A globally convergent algorithm to solve an inverse problem for waves with potential.

Maya de Buhan
Joint work with Lucie Baudouin and Sylvain Ervedoza

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Workshop: PDE, Optimal design and Numerics
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Outline

1. Introduction
   - The wave equation with potential
   - The inverse problem
   - Classical uniqueness and stability result
   - Classical resolution method
2. A Carleman estimate
3. Our algorithm
4. Numerical issues
5. Conclusion

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with potential

$$\begin{cases} \partial_t^2 w - \Delta w + pw = g, & \text{in } \Omega \times (0,T), \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega. \end{cases} \tag{1}$$

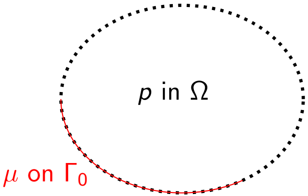
Here, w denotes the amplitude of the waves, p is a potential supposed to be in $L^\infty(\Omega)$, g is a source term for instance in $L^2(\Omega \times (0,T))$ and (w_0, w_1) are the initial data lying in $H_0^1(\Omega) \times L^2(\Omega)$.
D'Alembertian operator:

$$\Box = \partial_t^2 - \Delta.$$

Given the source term g and the initial data (w_0, w_1), can we determine the unknown potential $p(x)$, $\forall x \in \Omega$, from the additional knowledge of the flux

$$\mu = \partial_\nu w, \qquad \text{on } \Gamma_0 \times (0, T),$$

where Γ_0 is a part of $\partial\Omega$?

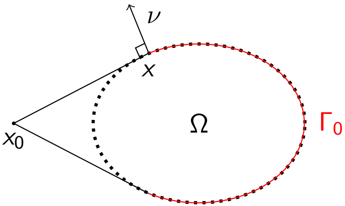


Uniqueness ? Stability ? Numerical resolution ?

Theorem (Baudouin-Puel)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0)\cdot \nu(x) \geq 0\},$$



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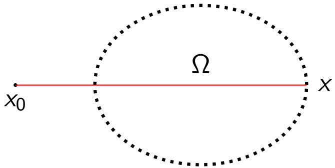

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Then for $m > 0$, *there exists a positive constant* $M = M(\Omega, T, x_0, m)$ *such that for all* p *and* q *in* $L^\infty_m(\Omega) = \{p \in L^\infty(\Omega), \|p\|_{L^\infty(\Omega)} \leq m\}$*:*

$$\|p - q\|_{L^2(\Omega)} \leq M \|\partial_t\left(\partial_\nu w[p] - \partial_\nu w[q]\right)\|_{L^2(\Gamma_0\times(0,T))},$$

where $w[p]$ *and* $w[q]$ *denote the corresponding solutions of* (1).

A classical method for solving this inverse problem consists in minimizing

$$J(q) = \left\| \partial_t \left(\partial_\nu w[q] - \mu \right) \right\|^2_{L^2(\Gamma_0 \times (0,T))},$$

where $\mu = \partial_\nu w[p]$ is the observation.

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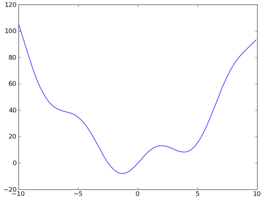

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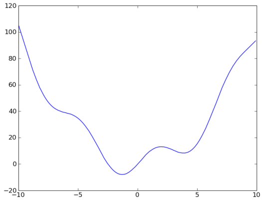


We propose a new algorithm to solve the inverse problem and prove its global convergence. It is based on Carleman estimates.

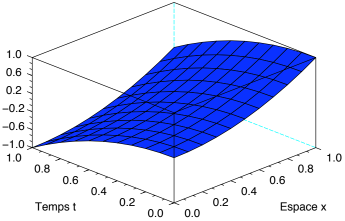
Outline

1. Introduction
2. A Carleman estimate
   - Carleman weight function for waves
   - An estimate with pointwise term in time 0
3. Our algorithm
4. Numerical issues
5. Conclusion

We define, for $(x,t) \in \Omega \times (0,T)$,

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0, \quad \text{and} \quad \varphi(x,t) = e^{\lambda \psi(x,t)},$$

where $\beta > 0$, $\lambda > 0$ and $C_0 > 0$ is chosen such that $\psi \geq 1$ in $\Omega \times (0,T)$.



