal{S}$: $L^2((0,1),\mathbb{C})$ unit sphere.
- $\lambda_{k,V} \in \mathbb{R}$ and $\varphi_{k,V} \in \mathcal{S}$ eigenvalues and eigenvectors of

$$A_V \psi := \left(-\partial_{xx}^2 + V\right)\psi, \quad D(A_V) := H^2 \cap H_0^1((0,1),\mathbb{C})$$

- Functional framework

$$H^s_{(V)} := D\left(A_V^{s/2}\right), \quad \|\cdot\|^2_{H^s_{(V)}} := \sum_{k=1}^{\infty} |k^s \langle \cdot, \varphi_{k,V}\rangle|^2, \quad \forall s > 0.$$

Ground state and invariant

- Let $\Phi_{k,V}(t,x) := e^{-i\lambda_{k,V}t}\varphi_{k,V}(x)$. $\left(\Phi_{1,V},\dots,\Phi_{N,V}\right)$ solution with $u \equiv 0$.
- Bold notations : $\boldsymbol{\psi} := (\psi^1,\dots,\psi^N)$, $\boldsymbol{H} := H^N$.
- Unique weak solution $C^0([0,T], H^3_{(V)})$ for $u \in L^2((0,T),\mathbb{R})$, $\boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(V)}$,

$$\boldsymbol{\psi}(\cdot,\boldsymbol{\psi}_0,u).$$

- Unitary equivalent vectors $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_f$: there exists $\mathcal{U} : L^2 \to L^2$ unitary map such that $\boldsymbol{\psi}_f = \mathcal{U}\boldsymbol{\psi}_0$ i.e.

$$\psi_f^j = \mathcal{U}\psi_0^j, \quad \forall j \in \{1,\dots,N\}.$$

Main result and strategy

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0,1),\mathbb{R})$, system $(\mathbf{S_N})$ is globally exactly controllable in $\boldsymbol{H}^4_{(V)}$, generically with respect to $\mu \in H^4((0,1),\mathbb{R})$. More precisely, there exists a set $\mathcal{Q}_V$ residual in $H^4((0,1),\mathbb{R})$ such that for every $\mu \in \mathcal{Q}_V$

$$\forall\, \boldsymbol{\psi}_0, \boldsymbol{\psi}_f \text{ unitarily equivalent, } \exists T > 0,\ \exists u \in L^2((0,T),\mathbb{R});$$
$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f.$$

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$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f.$$

Overall strategy :

- Global approximate controllability towards finite sums of eigenvectors
    - use of a suitable Lyapunov function

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- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
  - Coron's return method : local exact controllability around finite sums of eigenvectors

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$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f.$$

Overall strategy :

- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
    - Coron's return method : local exact controllability around finite sums of eigenvectors
    - Connectedness and compactness : exact controllability around $\boldsymbol{z}_0$ (initial conditions) and $\boldsymbol{z}_f$ (targets) with z_0^j, z_f^j finite sums of eigenvectors

Main result and strategy

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0,1),\mathbb{R})$, system $(\mathbf{S_N})$ is globally exactly controllable in $\boldsymbol{H^4_{(V)}}$, generically with respect to $\mu \in H^4((0,1),\mathbb{R})$. More precisely, there exists a set $\mathcal{Q}_V$ residual in $H^4((0,1),\mathbb{R})$ such that for every $\mu \in \mathcal{Q}_V$

$$\forall\, \boldsymbol{\psi}_0, \boldsymbol{\psi}_f \text{ unitarily equivalent, } \exists T > 0,\ \exists u \in L^2((0,T),\mathbb{R});$$
$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f.$$

Overall strategy :

- Global approximate controllability towards finite sums of eigenvectors
- Exact controllability around finite sums of eigenvectors
- Time reversibility

$$\boldsymbol{\psi}(T, \overline{\boldsymbol{\psi}_f}, u) = \overline{\boldsymbol{\psi}_0} \implies \boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u(T-\cdot)) = \boldsymbol{\psi}_f.$$

Perturbation and favourable hypotheses

Dealing with an arbitrary potential V. Consider the control $u(t) := \tilde{u}(t) - 1$.

$$\begin{cases} i\partial_t \tilde{\psi}^j = \left(-\partial^2_{xx} + V(x)\right)\tilde{\psi}^j - (\tilde{u}(t)-1)\mu(x)\tilde{\psi}^j, & \\ \qquad\quad = \left(-\partial^2_{xx} + V(x) + \mu(x)\right)\tilde{\psi}^j - \tilde{u}(t)\mu(x)\tilde{\psi}^j, & x \in (0,1), \\ \tilde{\psi}^j(t,0) = \tilde{\psi}^j(t,1) = 0, & j \in \{1, \dots, N\}, \end{cases}$$

'New potential' : $V + \mu$

- Study of global approximate and local exact controllability of $(\mathbf{S}_N)$ under favourable hypothesis on the potential for arbitrary V.

Previous results : finite dimension and approximate controllability

- Finite dimension
  - **Turinici, Rabitz** (2004)
    Control of the orientation of an ensemble of molecules (finite dimension)
  - **Silveira, Pereira da Silva, Rouchon** (2009)
    Stabilization of density matrices (finite dimension)
- Approximate controllability in infinite dimension
  - **Boscain, Caponigro, Chambrion, Mason, Sigalotti** (2009, 2012)
    Simultaneous approximate controllability in L^2.
    Approximate control of density matrices (through control of Galerkin approximations)
  - **Boussaïd, Caponigro, Chambrion** (2013)
    Higher Sobolev norms for 'weakly coupled' systems.

