Nonlinear approximation theory for Boltzmann equation

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Outline

Spectral Methods

$$\frac{\partial f}{\partial t} = Q(f, f), (v, t) \in (-R, R) \times \mathbb{R}_+,$$

where Q is a bilinear operator and f is periodic on $(-R, R)$.

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$$f = \sum_{k=-\infty}^{\infty} a_k \exp\left(-i\frac{\pi}{R}k.v\right) = \sum_{k=-\infty}^{\infty} a_k \Phi_{N,k},$$

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$$f_N = \sum_{k=-N}^{N} a_k \Phi_{N,k}.$$

Spectral Methods

ODEs system

$$\sum_{k=-N}^{N} \frac{\partial a_k}{\partial t} \Phi_k = Q\left(\sum_{i=-N}^{N} a_i \Phi_i, \sum_{j=-N}^{N} a_j \Phi_j\right)$$

$$= \sum_{i,j=-N}^{N} a_i a_j Q\left(\Phi_i, \Phi_j\right)$$

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$$= \sum_{i,j=-N}^{N} a_i a_j Q\left(\Phi_i, \Phi_j\right)$$

Solve $\{a_i\}_{i=1}^{N}$

$$\frac{\partial a_k}{\partial t} = \sum_{i,j=-N}^{N} a_i a_j < Q\left(\Phi_i, \Phi_j\right), \Phi_k >,$$

The Boltzmann Equation

$$\partial_t f + v.\nabla_x f = Q(f,f), t \geq 0, x \in \mathbb{T}^d, v \in \mathbb{R}^d,$$

Q is the quadratic Boltzmann collision operator, defined by

$$Q(f,f) = \int_{S^{N-1}} \int_{\mathbb{R}^N} (f' f'_* - f f_*) B(|v - v_*|, \sigma) d\sigma dv_*,$$

where $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f'_*(t,x,v'_*)$, $f' = f(t,x,v')$ in which

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma; v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \sigma \in \mathbb{S}^{N-1}.$$

This is the so called "σ-representation" of the Boltzmann collision operator.

The Boltzmann Equation

Just keep in mind...

$$\partial_t f = Q(f, f), t \geq 0, v \in \mathbb{R}^d,$$

Q is a bilinear form.

Properties of the solution

In one word...

f is 'similar' to a Gaussian.

Properties of the solution: the equilibrium state

The Equilibrium State: the Maxwellian

$$\mu(\rho, u, T)(v) = \rho(2\pi T)^{-d/2}\exp(-|u-v|^2/2),$$

where ρ, u, T are the density, mean velocity and temperature of the gas

$$\rho = \int_{\mathbb{R}^N} f_0(v)dv, u = \frac{1}{\rho}\int_{\mathbb{R}^N} vf_0(v)dv, T = \frac{1}{N\rho}\int_{\mathbb{R}^N} |u-v|^2 f_0(v)dv.$$

Desvillettes-Villani 2004

$$\|f(t)-\mu\|_{L^2} = O(t^{-\infty}).$$

Properties of the solution: Maxwellian lower bounds

Carleman 1933, Pulvirenti and Wennberg 1997

If the initial condition f_0 satisfies

$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty,$$

then

$$\forall t_0 > 0, \exists K_0 > 0, \exists A_0 > 0; t \geq t_0$$
$$\Longrightarrow \forall v \in \mathbb{R}^3, f(t,v) \geq K_0 \exp(-A_0|v|^2).$$

Properties of the solution: Propagation of polynomial moments

Povzner 1962, Desvillettes 1993, Wennberg 1997, Mischler and Wennberg 1999

If the initial condition f_0 satisfies

$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty,$$

then

$$\forall s \geq 2, \forall t_0 > 0, \sup_{t \geq 0} \int_{\mathbb{R}^3} f(t,v)(1+|v|^s) < +\infty.$$