Function ψ for $x_0 = 0$, $\beta = 1$ and $C_0 = 0$

$$\psi(t) \leq \psi(0), \quad \forall t \in (0,T).$$

Theorem

Assume the **geometric and time conditions**. *Suppose* $\beta \in (0,1)$ *and*

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Then with $m > 0$, *there exists a constant* $M > 0$ *such that for all* s *and* λ *large enough, for all* $q \in L^\infty_m(\Omega)$ *and for all* $z \in L^2(0,T; H^1_0(\Omega))$ *satisfying* $\Box z + qz \in L^2(\Omega \times (0,T))$, $\partial_\nu z \in L^2(\Gamma_0 \times (0,T))$ *and* $z(0) = 0$ *in* Ω *:*

$$\underbrace{s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx}_{\text{initial energy}}$$

$$\leq M \underbrace{\int_0^T \int_\Omega e^{2s\varphi} |\Box z + qz|^2 \, dxdt}_{\text{source}} + M \, s \underbrace{\int_0^T \int_{\Gamma_0} e^{2s\varphi} \, |\partial_\nu z|^2 \, d\gamma dt}_{\text{observations}}.$$

Outline

1. Introduction
2. A Carleman estimate
3. Our algorithm
   - Iterative loop
   - Convergence result
   - Proof of the convergence result
4. Numerical issues
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Our algorithm
Iterative loop

Initialization: $q^0 = 0$.

Iteration: Given q^k,

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Iteration: Given q^k,

1 - Compute $w[q^k]$ the solution of

$$\left\{\begin{array}{ll} \partial_t^2 w - \Delta w + q^k w = g, & \text{in } \Omega \times (0,T), \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{array}\right.$$

and set $\mu^k = \partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[p] \right)$ on $\Gamma_0 \times (0,T)$.

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2 - Introduce the functional

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,$$

on the space $\mathcal{T}^k = \{z \in L^2(0,T; H_0^1(\Omega)), z(0) = 0, \Box z + q^k z \in L^2(\Omega \times (0,T)), \partial_\nu z \in L^2(\Gamma_0 \times (0,T))\}$.

Theorem

Assume the **geometric and time conditions***. Then, for all* $s > 0$ *and* $k \in \mathbb{N}$*, the functional* J_0^k *is continuous, strictly convex and coercive on* $\mathcal{T}^k$ *endowed with a suitable weighted norm.*

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3 - Let Z^k be the unique minimizer of the functional J_0^k, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0},$$

where w_0 is the initial condition of (1).

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4 - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \operatorname{sign}(q)m, & \text{if } |q| \geq m. \end{cases}$$

Theorem

Assume **the geometric and time conditions, the regularity assumption and the positivity condition**. *Let* $p \in L^\infty_m(\Omega)$. *There exists a constant* $M > 0$ *such that for all* s *large enough and for all* $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)}(q^k - p)^2\,dx \le \left(\frac{M}{\sqrt{s}}\right)^k \int_\Omega e^{2s\varphi(0)} p^2\,dx.$$

In particular, if s *is large enough,* q^k *converges toward* p *when* k *goes to infinity.*

The algorithm is based on the fact that $z^k = \partial_t \left(w[q^k] - w[p] \right)$ solves

$$\begin{cases} \partial_t^2 z^k - \Delta z^k + q^k z^k = g^k, & \text{in } \Omega \times (0, T), \\ z^k = 0, & \text{on } \partial\Omega \times (0, T), \\ z^k(0) = 0, \quad \partial_t z^k(0) = z_1^k, & \text{in } \Omega, \end{cases}$$

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Moreover, by definition,

$$\mu^k = \partial_\nu z^k \text{ on } \Gamma_0 \times (0,T),$$

and we notice that z^k is the unique minimizer of the functional:

$$J_{g^k}^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z - g^k|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,$$

Let us write the Euler Lagrange equations satisfied by:
- Z^k minimizer of J_0^k:

$$\nabla J_0^k(Z^k, z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box Z^k + q^k Z^k)(\Box z + q^k z)\, dxdt$$
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for all $z \in \mathcal{T}^k$.

