# Cosmological perturbations in teleparallel LQC

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# Isotropic LQC 1

Gravitational part of the classical Hamiltonian in Einstein Cosmology (flat FLRW geometry) $\mathcal{H}_G = -\frac{3}{\gamma^2}\beta^2 V$.

1. $\beta = \gamma H$, where $\gamma \cong 0.23$ is the Barbero-Immirzi parameter.
2. $V = a^3$ is the volume.
3. $\beta$ and $V$ are canonically conjugated: $\{\beta, V\} = \gamma/2$

In LQC the discrete nature of space-time is assumed, this leads to a complicated Hilbert space: the quotient of Besicovitch's space (almost periodic functions) by its subspace of null functions. (Asthekar, Singh, Class. Quant. Grav. (2011), arXiv:1108.0893).

Besicovitch's space is the closure of trigonometric polynomials under the semi-norm (in the $\beta$-representation)

$$||\Psi||^2 = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |\Psi(\beta)|^2 d\beta.$$

The elements of this space have the expansion

$$\Psi(\beta) = \sum_{n\in\mathbb{Z}} \alpha_n e^{i\lambda_n\beta/2}$$

with $\lambda_n \in \mathbb{R}$ and $\alpha_n$ is a square-summable sequence.
$\hat{V} = -i\frac{\gamma}{2}\frac{d}{d\beta}$ is well-defined. However, $\beta$ does not admit a quantum operator because $\beta\Psi(\beta)$ has infinite norm!!!!!

REMARK: In many work, Besicovitch's space is named Bohr's (Harald Bohr, Niels's brother) space. However, Bohr introduced the space of **uniform** almost periodic functions, which is a Banach but not a Hilbert space.

The classical Hamiltonian is modified making the replacement $\beta \to \frac{\sin(\lambda\beta)}{\lambda}$, where $\lambda^2 = \frac{\sqrt{3}}{2}\gamma$ is the minimum eigenvalue of the area operator in LQG ( Haro, Elizalde EPL (2010)).

# Isotropic LQC 3

From this "holonomy corrected"Hamiltonian, using the corresponding Hamilton equations and the Hamiltonian constrain, the Friedmann equation is modified as follows:

$$H^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_c}\right), \tag{1}$$

where $\rho_c = \frac{3}{\lambda^2\gamma^2} \cong 0.4\rho_{Pl}$ is the so-called *critical density*.

Remarks:

1. Equation (1) depicts an ellipse in the plane $(H, \rho)$. Then, "big bang" and "big rip" singularities are forbidden.
2. In EC, Friedmann's equation depicts a parabola which allows the existence of this kind of singularities.
3. For a non-phantom universe, it moves clockwise along the ellipse and bounces at $\rho = \rho_c$.
4. When the universe reaches the bounce all its points are causally connected. The horizon problem doesn't exist in LQC.

# Isotropic $F(T)$ gravity 1

Teleparallel theories are based in the Weitzenböck space-time. To build this space-time, one chooses a global system of four orthonormal vector fields $\{e_i\}$ related to the vectors $\{\partial_\mu\}$ via the relation $e_i = e_i^\mu \partial_\mu$. In addition, one introduces a covariant derivative $\nabla$ that defines absolute parallelism with respect the global basis $\{e_i\}$, that is, $\nabla e_i = 0$. From this, one acquires the metric Weitzenböck connection $\Gamma_{\mu\nu}^\gamma = e_i^\gamma \partial_\nu e_\mu^i$. (Note that this connection is metric and therefore it satisfies $\nabla g = 0$.) (Haro, Amoros PRL (2013), arXiv:1211.5336)

Due the absolute parallelism this connexion is curvature-free (Riemann tensor vanishes) but has torsion!!!

The basic invariant is the *scalar torsion*, namely $T$. For a flat FLRW geometry, it is easily obtained from the scalar curvature because GR can also be built with an action linear in $T$. Then, removing the total derivative that appears in the Hilbert action one gets $T = -6H^2$.

