Stationary Lifshitz black holes of $R^2$-corrected gravity theory

Özgür Sarıoğlu

Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06800, Ankara, Turkey

September 10, 2013

Lifshitz spacetimes:
• emerged in applications of the AdS/CFT duality to non-relativistic, specifically, condensed matter systems.
• thought of as gravity duals of theories w/ nontrivial scaling properties.

Lifshitz black holes:
• BH solutions to some gravity (+ matter) theory that are asymptotically Lifshitz.
• needed for describing “finite temperature aspects” of these non-relativistic systems.

Observation: Few exact analytic Lifshitz BHs exist, and all known ones are static.
A closer look at static Lifshitz spacetimes:

• a typical feature in condensed matter systems is the “dynamical scaling” property:

\[ t \mapsto \lambda^z t, \quad \vec{x} \mapsto \lambda \vec{x}, \] where \( z \neq 1 \) is called the “dynamical exponent”, instead of the more familiar conformal scalings:

\[ t \mapsto \lambda t, \quad \vec{x} \mapsto \lambda \vec{x}. \]

• plus the following usual additional symmetries: spatial translations + temporal translations + spatial rotations + parity (P) symmetry + time reversal (T) symmetry
• distinguish one spatial coordinate, call it \( r \ (0 \leq r < \infty) \), and let the dynamical scaling transformations act as

\[
t \mapsto \lambda^z t, \quad \vec{x} \mapsto \lambda \vec{x}, \quad r \mapsto r/\lambda,
\]

where \( z \neq 1 \) again and \( \vec{x} \) denotes a \((D-2)\)-dimensional vector.

• Then one ends up with the \( D \)-dimensional static Lifshitz spacetime suitable for AdS/CFT games:

\[
ds^2 = -\frac{r^{2z}}{\ell^{2z}} \, dt^2 + \frac{\ell^2}{r^2} \, dr^2 + \frac{r^2}{\ell^2} \left( \sum_{i=1}^{D-2} dx_i^2 \right),
\]

(1)

• When \( z = 1 \), usual AdS\(_D\) metric with \( SO(D - 1, 2) \) symmetry.
• the length scale set by \( \ell > 0 \).
Stationary Lifshitz spacetimes:

- Replace the two separate “P symmetry” + “T symmetry” requirements with the weaker “PT symmetry” invariance, i.e. ask for the following symmetries:
  - dynamical scaling transformations + spatial translations + temporal translations + spatial rotations + PT symmetry

- Then one ends up with the $D$-dimensional stationary Lifshitz spacetime:

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} \ dt^2 + 2\omega \ \frac{r^{z+1}}{\ell^{z+1}} \ dt \ d\phi + \frac{r^2}{\ell^2} \ d\phi^2 + \frac{\ell^2}{r^2} \ dr^2 + \frac{r^2}{\ell^2} \left( \sum_{i=1}^{D-3} dx_i^2 \right),$$

(2)

- For later convenience, $x_{D-2} \equiv \phi$ distinguished from the remaining $x_i \ (1 \leq i \leq D - 3)$.
- the static Lifshitz spacetime is obtained when $\omega$, the dimensionless “rotation parameter”, is set to 0.
$R^2$-corrected gravity theory:

- The action:

$$ I = \int d^D x \sqrt{-g} \left( R + 2\Lambda + \alpha R^2 \right), \quad (3) $$

where $\Lambda$ is the cosmological constant and $\alpha$ is a coupling constant.