Applying these equations to $z = Z^k - z^k$ and subtracting the two identities, we obtain:

$$\int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt \\ = \int_0^T \int_\Omega e^{2s\varphi} g^k (\Box z + q^k z) \, dxdt.$$

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This implies ($2ab \leq a^2 + b^2$) that

$$\frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt \leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |g^k|^2 \, dxdt.$$

The left hand side precisely is the right hand side of the Carleman estimate. Hence, we deduce:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t z(0)|^2\,dx \le M\int_0^T\int_\Omega e^{2s\varphi}|g^k|^2\,dxdt,$$

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Moreover

$$\begin{aligned}
&\partial_t Z^k(0) = (\tilde{q}^{k+1} - q^k)w_0, \text{ by definition of } \tilde{q}^{k+1},\\
&\partial_t z^k(0) = z_1^k = (p - q^k)w_0,\\
&g^k = (p - q^k)\partial_t w[p].
\end{aligned}$$

Therefore, since $\varphi(t) \leq \varphi(0)$ for all $t \in (0, T)$ we have:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |w_0|^2 (\tilde{q}^{k+1} - p)^2 \, dx$$
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Using the positivity condition on w_0 and the fact that

$$|q^{k+1} - p| = |T_m(\tilde{q}^{k+1}) - T_m(p)| \le |\tilde{q}^{k+1} - p|$$

because T_m is Lipschitz and $T_m(p) = p$, we immediately deduce

$$\int_\Omega e^{2s\varphi(0)} (q^{k+1} - p)^2 \, dx \le \left(\frac{M}{\sqrt{s}} \right)^{k+1} \int_\Omega e^{2s\varphi(0)} (q^0 - p)^2 \, dx.$$

$\square$

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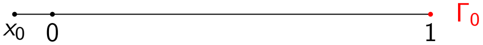
2. A Carleman estimate
3. Our algorithm
4. Numerical issues
   - Discretization of the problem
   - Discrete Carleman estimate
   - Examples in 1D
   - Example in 2D
5. Conclusion

Numerical issues

Discretization of the problem

- $\Omega = [0, 1]$, $x_0 = -0.1$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.5$, $\lambda = 0.1$, $s = 1$



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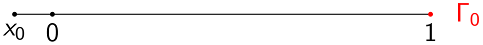
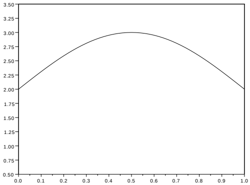


- finite differences in space $h = 0.02$, explicit Euler scheme in time $\tau = 0.01$
- $g = 0$, $w_1 = 0$, $w_0(x) = \sin(x\pi)$



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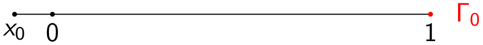

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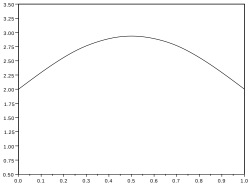

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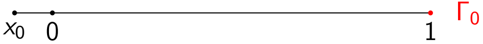

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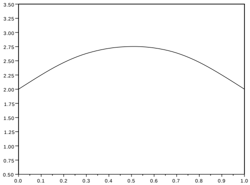

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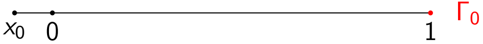

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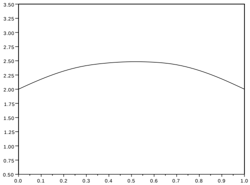

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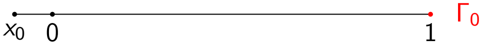

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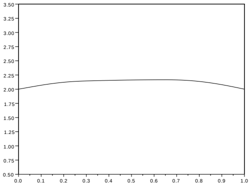

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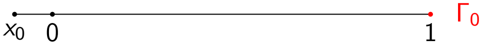

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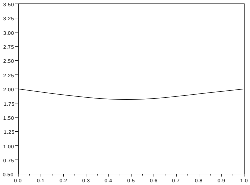