Previous results : a single particle ($N = 1$)

$V = 0$. $\mu \in H^3(0,1)$ satisfies $\exists c > 0$ such that

$$|\langle \mu\varphi_1, \varphi_k \rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*.$$

- **Beauchard Laurent** (2010), local exact controllability : $\forall T > 0$, $\exists \delta > 0$ such that

$$\forall \psi_f \in \mathcal{S} \cap H^3_{(0)} \quad \text{with} \quad ||\psi_f - \Phi_1(T)||_{H^3_{(0)}} < \delta,$$

there exists $u \in L^2((0,T), \mathbb{R})$ such that

$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi - u(t)\mu(x)\psi, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,\cdot) = \varphi_1, \end{cases} \qquad \Longrightarrow \qquad \psi(T) = \psi_f.$$

C^1 regularity of the map $\psi_f \mapsto u$.

- **Nersesyan** (2010), global exact controllability in $\mathcal{S} \cap H^{3+\epsilon}_{(0)}$ for generic μ.

M (2013) *Ann. Inst. H. Poincaré Anal. Non Linéaire*.

$V = 0$. $\mu \in H^3(0,1)$ satisfies $\exists c > 0$ such that

$$|\langle \mu\varphi_j, \varphi_k\rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \dots, N\}, \forall k \in \mathbb{N}^*.$$

$$(\psi_0^1, \dots, \psi_0^N) = (\varphi_1, \dots, \varphi_N).$$

- Unreachable targets with small controls in small time for $N \geq 2$.
- $N = 2$: local controllability in arbitrary time up to a global phase i.e. $\forall T > 0$, $\exists \theta \in \mathbb{R}$, $\exists \delta > 0$;

$$\forall (\psi_f^1, \psi_f^2) \in \left(\mathcal{S} \cap H^3_{(0)}\right)^2 \quad \text{with} \quad \langle \psi_f^1, \psi_f^2\rangle = 0 \text{ and}$$
$$||\psi_f^1 - e^{i\theta}\Phi_1(T)||_{H^3_{(0)}} + ||\psi_f^2 - e^{i\theta}\Phi_2(T)||_{H^3_{(0)}} < \delta,$$

$\exists u \in L^2((0,T),\mathbb{R})$ such that $(\psi^1, \psi^2)(T) = (\psi_f^1, \psi_f^2)$.

Previous results : a first step ($N = 2$ and $N = 3$) II

- $N = 2$: local exact controllability up to a global delay i.e. $\exists T^* > 0$; $\forall T \geq 0$, $\exists \delta > 0$;

$$\forall (\psi_f^1, \psi_f^2) \in \left(\mathcal{S} \cap H^3_{(0)}\right)^2 \quad \text{with} \quad \langle \psi_f^1, \psi_f^2 \rangle = 0 \text{ and}$$
$$||\psi_f^1 - \Phi_1(T)||_{H^3_{(0)}} + ||\psi_f^2 - \Phi_2(T)||_{H^3_{(0)}} < \delta,$$

$\exists u \in L^2((0, T^* + T), \mathbb{R})$ such that $(\psi^1, \psi^2)(T^* + T) = (\psi_f^1, \psi_f^2)$.

- $N = 3$: local controllability up to a global phase and a global delay i.e. $\exists T^* > 0$, $\exists \theta \in \mathbb{R}$; $\forall T \geq 0$, $\exists \delta > 0$;

$$\forall (\psi_f^1, \psi_f^2, \psi_f^3) \in \left(\mathcal{S} \cap H^3_{(0)}\right)^3 \quad \text{with} \quad \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ and}$$
$$||\psi_f^1 - e^{i\theta}\Phi_1(T)||_{H^3_{(0)}} + ||\psi_f^2 - e^{i\theta}\Phi_2(T)||_{H^3_{(0)}} + ||\psi_f^3 - e^{i\theta}\Phi_3(T)||_{H^3_{(0)}} < \delta,$$

$\exists u \in L^2((0, T^* + T), \mathbb{R})$ such that $(\psi^1, \psi^2, \psi^3)(T^* + T) = (\psi_f^1, \psi_f^2, \psi_f^3)$.

Approximate controllability towards finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^4((0,1),\mathbb{R})$ such that

(C₁) $\langle \mu \varphi_{j,V}, \varphi_{k,V} \rangle \neq 0$ for all $j \in \{1, \dots, N\}$, $k \in \mathbb{N}^*$.

(C₂) $\lambda_{j,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j \in \{1, \dots, N\}$, $k, p, q \in \mathbb{N}^*$ such that $\{j,k\} \neq \{p,q\}$ and $k \neq j$.

Theorem

Let $\mathcal{C}_M := \mathrm{Span}\{\varphi_{1,V}, \dots, \varphi_{M,V}\}$. Under Conditions **(C₁)** and **(C₂)**, for any $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ with $\langle \psi_0^j, \varphi_{j,V} \rangle \neq 0$, for all $j \in \{1, \dots, N\}$, there are $M \in \mathbb{N}^*$, $\boldsymbol{\psi}_f \in \boldsymbol{\mathcal{C}}_M$, sequences $T_n > 0$ and $u_n \in C_0^\infty((0, T_n), \mathbb{R})$ such that

$$\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) \underset{n \to \infty}{\longrightarrow} \boldsymbol{\psi}_f \quad \text{in } \boldsymbol{H}^3.$$

$N = M = 1$: **Nersesyan** (2010).