The first result:

Stationary Lifshitz spacetime (2) solves the field equations following from the action (3) for generic values of the parameters $z$ and $\omega$ in any $D \geq 3$, provided that $\alpha$ and $\Lambda$ are tuned as

$$ \alpha = \frac{1}{8\Lambda}, \quad \Lambda = \frac{2D^2 + 3(z - 1)^2 + 2D(2z - 3)}{8\ell^2} + \frac{(z - 1)^2}{8\ell^2(1 + \omega^2)}. \quad (4, 5) $$
The main result:
Provided that $\alpha$ and $\Lambda$ are precisely as in (4) and (5), the metric

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} h(r) \, dt^2 + \frac{r^2}{\ell^2} \left( d\phi + \omega \frac{\ell^2}{r^2} \, dt \right)^2 \right. 
+ \frac{\ell^2}{r^2} \, \frac{dr^2}{h(r)} + \frac{r^2}{\ell^2} \left( \sum_{i=1}^{D-3} dx_i^2 \right), \text{ where}$$

$$h(r) \equiv c + k \frac{\ell^{2(1+z)}}{r^2(1+z)} + M^- \frac{\ell^{p_-}}{r^{p_-}} + M^+ \frac{\ell^{p_+}}{r^{p_+}}, \text{ with}$$

$$c \equiv \frac{4 \ell^2 \Lambda}{2z^2 + (D - 2)(2z + D - 1)},$$

$$k \equiv \frac{2 \omega^2}{D^2 - 7D + 14 - 2z(D - 3)}, \text{ and}$$

$$p_{\pm} \equiv \frac{1}{2} \left( 3z + 2(D - 2) \pm \sqrt{z^2 + 4(D - 2)(z - 1)} \right), \text{ (10)}$$

solves the field equations of the action (3) for any $D \geq 3$. 
Remarks:
• Note that the coefficients $c$ and $k$ are completely determined by $z$ and $\omega$, whereas the integration constants $M^\pm$ are left as free parameters.
• (1) and the static version of the metric (6) [i.e. the one with $\omega = 0$ for which $c = 1$, $k = 0$, the relations (4) and (5) for $\alpha$ and $\Lambda$, and the metric function $h(r)$ in (7) are simplified accordingly] were first presented by E. Ayon-Beato, et al. in JHEP 1004, 030 (2010) [arXiv:1001.2361 [hep-th]].
• However with $\omega$ on, (2) and the stationary metric (6) (with the accompanying equations (4), (5) and (7)-(10), respectively) are clearly more general.
• In the conformal limit $z = 1$ with $D = 3$, the metric (6) becomes identical to the BTZ metric when one sets $M^+ = 0$, $M^- = -M < 0$ and $\omega = -j/2$. 
• The curvature scalars of the metrics (2) and (6) are both given by \( R = -4\Lambda \) precisely. This allows for the casting of the action (3) into the form

\[
I = \frac{1}{8\Lambda} \int d^Dx \, \sqrt{-g} \, (R + 4\Lambda)^2,
\]

and this theory cannot be mapped into a scalar-tensor theory by a conformal transformation of the metric.

• To have \( h(r) \) real, one needs

\[
z < z_- \quad \equiv \quad 4 - 2D - 2\sqrt{(D - 1)(D - 2)} < 0 \quad \text{or} \quad z > z_+ \quad \equiv \quad 4 - 2D + 2\sqrt{(D - 1)(D - 2)} > 0.
\]

For the metric (6) to describe a black hole, a careful analysis further chooses the branch \( z > z_+ > 0 \).
• One can construct analogous solution(s) of the form (6) for the critical value(s) of the dynamical exponent \( z = z_{\pm} \) with logarithmic \( h(r) \) function(s).

• In fact the region \( z \in (z_-, z_+) \) is not excluded either! However, let me skip the details of it here!

• A careful consideration shows that both the energy \( E \) and the entropy \( S \), as well as the angular momentum \( J \), vanish for this class of solutions: \( E = S = J = 0 \). Perhaps this is not so surprising after all, since the action \( I = 0 \) at the first place for these solutions! (Recall my previous remark following \( R = -4\Lambda \).)
Conclusions and open problems:

- As put forward by E. Ayon-Beato, et al., one may think of these solutions (or their Euclidean counterparts) as some kind of “gravitational instantons”.
- Stability of these solutions has not been analyzed yet.
- The implications of these solutions on the CMT side has not been studied yet.