Numerical issues

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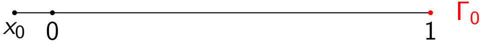

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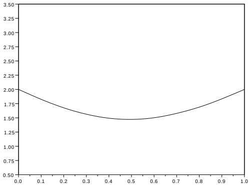

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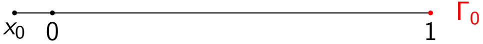

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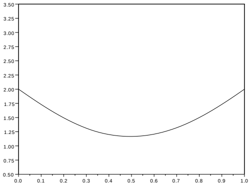

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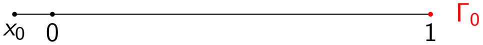

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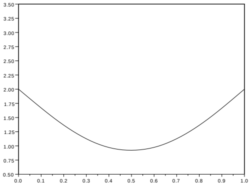

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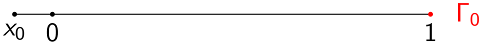

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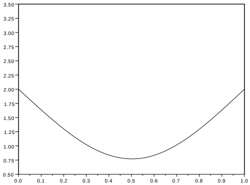

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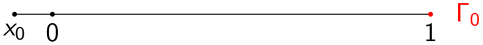

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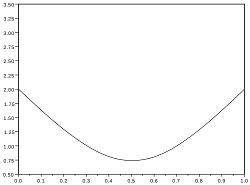

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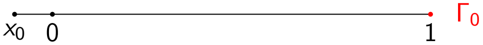

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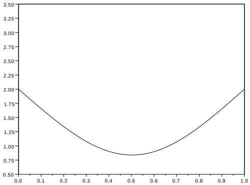

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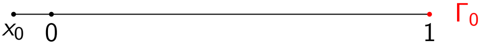

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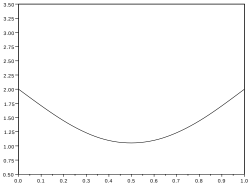

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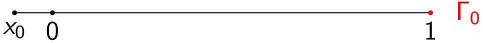

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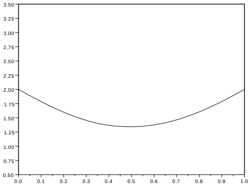

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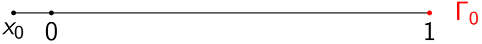

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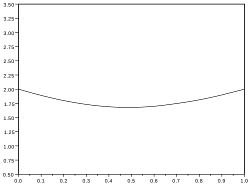

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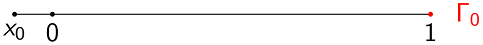

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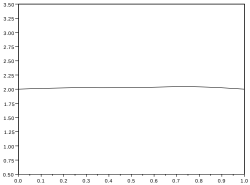

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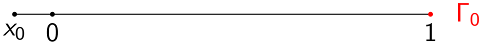

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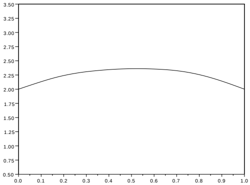

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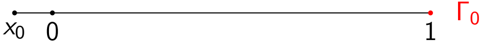

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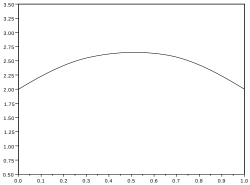

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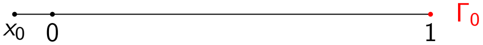

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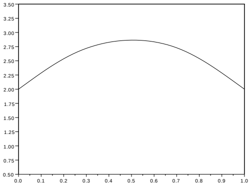

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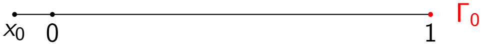

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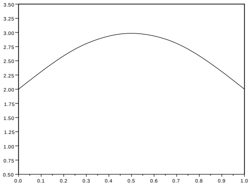

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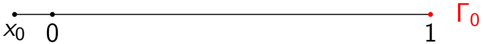

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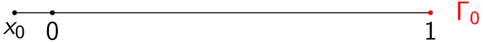

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- additional noise on the observation data:

$$\mu = (1 + \alpha\,\mathrm{Normal}(0, 0.5))\,\mu, \quad \alpha \geq 0.$$

Numerical issues

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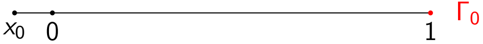