Properties of the solution: Maxellian upper bound

Bobylev, Gamba and Panferov 2004, Gamba, Panferov and Villani 2009, Alonso, Cañizo, Gamba and Mouhot 2013

Assume that the initial data satisfies for some $s \in [\gamma, 2]$

$$\int_{\mathbb{R}^3} f_0(v)(a_0|v|^s)dv \leq C_0,$$

then there are some constants $C, a > 0$ such that

$$\int_{\mathbb{R}^3} f(t,v)\exp(a|v|^s)dv < C_0.$$

Fourier Spectral Methods

Development

- Introduced by Perthame and Pareschi (1996) $\Longrightarrow$ developed by Pareschi, Villani, Toscani...
- Revisited in Mouhot's thesis under Villani (2004) $\Longrightarrow$ developed by Mouhot, Filbet, Pareschi...
- Theory: Mouhot-Filbet (2011)
- Extended to Discrete Velocity Methods in Rey's thesis under Mouhot and Filbet (2012).

Fourier Spectral Methods

Boltzmann equation

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Fourier Spectral Methods

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$$\frac{\partial f}{\partial t} = Q(f, f), (v, t) \in \mathbb{R}^3 \times \mathbb{R}.$$

In order to use the Spectral Method...

$$\frac{\partial f}{\partial t} = Q^R(f, f), (v, t) \in (-R, R)^3 \times \mathbb{R},$$

where Q^R is the truncated collision operator and f is periodic on $(-R, R)^3$.

$$f_N = \sum_{k=-(N,N,N)}^{(N,N,N)} a_k \exp\left(-i\frac{\pi}{R}k.v\right) = \sum_{k=-N}^{N} a_k \Phi_{N,k}.$$

Fourier Spectral Methods

ODEs system

$$\sum_{k=-N}^{N} \frac{\partial a_k}{\partial t} \Phi_{N,k} = Q^R \left(\sum_{i=-N}^{N} a_i \Phi_{N,i}, \sum_{j=-N}^{N} a_j \Phi_{N,j} \right)$$

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ODEs system

$$\frac{\partial a_k}{\partial t} = \sum_{i,j=-N}^{N} a_i a_j < Q^R \left(\Phi_{N,i}, \Phi_{N,j} \right), \Phi_{N,k} >$$

Fourier Spectral Methods

Advantages

- Easy to implement
- Fast, low cost

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Disadvantages

- Spreading support effect (Wennberg): if f supported in $(-R, R)^3 \Longrightarrow Q(f, f)$ supported in $(-\sqrt{2}R, \sqrt{2}R)^3$.
- Equilibrium state of the periodized equation is not the same with the original equation

$$\|f^{per}(t) - C\|_{L^2} = O(\exp(-t)).$$

$$\|f(t) - \mu\|_{L^2} = O(t^{-\infty}).$$

- Physical properties of the solution are destroyed (positivity, propagation of moments,...)

Discrete Velocity Models

Discrete Velocity Models

They are based on a Cartesian grid in velocity and a discrete collision operator, which is a nonlinear system of conservation laws

$$\frac{\partial f_i}{\partial t} = Q_i(f,f), (x,t) \in \Omega \times \mathbb{R}, v_i \in V,$$

where Q_i is the discrete collision operator. The velocity set V is assume to be a part of the regular grid

$$\mathcal{Z}_\Delta = \Delta\mathbb{Z}^3 = \{\Delta(i_1, i_2, i_3) \mid (i_1, i_2, i_3) \in \mathbb{Z}^3\},$$

contained in a truncated set

$$\mathcal{V}_\Delta^R = \{\Delta(i_1, i_2, i_3) \mid (i_1, i_2, i_3) \in \mathbb{Z}^3; |i_1|, |i_2|, |i_3| < R\}.$$

