Weight one modular forms of $\mathcal{S}_4$-type

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Introduction: Deligne-Serre

There is a one-to-one correspondence between:

representations $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2,\mathbb{C})$ continuous odd irreducible

and

weight one modular newforms $F(z) = \sum_{n=1}^{\infty} a_n q^n \in S_1(N,\varepsilon)$.

That correspondence satisfies: for all primes $p \nmid N$

$$\det(1 - \rho(\mathrm{Frob}_p)T) = T^2 - a_p T + \varepsilon(p).$$

Introduction

Given a Galois representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2,\mathbb{C})$ continuous, odd, and irreducible, the image of its associate projective representation

$$\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}(2,\mathbb{C})$$

has to be

$$\mathrm{Image}(\overline{\rho}) \simeq \mathcal{D}_n\,, \mathcal{A}_4\,, \mathcal{A}_5\,, \mathcal{S}_4\,.$$

The corresponding newform splits in

$$\begin{cases} \text{Theta series case:} & \text{when } \mathrm{Image}(\overline{\rho}) \simeq \mathcal{D}_n \\ \text{Exotic cases:} & \text{when } \mathrm{Image}(\overline{\rho}) \simeq \mathcal{A}_4\,, \mathcal{A}_5\,, \mathcal{S}_4\,. \end{cases}$$

Goal (25 years ago) was to compute $F(z) = \sum_{n=1}^{\infty} a_n q^n \in S_1(N,\varepsilon)$ arising from the exotic octahedral case $\mathcal{S}_4$.

Octahedral weight one modular forms

- Input: Given $K/\mathbb{Q}$ an $\mathcal{S}_4$-extension, together with

$$\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(K/\mathbb{Q}) \simeq \mathcal{S}_4 \hookrightarrow \mathrm{PGL}(2,\mathbb{C})$$

and an integer $m \geq 2$.

- Output: Decide whether an odd lifting $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2,\mathbb{C})$ of $\overline{\rho}$ exists with a certain image. If so, then return the level N and the sequence

$$\{a_1, a_2, a_3, a_4, \ldots, a_m\}$$

such that $F(z) = \sum_{n=1}^{\infty} a_n q^n \in S_1(N,\varepsilon)$ is the exotic octahedral modular newform attached to ρ.

Liftings

Given a projective representation

$$\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{PGL}(2,\mathbb{C}),$$

in case that there exists a linear lifting

$$\begin{tikzcd} & \mathrm{GL}(2,\mathbb{C}) \arrow[d] \\ \overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[ur, "\rho"] & \mathrm{PGL}(2,\mathbb{C}) \end{tikzcd}$$

all the others liftings are $\rho\otimes\chi$, where $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^*$ is any character.

Thanks to a theorem of Tate, $\overline{\rho}$ has always linear liftings. But if we impose a certain group as image of ρ, then there is an obstruction to its existence.

Serre's obstruction

We consider the Galois embedding problem

$$
\begin{array}{ccccccccc}
 & & & & & & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & & \\
 & & & & & & \big\downarrow \overline{\rho} & & \\
1 & \longrightarrow & C_2 & \longrightarrow & \tilde{\mathcal{S}}_4 & \xrightarrow{\pi} & \mathcal{S}_4 & \longrightarrow & 1
\end{array}
$$

where $\tilde{\mathcal{S}}_4$ is the 2-cover of $\mathcal{S}_4$ such that the preimage of transpositions are involutions. There are isomorphisms $\tilde{\mathcal{S}}_4 \simeq \mathrm{GL}(2,\mathbb{F}_3)$ and $\mathcal{S}_4 \simeq \mathrm{PGL}(2,\mathbb{F}_3)$. It turns out that $\tilde{\mathcal{S}}_4$ admits two (complex conjugate) irreducible linear representations.

A solution to the embedding problem (if exists) is a morphism

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \tilde{\mathcal{S}}_4$$

with $\overline{\rho} = \pi \circ \rho$. Serre gives a formula to compute the obstruction.

Serre's obstruction

Let $f \in \mathbb{Q}[x]$ be a degree 4 polynomial with Galois group $\mathrm{Gal}(f) \simeq \mathcal{S}_4$. Let x_i be the roots of f, and denote

$$K_1 = \mathbb{Q}(x_1), \qquad K = K_1^{\mathrm{gal}} = \mathbb{Q}(x_1, x_2, x_3, x_4).$$

The embedding problem $\tilde{\mathcal{S}}_4 \to \mathcal{S}_4 \simeq \mathrm{Gal}(K/\mathbb{Q})$ is solvable if and only if the element in the Brauer group of $\mathbb{Q}$ given by

$$\mathrm{e}_K := w(\mathrm{Tr}_{K_1/\mathbb{Q}}(X^2)) \otimes (2, d)$$

is trivial. Here, w denotes the Witt invariant and $(2, d)$ the Hilbert symbol where $d = \mathrm{disc}(K_1/\mathbb{Q})$.