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- minimization of J by a conjugate gradient

Baudouin-Ervedoza

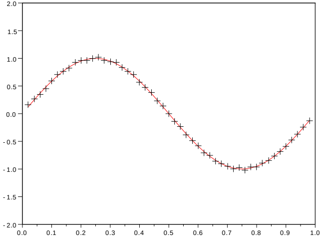
A regularization term must be add to make the Carleman estimates uniform with respect to the discretization parameter h.

$$J^k_{0,h}(z_h) = \int_0^T \int_0^1 e^{2s\varphi} |\Box_h z_h + q^k z_h|^2 dt + s\int_0^T e^{2s\varphi(t,1)} \left|\partial_h^- z_h(t,1) - \mu^k\right|^2 dt$$
$$+ s\int_0^T \int_0^1 e^{2s\varphi} |h\partial_h^+ \partial_t z_h|^2\, dt.$$

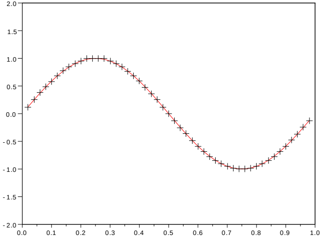
This term is needed due to spurious waves created by the discretization process (Ervedoza-Zuazua).

$$p(x) = \sin(2\pi x)$$

without regularization



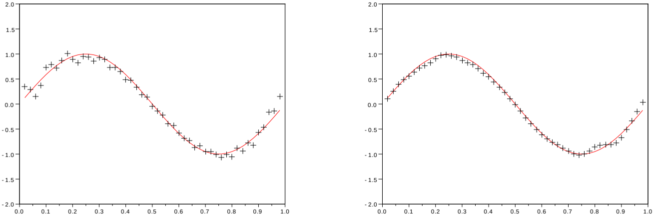
with regularization



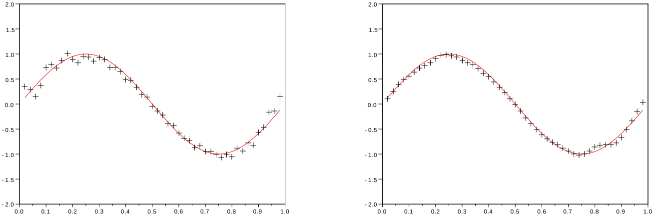
noise $\alpha = 0\%$

Numerical issues

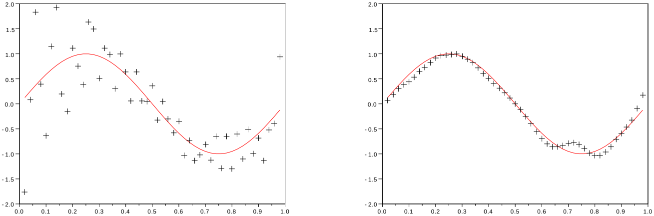
Examples in 1D



2% noise

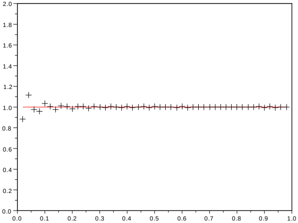


2% noise

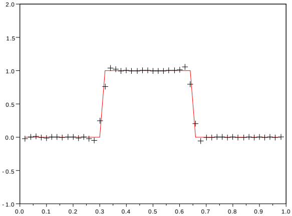


10% noise

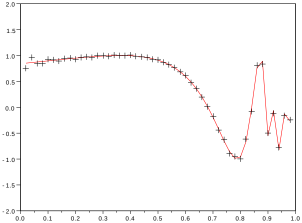
Numerical issues
Examples in 1D



$$p(x) = 1$$



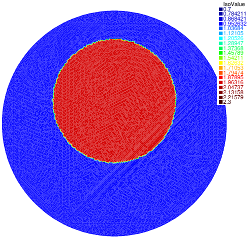
$$p(x) = 0 \text{ or } 1$$



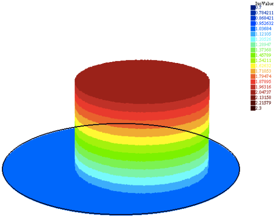
$$p(x) = \sin(1 - 1/x)$$

Numerical issues

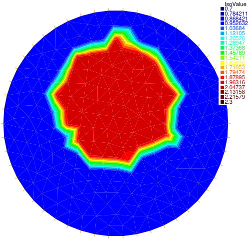
Example in 2D



p in 2D-view

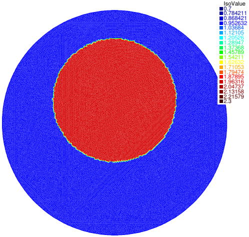


p in 3D-view

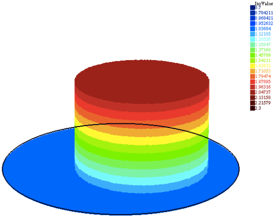


Numerical issues

Example in 2D



p in 2D-view



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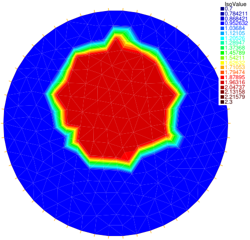
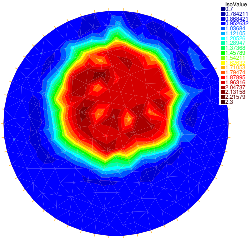


p_h



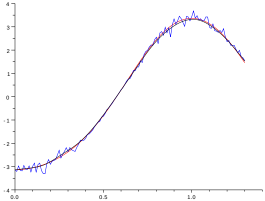
q_h

Outline

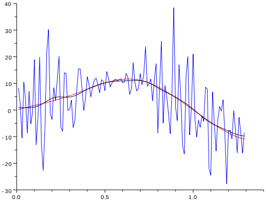
1. Introduction
2. A Carleman estimate
3. Our algorithm
4. Numerical issues
5. Conclusion
   - Drawbacks of the method
   - Prospects

- We have to derive in time the observation flux: $\partial_t(\partial_\nu w[p])$