# Isotropic $F(T)$ gravity 2

The Lagrangian in $F(T)$ gravity (flat FLRW geometry) is

$$\mathcal{L}_T = VF(T) + \mathcal{L}_M, \tag{2}$$

being $\mathcal{L}_M$ the matter Lagrangian density.

Legendre's transformation leads to the following Hamiltonian density

$$\mathcal{H}_T = \left(2T\frac{dF(T)}{dT} - F(T) + \rho\right) V. \tag{3}$$

The Friedmann equation is obtained from the constrain $\mathcal{H}_T = 0$

$$\rho = -2\frac{dF(T)}{dT}T + F(T) \equiv G(T). \tag{4}$$

Conversely, given a curve of the form $\rho = G(T)$ one can find the Lagrangian density $\mathcal{L}_T$, integrating (4) obtaining as a result

# Isotropic $F(T)$ gravity 3

$$F(T) = -\frac{\sqrt{-T}}{2} \int \frac{G(T)}{T\sqrt{-T}} dT. \tag{5}$$

Splitting the ellipse in two pieces $\rho = G_-(T)$ (the branch where $\dot{H} < 0$) and $\rho = G_+(T)$ (the branch where $\dot{H} > 0$), where

$$G_\pm(T) = \frac{\rho_c}{2}\left(1 \pm \sqrt{1 + \frac{2T}{\rho_c}},\right). \tag{6}$$

the modified Friedmann equation in LQC could be obtained using the following function

$$F_\pm(T) = \pm\sqrt{-\frac{T\rho_c}{2}} \arcsin\left(\sqrt{-\frac{2T}{\rho_c}}\right) + G_\pm(T), \tag{7}$$

which is the basis of the teleparallel formulation of LQC. (Bamba, Haro, Odinsov, JCAP (2013), arXiv:1211.2968).

# Scalar perturbation in teleparallel LQC 1

Longitudinal gauge $ds^2 = (1+2\Phi)dt^2 - a^2(1-2\Phi)d\mathbf{x}^2$.
Lagrangian matter density $\mathcal{L}_M = \left(\frac{1}{2}\dot{\varphi} - V(\varphi)\right) V$, with $\varphi = \bar{\varphi} + \delta\varphi$ ($\bar{\varphi}$ being the homogeneous part of the field).
Equation of evolution for the newtonian potential (Haro, arXiv:1309.0352)

$$\Phi'' - \tilde{c}_s^2 \Delta\Phi + 2\left(\mathcal{H} - \left(\frac{\bar{\varphi}''}{\bar{\varphi}'} + \epsilon\right)\right)\Phi' + 2\left(\mathcal{H}' - \mathcal{H}\left(\frac{\bar{\varphi}''}{\bar{\varphi}'} + \epsilon\right)\right)\Phi = 0. \quad (8)$$

Where we have introduced the notation

1. $\Omega = 1 - \frac{2\rho}{\rho_c}$ and $\epsilon = \frac{1}{2}\frac{\Omega'}{\Omega}$
2. $\tilde{c}_s^2 = 2|\Omega| \left|\frac{dF_\pm(T)}{dT}\right|$, with

$$\left|\frac{dF_\pm(T)}{dT}\right| = \frac{\rho_c}{4\rho\sqrt{\frac{\rho_c}{\rho} - 1}} \arcsin\left(\frac{2\rho}{\rho_c}\sqrt{\frac{\rho_c}{\rho} - 1}\right). \quad (9)$$

The dynamical equation for the newtonian potential eq. (8) is obtained combining the equations:

$$\tilde{c}_s^2\Delta\Phi - 3\mathcal{H}\Phi' - \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi = \frac{\Omega}{2}\left(\delta\varphi'\bar{\varphi}' + \delta\varphi\frac{\partial V(\bar{\varphi})}{\partial\varphi}a^2\right) \quad (10)$$

$$\Phi' + \mathcal{H}\Phi = \frac{\Omega}{2}\delta\varphi\bar{\varphi}' \quad (11)$$

$$\Phi'' + (3\mathcal{H} - 2\epsilon)\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2 - 2\mathcal{H}\epsilon\right)\Phi = \frac{\Omega}{2}\left(\delta\varphi'\bar{\varphi}' - \delta\varphi\frac{\partial V(\bar{\varphi})}{\partial\varphi}a^2\right) \quad (12)$$

REMARK: These equations only differ from the classical ones in the velocity of sound, the factor $\Omega$ that appears in the right hand side and with the terms containing $\epsilon$ in the third equation.