Discrete Velocity Models

Historical notes

- Started by Carleman (1957)
- Existence results: Tartar (1976, 1977), Bony (1987, 1991)
- Weak convergence as $R \to \infty$ and $\Delta \to 0$ for some cases: Desvillettes, Mischler (1996, 1997), Panferov, Heintz (2002) based on Diperna-Lions Theory (averaging lemma)
- Consistency: Palczewsk, Schneideri, Bobylev (1997), Fainsilber, Kurlberg, Wennberg (2006)

$$\|Q_i(f,f) - Q(f,f)\| \to 0 \text{ on } (-R,R)^3$$

$\Longrightarrow$ **Fourier Spectral Methods is more consistent**

- Hundreds of numerical results

Discrete Velocity Models

Disadvantages

- Difficult to implement.
- No full, strong convergence proof. No error estimate: We do not know how large we should truncate the domain i.e. how we could choose R.
- Very expensive even for small R.
- No preservation of important properties of the Boltzmann equation.

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Fourier Spectral Methods $\rightarrow$ Discrete Velocity Methods: Rey's thesis under Mouhot and Filbet (July 2012).

Discrete Velocity Models

Compare FSMs and DVMs

- For DVMs, we have to remove binary collisions which spreads outside the bounded velocity space. This truncation breaks down the convolution structure of the collision operators.
- For FSMs, the convolution structure is perfectly preserved however we need to add nonphysical binary collisions by a periodized process.

The main idea

Problem

$$\frac{\partial f_1}{\partial t} = Q(f_1, f_1), (v, t) \in \mathbb{R}^3 \times \mathbb{R},$$

$$\frac{\partial f_2}{\partial t} = Q^R(f_2, f_2), (v, t) \in (-R, R)^3 \times \mathbb{R},$$

$$f_1 \backsim f_2.$$

The main idea

Absorbing boundary condition

$$\mathcal{L}u = 0 \text{ on } \mathbb{R}^3,$$

$$\begin{aligned} \mathcal{L}u &= 0 \text{ in } \Omega \\ \mathcal{B}u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The main idea

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The main idea

The change of variable mapping

$$\varphi : \mathbb{R}^3 \to (-1,1)^3,$$

$$\varphi(v) = (\varphi_1(v_1), \varphi_2(v_2), \varphi_3(v_3)) = \left(\frac{v_1}{1+|v|}, \frac{v_2}{1+|v|}, \frac{v_3}{1+|v|} \right),$$

the prize to pay: a weight of $< v >^{-4}$ in the norms used in the proofs.

The new formulation of Boltzmann

$$
\begin{aligned}
&\partial_t g(t,\bar{v}) \\
= \; & \int_{(-1,1)^3}\int_{\mathbb{S}^2} \frac{B(|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|,\sigma)}{(1-|\bar{v}_*|)^4} \\
&\times\left[g\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right. \\
&\times g\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) \\
&\left.-g(\bar{v})g(\bar{v}_*)\right]d\sigma d\bar{v}_*,
\end{aligned}
$$

The spectral algorithm

Expand g

$$g_N = \sum_{k=-N}^{N} a_k \Phi_{N,k}.$$

The spectral algorithm

Expand g

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The spectral algorithm

A horrible algorithm

$$
\begin{aligned}
\partial_t a_k = \int_{(-1,1)^3} \Phi_{N,k} \Bigg\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B} \\
\times \Bigg[\mathcal{C} \left(\sum_{l=0}^{N-1} a_l \Phi_{N,l} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right. \right. \\
\times \left(\sum_{l'=0}^{N-1} a_{l'} \Phi_{N,l'} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right) \\
- \left(\sum_{l=0}^{N-1} a_l \Phi_{N,l}(\bar{v}) \right) \left(\sum_{l'=0}^{N-1} a_{l'} \Phi_{N,l'}(\bar{v}_*) \right) \eta(\bar{v}_*) \Bigg] d\sigma d\bar{v}_* \Bigg\} d\bar{v},
\end{aligned}
$$

where $\mathcal{B}$ and $\mathcal{C}$ depend on $\bar{v}$, $\bar{v}_*$ and σ.