observation at $x = 1$



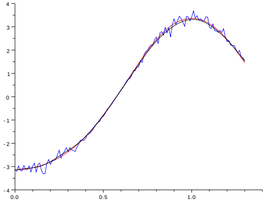
time derivative



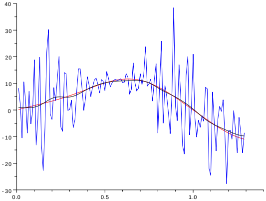
$\Longrightarrow$ we regularize the signal by convolutions with a gaussian.

- We have to derive in time the observation flux: $\partial_t(\partial_\nu w[p])$

observation at $x = 1$



time derivative



$\Longrightarrow$ we regularize the signal by convolutions with a gaussian.

- For $\lambda = 1$ and $s = 3$, $\max(\exp(2s\varphi))/\min(\exp(2s\varphi)) = 10^{110}$!
  $\Longrightarrow$ we tried to work with the conjugate variable $\tilde{z} = e^{s\varphi}z$,
  $\Longrightarrow$ we are trying to change the weights (coming soon...hopefully).

- Recovery of the wave propagation speed $c(x)$

$$\begin{cases} \partial_t^2 w - \nabla \cdot (c^2 \nabla w) = g, & \text{in } \Omega \times (0,T), \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega. \end{cases}$$

Application to medical imaging or radar.

$\star\star\star$

Back

Theorem

Assume the **geometric and time conditions**. *Define the weight functions* φ *with* $\beta \in (0,1)$ *being such that*

$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$

Back

Theorem

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$$\beta T > \sup_{x \in \Omega} |x - x_0|.$$

Then there exist a constant $M > 0$ *such that for all* s *and* λ *large enough:*

$$s \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(|\partial_t z|^2 + |\nabla z|^2 \right) \, dxdt + s^3 \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |z|^2 \, dxdt \\ \leq M \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\Box z|^2 \, dxdt + Ms \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi} \left| \partial_\nu z \right|^2 \, d\gamma dt,$$

for all $z \in L^2((-T,T); H_0^1(\Omega))$ *satisfying* $\Box z = \partial_t^2 z - \Delta z \in L^2(\Omega \times (-T,T))$ *and* $\partial_\nu z \in L^2(\partial\Omega \times (-T,T))$.