Sketch of proof I

- Lyapunov strategy.

$$\mathcal{L}(\boldsymbol{z}) := \alpha \sum_{j=1}^{N} \| \left(-\partial_{xx}^2 + V\right)^2 \mathcal{P}_N z^j \|_{L^2}^2 + 1 - \prod_{j=1}^{N} |\langle z^j, \varphi_{j,V}\rangle|^2,$$

with $\mathcal{P}_N$ orthogonal projection in L^2 onto $\overline{\text{Span}\{\varphi_{k,V}\,;\, k \geq N+1\}}$.

- Decrease : $\boldsymbol{z} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ with $\langle z^j, \varphi_{j,V}\rangle \neq 0$, for all $j \in \{1, \dots, N\}$. Either

$$\boldsymbol{z} \in \bigcup_{M \in \mathbb{N}^*} \boldsymbol{\mathcal{C}}_M,$$

or $\exists T > 0$, $\exists u \in C_0^\infty((0,T), \mathbb{R})$ such that

$$\mathcal{L}\big(\psi(T, \boldsymbol{z}, u)\big) < \mathcal{L}(\boldsymbol{z}).$$

Sketch of proof II

idea : existence of T and $w \in C_0^\infty((0,T),\mathbb{R})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{L}\big(\psi(T,\psi_0,\sigma w)\big)\Big|_{\sigma=0} \neq 0.$$

- We define

$$\mathcal{K} := \Big\{\psi \in \boldsymbol{H}^4_{(V)}\,;\ \psi(T_n,\psi_0,u_n) \underset{n\to\infty}{\longrightarrow} \psi \text{ in } \boldsymbol{H}^3,\ \text{for } T_n \geq 0,\ u_n \in C_0^\infty((0,T_n),\mathbb{R})\Big\}.$$

- $\boldsymbol{e} \in \mathcal{K}$ such that $\mathcal{L}(\boldsymbol{e}) = \inf\limits_{\psi\in\mathcal{K}} \mathcal{L}(\psi)$. Then

$$\boldsymbol{e} \in \bigcup_{M\in\mathbb{N}^*} \mathcal{C}_M.$$

Local exact controllability around finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0,1), \mathbb{R})$ such that

(C_3) there exists $c > 0$ such that
$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \dots, N\}, \forall k \in \mathbb{N}^*,$$

(C_4) $\lambda_{k,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{n,V}$ for all $j, n \in \{1, \dots, N\}$, $k \geq j+1$, $p \geq n+1$ with $\{j,k\} \neq \{p,n\}$,

(C_5) $1, \lambda_{1,V}, \dots, \lambda_{N,V}$ are rationally independent.

Theorem

Let $C_0, C_f \in U_N$ and $\boldsymbol{z}_0 := C_0\boldsymbol{\varphi}_V$, $\boldsymbol{z}_f := C_f\boldsymbol{\varphi}_V$. Under Conditions **($C_3$)**-**($C_5$)**, there exists $T > 0$, $\delta > 0$ such that, if

$$\mathcal{O}_{\delta,C} := \left\{\phi \in \boldsymbol{H}^3_{(V)} \,;\, \langle \phi^j, \phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \|\phi^j - (C\boldsymbol{\varphi}_V)^j\|_{H^3_{(V)}} < \delta\right\},$$

for every $\psi_0 \in \mathcal{O}_{\delta,C_0}$, $\psi_f \in \mathcal{O}_{\delta,C_f}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that the associated solution satisfies $\psi(T) = \psi_f$.

Local controllability around eigenvectors up to global phases

Auxiliary controllability result

Proposition

Assume Conditions $(\mathbf{C_3})$-$(\mathbf{C_4})$.

- $T > 0$, there are $\theta_1, \dots, \theta_N \in \mathbb{R}$, $\delta > 0$;

$$\forall \boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(V)};\ \langle \psi_0^j, \psi_0^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\psi_0^j - \varphi_{j,V}\|_{H^3_{(\mathbf{V})}} < \delta,$$

$$\forall \boldsymbol{\psi}_f \in \boldsymbol{H}^3_{(V)};\ \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\psi_f^j - e^{i\theta_j}\varphi_{j,V}\|_{H^3_{(\mathbf{V})}} < \delta,$$

  there exists $u \in L^2((0,T),\mathbb{R})$ such that $\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f$.
- C^1 regularity of the map $(\boldsymbol{\psi}_0, \boldsymbol{\psi}_f) \mapsto u$.

Similar to **M** (2013) for $N = 2, 3$. No condition on the phase terms θ_j.