The spectral algorithm

A "nicer" algorithm

$$\partial_t a_k = \sum_{l,l'=0}^{N-1} a_l a_{l'} \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)\left[\Phi_{N,l}(\varphi(v'_*))\Phi_{N,l'}(\varphi(v')) - \Phi_{N,l}(\varphi(v_*))\Phi_{N,l'}(\varphi(v))\right]\Phi_{N,k}(\varphi(v))\,d\sigma\, dv_*\, dv,$$

$\Phi_{N,k}(\varphi(v))$: a nonlinear wavelet basis.

The spectral algorithm

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$$\partial_t a_k = \sum_{l,l'=0}^{N-1} a_l a_{l'} \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)\left[\Phi_{N,l}(\varphi(v_*'))\Phi_{N,l'}(\varphi(v')) - \Phi_{N,l}(\varphi(v_*))\Phi_{N,l'}(\varphi(v))\right]\Phi_{N,k}(\varphi(v))\,d\sigma\, dv_*\, dv,$$

$\Phi_{N,k}(\varphi(v))$: a nonlinear wavelet basis.

Ron Devore 2007

The fundamental problem of approximation theory is to resolve a complicated function, by simpler, easier to compute functions called "the approximants". The main idea of nonlinear approximation is that the approximants do not come from linear spaces but rather from nonlinear manifolds.

The spectral algorithm

Compare with Fourier Spectral Methods

$$
\begin{aligned}
\frac{\partial a_k}{\partial t} &= \sum_{l,l'=0}^{N-1} a_i a_j \int_{(-R,R)^6\times\mathbb{S}^2} B\left[\exp(-i\frac{\pi}{R}l.v'_*)\exp(-i\frac{\pi}{R}l'.v')\right.\\
&\left.-\exp(-i\frac{\pi}{R}l.v_*)\exp(-i\frac{\pi}{R}l'.v)\right]\exp(-i\frac{\pi}{R}k.v)d\sigma dv_* dv,
\end{aligned}
$$

Compare with Discrete Velocity Models

- Discrete Velocity Models: Nonadaptive, impossible to get an error estimate
- Adaptive Discrete Velocity Models=Adaptive Spectral Method + Haar Basis: The convolution structure of the collision operator is perfectly preserved and the accuracy is spectral.

A filtering technique

Propagation of polynomial moments

$$\forall s \geq 2, \forall t_0 > 0, \sup_{t \geq 0} \int_{\mathbb{R}^3} f(t,v)(1+|v|^s) < +\infty,$$

or

$$\forall s \geq 2, \forall t_0 > 0, \sup_{t \geq 0} \int_{(-1,1)^3} g(t,\bar{v})(1-\bar{v})^s < +\infty,$$

A filtering technique

Why Wavelets?

Suppose that we approximate g by its truncated Fourier series

$$g_N = \sum_{-N}^{N} \hat{g}_k \exp(i\pi k.\bar{v}).$$

The approximate solution g_N will never satisfy the properties that we mention above no matter how good f is: all components of the Fourier basis, i.e. the sin and cos functions are globally and smoothly defined on the whole interval $[-1, 1]$ and they encounter singular problems at the extremes -1 and $1 \Longrightarrow$ we need to remove compactly supported wavelets which contain the singular points -1 and 1. We also apply Zuazua's filter into our method.

- Our new adaptive spectral algorithm is proved to strongly converge.
- The algorithm preserves all of the important properties of the equation while previous strategies could not, even positivity property, conservation laws or convergence to equilibrium.
- Fast and easy to handle
- A unified point of view for both Fourier Spectral Methods and Discrete Velocity Models.
- A complete theory is supplied while for previous strategies, only partial results were proved (partial convergence, partial consistency)
- The algorithm has a spectral accuracy while DVMs do not have.

THANK YOU FOR YOUR ATTENTION!