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- Introduce the conjugate operator:

$$Pw = e^{s\varphi}\Box(e^{-s\varphi}w) = \partial_t^2 w - \Delta w + s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)w$$
$$-\, s(\partial_t^2\varphi - \Delta\varphi)w + s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)w - 2s\partial_t w\partial_t\varphi + 2s\nabla w\cdot\nabla\varphi.$$

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- Show that the terms in $|w|^2$, $|\nabla w|^2$ and $|\partial_t w|^2$ can be bounded by below when s is large enough.
- Finally, come back to the initial variable z and absorb the residual terms thanks to the weights s.

Proofs

Proof of the estimate with pointwise term in time 0

- Since $z(0) = 0$ in Ω, we can extend the function z by $z(t) = z(-t)$ for $t \in (-T, 0)$ and apply the Carleman estimate to this extended function z. Of course, since each term is odd or even, the integrals on $(-T, T)$ simply are twice the integrals on $(0, T)$.

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- The Carleman estimate for the operator $\Box + p$ with $p \in L^\infty_{\leq m}(\Omega)$ is a direct consequence noticing that in $\Omega \times (0, T)$,

$$|\Box z|^2 \leq 2|\Box z + pz|^2 + 2\|p\|^2_{L^\infty(\Omega)}|z|^2 \leq 2|\Box z + pz|^2 + 2m^2|z|^2.$$

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Then choosing s large enough, one can absorb the term

$$2Mm^2 \int_0^T \int_\Omega e^{2s\varphi}|z|^2 \, dxdt,$$

by the left hand side.

Proofs

Proof of the estimate with pointwise term in time 0

- We set

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$$\begin{aligned}\int_{-T}^{0}\int_\Omega P_1 w\, \partial_t w \, dxdt &= \int_{-T}^{0}\int_\Omega (\partial_t^2 w - \Delta w + s^2 w(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\, \partial_t w \, dxdt \\ &\geq \frac{1}{2}\int_\Omega |\partial_t w(0)|^2 \, dx - Ms^2 \int_{-T}^{0}\int_\Omega |w|^2 \, dxdt,\end{aligned}$$

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$$\begin{aligned}\int_{-T}^{0}\int_{\Omega} P_1 w\,\partial_t w\,dxdt &= \int_{-T}^{0}\int_{\Omega}(\partial_t^2 w - \Delta w + s^2 w(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\,\partial_t w\,dxdt \\ &\geq \frac{1}{2}\int_{\Omega}|\partial_t w(0)|^2\,dx - Ms^2\int_{-T}^{0}\int_{\Omega}|w|^2\,dxdt,\end{aligned}$$

implying in particular, by Cauchy-Schwarz, that

$$\begin{aligned}s^{1/2}\int_{\Omega}|\partial_t w(0)|^2\,dx \leq \int_{-T}^{T}\int_{\Omega}|P_1 w|^2\,dxdt + s\int_{-T}^{T}\int_{\Omega}|\partial_t w|^2\,dxdt \\ + Ms^{5/2}\int_{-T}^{T}\int_{\Omega}|w|^2\,dxdt.\end{aligned}$$

We can use the Carleman estimate on w and, bounding each term from above and from below, we get:

$$s^{1/2}\int_\Omega |\partial_t w(0)|^2\,dx + s\int_{-T}^{T}\int_\Omega (|\partial_t w|^2 + |\nabla w|^2)\,dxdt + s^3\int_{-T}^{T}\int_\Omega |w|^2\,dxdt$$
$$+\int_{-T}^{T}\int_\Omega |P_1 w|^2\,dxdt \ \leq M\int_{-T}^{T}\int_\Omega |Pw|^2\,dxdt + Ms\int_{-T}^{T}\int_{\Gamma_0} |\partial_\nu w|^2\ d\gamma dt.$$

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Coming back to the initial variable z, we obtain

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t z(0)|^2\,dx$$
$$\leq M\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\Box z|^2\,dxdt + Ms\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu z|^2\ d\gamma dt.$$

$\square$

Back