Proof : rotation

1. Proof in the case $C_0 = C_f = I_N$. $\psi_0, \psi_f \approx \boldsymbol{\varphi}_V$.

- Use of the proposition.

$$\psi_0 \approx \boldsymbol{\varphi}_V \quad \overset{T^*,u}{\rightsquigarrow} \quad \left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right).$$

- Rotation and rational independence of eigenvalues : Condition $(\mathbf{C_5})$.

$$\left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right) \overset{T_r,u=0}{\rightsquigarrow} \zeta := \left(e^{i(\theta_1-\lambda_{1,V}T_r)}\varphi_{1,V}, \dots, e^{i(\theta_N-\lambda_{N,V}T_r)}\varphi_{N,V}\right) \approx \left(e^{-i\theta_1}\varphi_{1,V}, \dots, e^{-i\theta_N}\varphi_{N,V}\right)$$

- Use of the proposition.

$$\overline{\psi_f} \approx \boldsymbol{\varphi}_V \quad \overset{T^*,v}{\rightsquigarrow} \quad \overline{\zeta}.$$

- Conclusion : time-reversibility

$$\zeta \quad \overset{T^*,v(T^*-\cdot)}{\rightsquigarrow} \quad \psi_f.$$

Proof : linearity

2. Proof in the case $C_0 = C_f = C \in U_N$**.** Let $\boldsymbol{z} := C\varphi_V$. $\psi_0, \psi_f \approx \boldsymbol{z}$.

- Let $\delta_{\boldsymbol{z}} > 0$ such that

$$C^*\left(B_{H^3_{(\boldsymbol{V})}}(\boldsymbol{z}, \delta_{\boldsymbol{z}})\right) \subset B_{H^3_{(\boldsymbol{V})}}(\varphi_V, \delta),$$

  and

$$\widetilde{\psi}_0 := C^*\psi_0, \quad \widetilde{\psi}_f := C^*\psi_f.$$

- Step **1.** : $\widetilde{T} := 2T^* + T_r$, $\exists u \in L^2((0, \widetilde{T}), \mathbb{R})$ such that

$$\widetilde{\psi}_0 \quad \overset{\widetilde{T}, u}{\rightsquigarrow} \quad \widetilde{\psi}_f.$$

- Linearity of $(\mathbf{S}_{\boldsymbol{N}})$ with respect to the state

$$\psi(\widetilde{T}, \psi_0, u) = \psi(\widetilde{T}, C\widetilde{\psi}_0, u) = C\psi(\widetilde{T}, \widetilde{\psi}_0, u) = C\widetilde{\psi}_f = \psi_f.$$

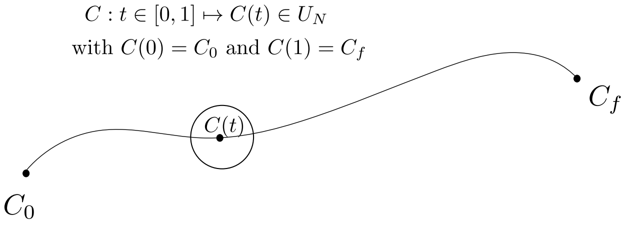
Proof : connectedness and compactness

3. Conclusion : $C_0, C_f \in U_N$.

- Connectedness in the set of unitary matrices and compactness.

Proof : connectedness and compactness

3. **Conclusion :** $C_0, C_f \in U_N$.
   - Connectedness in the set of unitary matrices and compactness.



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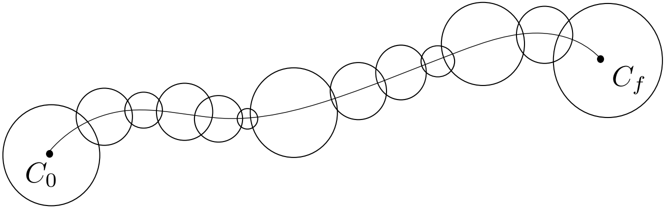
   - Connectedness in the set of unitary matrices and compactness.



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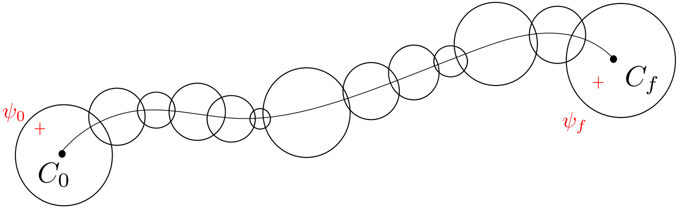

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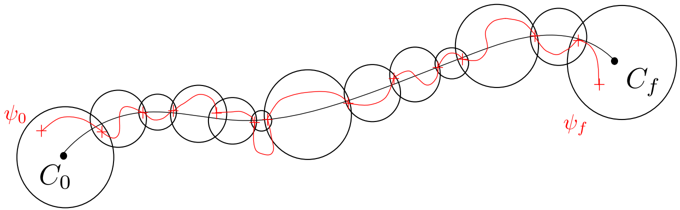


Local controllability around eigenvectors up to global phases

Proposition

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0,1),\mathbb{R})$ such that

$(\mathbf{C_3})$ there exists $c > 0$ such that

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1,\dots,N\}, \forall k \in \mathbb{N}^*,$$

$(\mathbf{C_4})$ $\lambda_{k,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{n,V}$ for all $j, n \in \{1,\dots,N\}$, $k \geq j+1$, $p \geq n+1$ with $\{j,k\} \neq \{p,n\}$,

$T > 0$, there are $\theta_1,\dots,\theta_N \in \mathbb{R}$, $\delta > 0$;

$$\forall \boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(V)};\ \langle \psi_0^j, \psi_0^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\psi_0^j - \varphi_{j,V}\|_{H^3_{(V)}} < \delta,$$

$$\forall \boldsymbol{\psi}_f \in \boldsymbol{H}^3_{(V)};\ \langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\psi_f^j - e^{i\theta_j}\varphi_{j,V}\|_{H^3_{(V)}} < \delta,$$

there exists $u \in L^2((0,T),\mathbb{R})$ such that $\boldsymbol{\psi}(T,\boldsymbol{\psi}_0,u) = \boldsymbol{\psi}_f$.

