The cost of controlling the Stokes system

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Given $u_0 \in L^2(\Omega)$, it is well known that there exists $f \in L^2(\omega \times (0,T))$ such that the associated solution v to the heat equation

$$\left| \begin{array}{l} v_t - \Delta u = f 1_\omega \quad \text{in} \quad \Omega \times (0,T), \\ v = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \\ v(0) = v_0 \quad \text{in} \quad \Omega \end{array} \right. \tag{1}$$

satisfies:

$$v(T) = 0. \tag{2}$$

Moreover, one also has the following estimate:

$$||f 1_\omega||_{L^2(Q)} \leq C_h ||v_0||_{L^2(\Omega)}, \tag{3}$$

for a constant C_h, the *cost of controllability for the heat equation*, of the form $e^{C(\Omega,\omega)(1+1/T)}$.

The heat equation has a cost of controllability of order $e^{C/T}$.

The main reason for the form of the of the constant C_h is due to the fact that the fundamental solution of the heat equation in $\mathbb{R}^N$ is given by

$$\Phi(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{4}$$

Consider now the Stokes system:

$$\left| \begin{array}{l} y_t - \Delta y + \nabla p = g 1_\omega \ \text{ in } \Omega \times (0,T), \\ div \ y = 0 \ \text{ in } \Omega \times (0,T), \\ y = 0, \ \text{ on } \partial\Omega \times (0,T), \\ y(0) = y_0 \ \text{ in } \ \Omega, \end{array} \right. \tag{5}$$

We also have that, given $y_0 \in L^2(\Omega)$ with $div \ y_0 = 0$, there exists $g \in L^2(\omega \times (0,T))$ such that the associated solution y_0 to (5) satisfies:

$$y(T) = 0.$$

Nevertheless, for the Stokes system one has

$$||g1_\omega||_{L^2(Q)} \leq C_S||y_0||_{L^2(\Omega)}, \tag{6}$$

for a constant C_S, the *cost of controllability for the Stokes equation*, of the form $e^{C(\Omega,\omega)(1+1/T^4)}$.

The Stokes system has a cost of controllability of order e^{C/T^4}.

For $N = 2$, the fundamental solution of the Stokes system can be written as

$$\Gamma(z; x, t) = -\Delta\Psi(z; x, t)I + Hess\Psi(z; x, t),$$

where, for each $z \in \mathbb{R}^2$, $t > 0$, Ψ satisfies

$$-\Delta\Psi(z; x, t) = \Phi(z - x, t).$$

For $N = 3$ we have a similar (much more complicated!) formula.

OPEN PROBLEM

Are the cost of the controllability for the heat and the Stokes equation of the same order?