Bayer-Frey give a convenient formula to compute the local invariants $\mathrm{e}_{K,p}$ for odd primes in terms of the ramification type of p in K_1.

Crespo's construction

Assume that the above embedding problem is solvable. Then, a field solution is

$$\tilde{K} := K(\sqrt{\gamma}),$$

where γ is a non-zero coordinate of the spinor norm of an element in a suitable Clifford algebra. More precisely,

$$\gamma = \det\left(\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \cdot P \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/2 & 1/2 \\ & & \sqrt{d}/2 & -\sqrt{d}/2 \end{pmatrix} + \mathrm{Id}\right) \neq 0\,.$$

Here, $P \in \mathcal{M}_4(\mathbb{Q})$ such that $P^t\,\mathrm{Tr}(x_i x_j)P = \mathrm{diag}(1,1,2,2d)$. Moreover, all solutions are $\tilde{K} = K(\sqrt{c\gamma})$ with $c \in \mathbb{Q}^*$.

The newform

Let $p \nmid d$ be a prime. Denote by $\tilde{\pi}_p$ the conjugacy class of the Frobenius elements for p in $\tilde{\mathcal{S}}_4$ which is mapped to the conjugacy class π_p in $\mathcal{S}_4$. Then a_p^2 is determined by π_p and the following table holds:

$p\mathcal{O}_{K_1}$	π_p	a_p^2	$\tilde{\pi}_p$
$\mathfrak{P}_1\mathfrak{P}_1'\mathfrak{P}_1''\mathfrak{P}_1'''$	$1A$	4	$1\tilde{A}, 2\tilde{A}$
$\mathfrak{P}_2\mathfrak{P}_2'$	$2A$	0	$4\tilde{A}$
$\mathfrak{P}_1\mathfrak{P}_1'\mathfrak{P}_2$	$2B$	0	$2\tilde{B}$
$\mathfrak{P}_1\mathfrak{P}_3$	$3A$	1	$3\tilde{A}, 6\tilde{A}$
$\mathfrak{P}_4$	$4A$	-2	$8\tilde{A}, 8\tilde{B}$

The character table of $\tilde{\mathcal{S}}_4$ gives:

$\tilde{\pi}_p$	$1\tilde{A}$	$2\tilde{A}$	$2\tilde{B}$	$3\tilde{A}$	$4\tilde{A}$	$6\tilde{A}$	$8\tilde{A}$	$8\tilde{B}$
a_p	2	-2	0	-1	0	1	$i\sqrt{2}$	$-i\sqrt{2}$

The newform

The element γ helps to determine the sign.

$p\mathcal{O}_{K_1}$	π_p	a_p	$\tilde{\pi}_p$
$\mathfrak{P}_1\mathfrak{P}_1'\mathfrak{P}_1''\mathfrak{P}_1'''$	$1A$	$2,-2$	$1\tilde{A},2\tilde{A}$
$\mathfrak{P}_1\mathfrak{P}_3$	$3A$	$-1,1$	$3\tilde{A},6\tilde{A}$
$\mathfrak{P}_4$	$4A$	$i\sqrt{2},-i\sqrt{2}$	$8\tilde{A},8\tilde{B}$

- $\tilde{\pi}_p = 1\tilde{A}$, $3\tilde{A}$ if and only if $\left(\dfrac{\gamma}{\mathfrak{P}_1}\right) = 1$
- $\tilde{\pi}_p = 8\tilde{A}$ if and only if $\gamma^{(p-1)/2} \equiv b_s \bmod \mathfrak{P}_4$, where

$$b_s = -\frac{1}{2} + \left(\gamma^{(321)} - \gamma^{(123)}\right)/2\gamma.$$

Example

The problem is solvable for the splitting field $K = \mathbb{Q}(x_1, x_2, x_3, x_4)$ of $f = x^4 - 2x - 1$. The element γ can be taken as

$$\gamma = 3(x_1^3 x_2^3 - x_2^2 - x_1^2 x_2 + x_1 x_2 + x_2) + x_1^3 - 2x_1^2 + 4x_1 \,.$$

The corresponding newform F has level $2^3 \cdot 43$.

p	$\tilde{\pi}_p$	a_p
$487, 619, 719$	$1\tilde{A}$	2
$173, 827, 857$	$2\tilde{A}$	-2
$19, 37, 71, 113, 137, 149, 157, 191, 199$	$2\tilde{B}$	0
$11, 17, 53, 67, 97, 101, 103, 127, 193$	$3\tilde{A}$	-1
$47, 59, 79, 107, 181, 197$	$4\tilde{A}$	0
$13, 23, 31, 41, 83, 109, 139, 167$	$6\tilde{A}$	1
$7, 29, 61, 89, 179$	$8\tilde{A}$	$i\sqrt{2}$
$3, 5, 73, 151, 163$	$8\tilde{B}$	$-i\sqrt{2}$