Natural strategy : linear test

- Linearized system around $\left(\Phi_{1,V},\dots,\Phi_{N,V},u\equiv 0\right)$

$$\begin{cases} i\partial_t \Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_{j,V}, \quad j\in\{1,\dots,N\} \\ \Psi^j(t,0)=\Psi^j(t,1)=0, \\ \Psi^j(0,x)=0. \end{cases}$$

$$\Psi^j(T) = i\sum_{k=1}^{\infty}\langle \mu\varphi_{j,V},\varphi_{k,V}\rangle \int_0^T v(t)e^{i(\lambda_{k,V}-\lambda_{j,V})t}\mathrm{d}t\,\Phi_{k,V}(T).$$

Natural strategy : linear test

- Linearized system around $\left(\Phi_{1,V}, \dots, \Phi_{N,V}, u \equiv 0\right)$

$$\begin{cases} i\partial_t \Psi^j = -\partial_{xx}^2 \Psi^j - v(t)\mu(x)\Phi_{j,V}, \quad j \in \{1, \dots, N\} \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, \\ \Psi^j(0,x) = 0. \end{cases}$$

$$\Psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t \, \Phi_{k,V}(T).$$

- Gap condition + null upper density (Conditions **(C₃)**-**(C₄)**) $\rightsquigarrow$ Solution of moment problem for non redundant frequencies

$$\left\{\lambda_{k,V} - \lambda_{j,V} \, ; \, j \in \{1, \dots, N\}, k \geq j+1 \text{ and } k = j = N\right\}.$$

Natural strategy : linear test

- Linearized system around $\left(\Phi_{1,V}, \dots, \Phi_{N,V}, u \equiv 0\right)$

$$\begin{cases} i\partial_t \Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_{j,V}, & j \in \{1, \dots, N\} \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, \\ \Psi^j(0,x) = 0. \end{cases}$$

$$\Psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t \, \Phi_{k,V}(T).$$

- Gap condition + null upper density (Conditions $(\mathbf{C_3})$-$(\mathbf{C_4})$) $\rightsquigarrow$ Solution of moment problem for non redundant frequencies

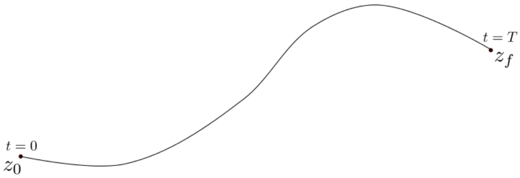
$$\left\{\lambda_{k,V} - \lambda_{j,V} \, ; \, j \in \{1, \dots, N\}, k \geq j+1 \text{ and } k = j = N\right\}.$$

- Lost directions.

$$\frac{\langle \Psi^j(T), \Phi_{j,V}(T)\rangle}{\langle \mu\varphi_{j,V}, \varphi_{j,V}\rangle} = \frac{\langle \Psi^k(T), \Phi_{k,V}(T)\rangle}{\langle \mu\varphi_{k,V}, \varphi_{k,V}\rangle}, \quad \forall j,k \in \{1, \dots, N\}.$$

The return method

Introduced by **Coron** (1992). Controllability of nonlinear systems with non-controllable linearized system.



The return method

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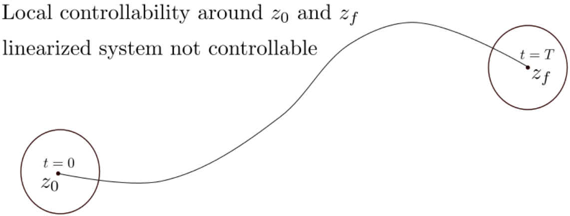

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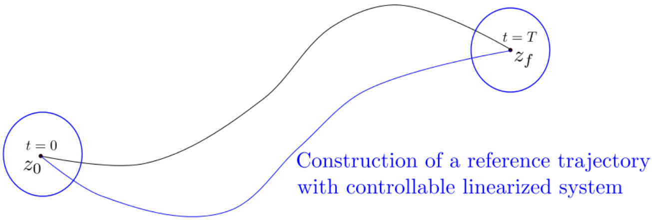


The reference trajectory

Let $T > 0$ and $0 < \varepsilon_1 < \cdots < \varepsilon_{N-1} < T$.