Following Mukhanov's book (pages 336-337) we introduce the variables

$$v = a\frac{\sqrt{|\Omega|}}{\tilde{c}_s}(\delta\varphi + \frac{\bar{\varphi}'}{\mathcal{H}}\Phi); \quad z = \frac{a\sqrt{|\Omega|}\bar{\varphi}'}{\tilde{c}_s\mathcal{H}} \tag{13}$$

and

$$u_\pm = \mp\frac{2a\Phi}{\sqrt{|\Omega|}\bar{\varphi}'}; \quad \theta = \frac{1}{\tilde{c}_s z}, \tag{14}$$

from equations (10) and (11) one gets

$$\tilde{c}_s \Delta u_\pm = z\left(\frac{v}{z}\right)'; \quad \theta\left(\frac{u_\pm}{\theta}\right)' = \tilde{c}_s v. \tag{15}$$

# Mukhanov-Sasaki equation for scalar perturbations 2

Performing the Laplacian in the second equation and using the first one, one gets the Mukhanov-Sasaki equation

$$v'' - \tilde{c}_s^2 \Delta v - \frac{z''}{z} v = 0. \tag{16}$$

Note alseo that, in LQC as a $F(T)$ theory, the variable $v$ is related with the curvature fluctuation in co-moving coordinates

$$\zeta = \Phi - \frac{H}{\dot{H}}(\dot{\Phi} + H\Phi), \tag{17}$$

by the relation $v = z\zeta$.

REMARK: In many works is stated that equation of type (16) comes from the dynamical equation of the newtonian potential eq. (8). I've spent many hours trying to do it, but I couldn't!!!

# Power spectrum in a matter dominated universe 1

For a matter dominated universe the scalar factor is given by $a(t) = \left(\frac{3}{4}\rho_c t^2 + 1\right)^{1/3}$.
When holonomy correction are small ($\rho \ll \rho_c$) the classical Mukhanov-Sasaki equation for a matter-dominated universe is recovered,

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \Longleftrightarrow v_k'' + \left(k^2 - \frac{2}{\eta^2}\right) v_k = 0. \qquad (18)$$

At early times the universe must be in the Bunch-Davies vacuum, i.e., when $\eta \to -\infty$

$$v_k(\eta) = \sqrt{\frac{-\pi\eta}{4}} H^{(1)}_{3/2}(-k\eta). \qquad (19)$$

When time moves forward the modes leave the Hubble radius.
For a matter-dominated universe in EC, modes well outside of the Hubble radius are characterized by the condition

# Power spectrum in a matter dominated universe 2

$$k^2\eta^2 \ll 1, \Longleftrightarrow k^2 \ll \left|\frac{a''}{a}\right| \Longleftrightarrow k^2 \ll \left|\frac{1}{\tilde{c}_s^2}\frac{z''}{z}\right|. \tag{20}$$

When holonomy effects are not important, for modes well outside, the M-S equation becomes

$$v_k'' - \frac{z''}{z}v_k = 0, \tag{21}$$

which solution is obtained using the method of reduction

$$v_k(\eta) = B_1(k)z(\eta) + B_2(k)z(\eta)\int_{-\infty}^{\eta}\frac{d\bar{\eta}}{z^2(\bar{\eta})}. \tag{22}$$