Under Conditions $(\mathbf{C_3})$ and $(\mathbf{C_4})$, there exist $\overline{\eta} > 0$, $C > 0$ such that $\forall \eta \in (0, \overline{\eta})$, $\exists \theta_1^\eta, \dots, \theta_N^\eta \in \mathbb{R}$, $\exists u_{ref}^\eta \in L^2((0,T),\mathbb{R})$ with

$$\|u_{ref}^\eta\|_{L^2} \leq C\eta,$$

such that $\forall j \in \{1, \dots, N\}$, $\forall k \in \{1, \dots, N-1\}$,

$$\langle \mu \psi_{ref}^{j,\eta}(\varepsilon_k), \psi_{ref}^{j,\eta}(\varepsilon_k) \rangle = \langle \mu \varphi_{j,V}, \varphi_{j,V} \rangle + \eta \delta_{j=k},$$

and

$$\boldsymbol{\psi}_{ref}^\eta(T) = \left(e^{i\theta_1^\eta} \varphi_{1,V}, \dots, e^{i\theta_N^\eta} \varphi_{N,V} \right).$$

Main ideas : Small perturbations + partial control results (moment problem and invariants)

$$\boldsymbol{\psi}_{ref}^\eta(T) = \left(e^{i\theta_1^\eta} \varphi_{1,V}, \dots, e^{i\theta_N^\eta} \varphi_{N,V} \right) \Longleftrightarrow \langle \psi_{ref}^{j,\eta}(T), \Phi_{k,V}(T) \rangle = 0, \forall k \geq j+1.$$

skip proof

Proof of the construction of the reference trajectory

- $[0, \varepsilon_{N-1}]$: Small perturbation (partial control result) such that

$$\langle \mu \psi_{ref}^{j,\eta}(\varepsilon_k), \psi_{ref}^{j,\eta}(\varepsilon_k) \rangle = \langle \mu \varphi_{j,V}, \varphi_{j,V} \rangle + \eta \delta_{j=k}, \forall j \in \{1, \dots, N\}, \forall k \in \{1, \dots, N-1\}.$$

- $[\varepsilon_{N-1}, T]$: Reaching the target.

$$\boldsymbol{\psi}_{ref}^{\eta}(T) = \left(e^{i\theta_{\mathbf{1}}^{\eta}} \varphi_{1,V}, \dots, e^{i\theta_{\mathbf{N}}^{\eta}} \varphi_{N,V} \right) \iff \mathcal{P}_j\left(\psi_{ref}^{j,\eta}(T)\right) = 0, \quad \forall j \in \{1, \dots, N\},$$

where

$$\mathcal{P}_j(\psi) = \sum_{k \geq j+1} \langle \psi, \varphi_{k,V} \rangle \varphi_{k,V}.$$

Inverse mapping theorem at $\left(0, \Phi_{1,V}(\varepsilon_{N-1}), \dots, \Phi_{N,V}(\varepsilon)\right)$ to

$$\Theta\left(u, \boldsymbol{\psi}_0\right) := \left(\boldsymbol{\psi}_0, \mathcal{P}_1\left(\psi^1(T)\right), \dots, \mathcal{P}_N\left(\psi^N(T)\right) \right)$$

Continuous right inverse of $\mathrm{d}\Theta\left(0, \Phi_{1,V}(\varepsilon_{N-1}), \dots, \Phi_{N,V}(\varepsilon)\right)$: solve a trigonometric moment problem with frequencies

$$\{\lambda_{k,V} - \lambda_{j,V} \,;\, j \in \{1, \dots, N\}, k \geq j+1\}.$$

Controllability of the linearized system around the reference trajectory

$$\begin{cases} i\partial_t \Psi^{j,\eta} = \left(-\partial^2_{xx} + V(x)\right)\Psi^{j,\eta} - u^\eta_{ref}(t)\mu(x)\Psi^{j,\eta} - v(t)\mu(x)\psi^{j,\eta}_{ref}, \\ \Psi^{j,\eta}(t,0) = \Psi^{j,\eta}(t,1) = 0, \\ \Psi^{j,\eta}(0,x) = \Psi^{j,\eta}_0(x). \end{cases}$$

Linearization of the invariants :

$$\mathrm{Re}(\langle \Psi^{j,\eta}, \psi^{j,\eta}_{ref}(t)\rangle) = 0, \quad \forall\, 1 \leq j \leq N,$$
$$\langle \Psi^{j,\eta}, \psi^{k,\eta}_{ref}(t)\rangle + \overline{\langle \Psi^{k,\eta}, \psi^{j,\eta}_{ref}(t)\rangle} = 0, \quad \forall\, 1 \leq k < j \leq N.$$

Controllability : There exists $\hat{\eta} \in (0, \overline{\eta})$ such that for any $\eta \in (0, \hat{\eta})$, for any suitable $(\mathbf{\Psi}_0, \mathbf{\Psi}_f) \in \boldsymbol{H}^3_{(V)}$, there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution initiated from $\mathbf{\Psi}_0$ satisfies

$$\mathbf{\Psi}^\eta(T) = \mathbf{\Psi}_f.$$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V} \rangle$	1	2	$\ldots$	N	$\ldots$
$\Psi^{1,\eta}$					$\ldots$
$\Psi^{2,\eta}$					$\ldots$
$\vdots$	$\vdots$		$\ddots$		$\vdots$
$\Psi^{N,\eta}$					$\ldots$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\dots$		

- Choice of η small enough + moment problem
  - For $\eta = 0$: $j \in \{1,\dots,N\}$, $k \geq j+1$ and $k = j = N$

$$\langle \Psi^{j,0}(T), \Phi_{k,V}(T)\rangle = i\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V}-\lambda_{j,V})t} \mathrm{d}t,$$

solve a trigonometric moment problem (Conditions (C_3) and (C_4)).

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\dots$		

- Choice of η small enough + moment problem
  - For $\eta = 0$: $j \in \{1, \dots, N\}$, $k \geq j+1$ and $k = j = N$

$$\langle \Psi^{j,0}(T), \Phi_{k,V}(T)\rangle = i\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t,$$

    solve a trigonometric moment problem (Conditions **(C_3)** and **(C_4)**).
  - Choice of η sufficiently small $\Longrightarrow$ controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$.

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\dots$		

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
  - For $\eta = 0$:

$$\langle \Psi^{j,0}(T), \Phi_{j,V}(T)\rangle \rightsquigarrow \langle \mu\varphi_{j,V}, \varphi_{j,V}\rangle \int_0^T v(t)\mathrm{d}t, \quad \forall j \in \{1,\dots,N\}.$$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \ldots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V} \rangle$	1	2	$\ldots$	N	$\ldots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\ldots$		

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
  - For $\eta = 0$:
    $$\langle \Psi^{j,0}(T), \Phi_{j,V}(T) \rangle \rightsquigarrow \langle \mu\varphi_{j,V}, \varphi_{j,V} \rangle \int_0^T v(t)\mathrm{d}t, \quad \forall j \in \{1, \ldots, N\}.$$
  - For $\eta > 0$:
    $$\langle \Psi^{j,\eta}(T), \Phi_{j,V}(T) \rangle \rightsquigarrow \int_0^T v(t) \langle \mu\psi_{ref}^{j,\eta}(t), \psi_{ref}^{j,\eta}(t) \rangle \mathrm{d}t, \quad \forall j \in \{1, \ldots, N\}.$$
    Independence condition on $\langle \mu\psi_{ref}^{j,\eta}(t), \psi_{ref}^{j,\eta}(t) \rangle$ in the construction of ψ_{ref}^{η}.

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$		$\ddots$	$\ddots$		$\ddots$
$\Psi^{N,\eta}$					

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
- Invariants

$$\langle \Psi^{j,\eta}, \psi_{ref}^{k,\eta}(t)\rangle + \overline{\langle \Psi^{k,\eta}, \psi_{ref}^{j,\eta}(t)\rangle} = 0, \quad \forall\, 1 \leq k < j \leq N.$$

Global exact controllability under favourable hypotheses

$V, \mu \in H^4((0,1),\mathbb{R})$ such that

(C₆) for any $j \in \mathbb{N}^*$, $\exists c_j > 0$;

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

(C₇) $\{1, (\lambda_{j,V})_{j\in\mathbb{N}^*}\}$ are rationally independent : $\forall M \in \mathbb{N}^*$, $\forall \boldsymbol{r} \in \mathbb{Q}^{M+1}\backslash\{\mathbf{0}\}$,

$$r_0 + \sum_{j=1}^{M} r_j \lambda_{j,V} \neq 0.$$

Conditions $(\mathbf{C_6})$-$(\mathbf{C_7})$ $\Longrightarrow$ Conditions $(\mathbf{C_1})$-$(\mathbf{C_5})$, for any $N \in \mathbb{N}^*$.

Theorem

Let $N \in \mathbb{N}^*$. Under Conditions $(\mathbf{C_6})$-$(\mathbf{C_7})$, for any unitarily equivalent vectors $\psi_0, \psi_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$, there are $T > 0$, $u \in L^2((0,T),\mathbb{R})$ such that

$$\psi(T, \psi_0, u) = \psi_f.$$

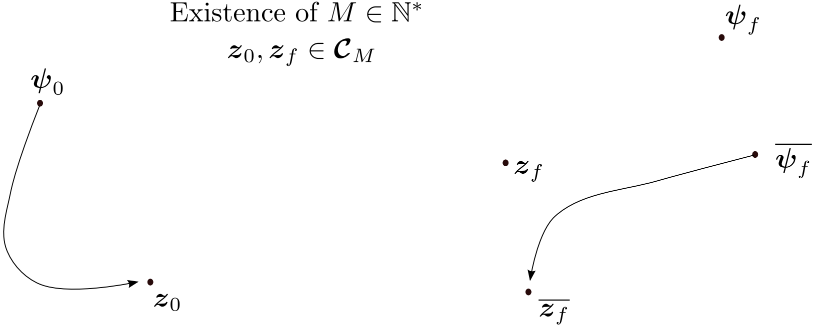
Sketch of proof

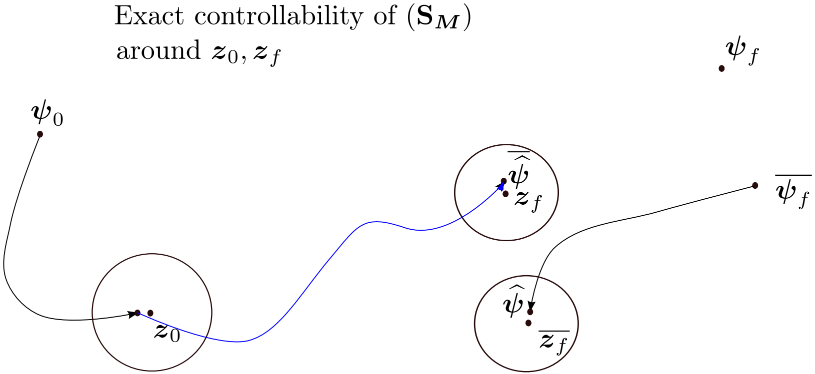
Sketch of proof

Global approximate controllability

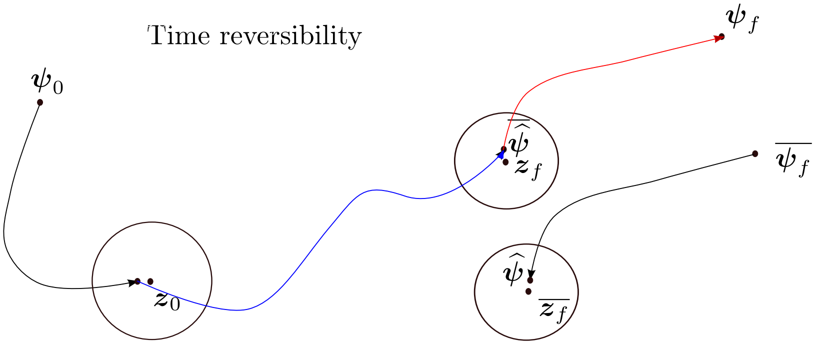
Existence of $M \in \mathbb{N}^*$

$\boldsymbol{z}_0, \boldsymbol{z}_f \in \boldsymbol{\mathcal{C}}_M$




Exact controllability of $(\mathbf{S}_{\boldsymbol{M}})$
around $\boldsymbol{z}_0, \boldsymbol{z}_f$
$\boldsymbol{\psi}_f$
$\boldsymbol{\psi}_0$
$\overline{\widehat{\boldsymbol{\psi}}}$
$\boldsymbol{z}_f$
$\overline{\boldsymbol{\psi}_f}$
$\boldsymbol{z}_0$
$\widehat{\boldsymbol{\psi}}$
$\overline{\boldsymbol{z}_f}$

Sketch of proof



Dealing with an arbitrary potential I

$V \in H^4((0,1),\mathbb{R})$ arbitrary

$$\begin{cases} i\partial_t \psi^j = -\left(\partial^2_{xx} + V(x) + \mu(x)\right)\psi^j - u(t)\mu(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \end{cases} \qquad (\tilde{\mathbf{S}}_{\boldsymbol{N}})$$

Link between propagators of $(\mathbf{S}_{\boldsymbol{N}})$ and $(\tilde{\mathbf{S}}_{\boldsymbol{N}})$:

$$\tilde{\psi}(T,\psi_0,u) = \psi(T,\psi_0,u-1).$$

- $\mathcal{Q}_V$: set of $\mu \in H^4((0,1),\mathbb{R})$ such that Conditions $(\mathbf{C_6})$ and $(\mathbf{C_7})$ are satisfied for V replaced by $V+\mu$ i.e.

$$\forall j \in \mathbb{N}^*,\ \exists c_j > 0;\ |\langle \mu\varphi_{j,V+\mu}, \varphi_{k,V+\mu}\rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

$$\{1, (\lambda_{j,V+\mu})_{j\in\mathbb{N}^*}\} \text{ are rationally independent.}$$

- $\mu \in \mathcal{Q}_V$: global exact controllability of $(\tilde{\mathbf{S}}_{\boldsymbol{N}})$ in $\boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}$.

Dealing with an arbitrary potential II

- Assume $\mu \in \mathcal{Q}_V$. Let $\psi_0, \psi_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$.
  Let $u_1 \in H^1((0,1),\mathbb{R})$ with $u_1(0) = 0$, $u_1(1) = -1$. Then,

$$\widetilde{\psi}_0 := \psi(1, \psi_0, u_1), \quad \overline{\widetilde{\psi}_f} := \psi(1, \overline{\psi_f}, u_1) \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}.$$

  - Reaching the 'right space' : $\psi_0 \quad \overset{1,u_1}{\rightsquigarrow} \quad \widetilde{\psi}_0$, for $(\mathbf{S}_N)$,
  - Global exact controllability of $(\tilde{\mathbf{S}}_N)$: $\exists \widetilde{T} > 0$, $\exists \tilde{u} \in L^2((0,\widetilde{T}),\mathbb{R})$ such that

$$\widetilde{\psi}_0 \quad \overset{\widetilde{T},\tilde{u}}{\rightsquigarrow} \quad \widetilde{\psi}_f, \qquad \text{for } (\tilde{\mathbf{S}}_N),$$

  i.e.

$$\widetilde{\psi}_0 \quad \overset{\widetilde{T},\tilde{u}-1}{\rightsquigarrow} \quad \widetilde{\psi}_f, \qquad \text{for } (\mathbf{S}_N).$$

  - Time reversibility : $\widetilde{\psi}_f \quad \overset{1,u_1(1-\cdot)}{\rightsquigarrow} \quad \psi_f$, for $(\mathbf{S}_N)$.
- $\mathcal{Q}_V$ is residual in $H^4((0,1),\mathbb{R})$.

Open problems and perspectives

Conclusion

- Global exact controllability
- Arbitrary number of equations
- No restriction on the potential

Open problems

- Large time : Lyapunov strategy, rotation (Kronecker diophantine approximation), compactness argument.
- Optimal spaces : $\boldsymbol{H}^4_{(V)}$, $\boldsymbol{H}^3_{(V)}$ (Lyapunov strategy in infinite dimension)

Open problems and perspectives

Conclusion

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Open problems

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Thank you for your attention.

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