Matching this solution with (19) one gets

$$B_1(k) = \sqrt{\frac{8}{3}}\frac{k^{3/2}}{\rho_c} \quad B_2(k) = i\sqrt{\frac{3}{8}}\frac{\rho_c}{2k^{3/2}}. \tag{23}$$

# Power spectrum in a matter dominated universe 3

Now we consider modes that in the contracting phase leave the Hubble radius, then evolve satisfying $k^2 \ll \left|\frac{1}{c_{s,\pm}^2}\frac{z''}{z}\right|$, and in the classical regime are still well outside. For these modes one has

$$v_k(\eta) = (B_1(k) + B_2(k)R)z(\eta), \tag{24}$$

where $R \cong \int_{-\infty}^{\infty} \frac{d\bar{\eta}}{z^2(\bar{\eta})}$, becase $\eta$ is large enough.
The scalar power spectrum is scale invariant and is given by

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2}|\zeta_k(\eta)|^2 = \frac{k^3}{2\pi^2}\left|\frac{v_k(\eta)}{z(\eta)}\right|^2 \cong \frac{3\rho_c^2}{64\pi^2}R^2 = \frac{\rho_c}{36\pi^2}\mathcal{C}^2, \tag{25}$$

where $\mathcal{C} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = 0.915965\ldots$ is the Catalan's constant.

# Power spectrum in a matter dominated universe 4

Some remarks:

1. Since $\rho_{pl} = 64\pi^2$, to agree with the observed value $\mathcal{P}_\zeta(k) \sim 10^{-9}$ one has to take $\rho_c \sim 10^{-9}\rho_{pl}$, which is in contradiction with its current value $\rho_c \cong 0.4\rho_{pl}$. In fact, since $\rho_c = \frac{2\sqrt{3}}{\gamma^3}$, to get $\rho_c \sim 10^{-9}\rho_{pl}$, one has to take as a value of the Barbero-Immirzi parameter $\gamma \sim 10^{-3}$, which is smaller than its current value 0.2375 obtained relating the black hole entropy in LQC with the Bekenstein-Hawking entropy formula (Miessner, Class.Quant.Grav. (2004) arXiv:0407052).
2. This contradiction does not appears in our version of LQC, where $\rho_c$ could be understood as a parameter, that seems to be close to $10^{-9}\rho_{pl}$.
3. For this value of the critical density, geometric quantum effects does not affect the evolution of the universe in teleparallel LQC.

# Tensor perturbations in teleparallel LQC 1

In teleparallel LQC, the equation of tensor perturbations can be obtained from the equation (Cai, Cheng, Dent, Dutta, Saridakis, Class.Quant.Grav. (2011) arXiv:1104.4349)

$$\frac{dF_{\pm}(T)}{dT}\left(\ddot{h}_i^a - \frac{\Delta h_i^a}{a^2} + 3H\dot{h}_i^a\right) + \dot{T}\dot{h}_i^a \frac{d^2F_{\pm}(T)}{dT^2} = 0. \quad (26)$$

Performing the change of variables

$$v_t \equiv \frac{ah}{\sqrt{\left|\frac{dF_{\pm}(T)}{dT}\right|}}, \quad z_t \equiv \frac{a}{\sqrt{\left|\frac{dF_{\pm}(T)}{dT}\right|}}, \quad (27)$$

where $h$ represents the two degrees of freedom of $h_i^a$, we have obtained the following M-S equation for tensor perturbations

$$v_t'' - \Delta v_t - \frac{z_t''}{z_t} v_t = 0. \quad (28)$$

We have calculated the ratio of tensor to scalar perturbations for a matter dominated universe, obtaining

$$r \equiv \frac{\mathcal{P}_h(k)}{\mathcal{P}_\zeta(k)} \cong 6. \tag{29}$$

Remarks:

1. This ratio is of the order 1, which coincides with the current calculations in $F(T)$ gravity.
2. However it does not agree with the current CMB bound $r \lesssim 0.2$.
3. Assuming a matter-dominated at very early time (to preserve the scale invariant spectrum), but for the major part of the evolution considering a non-phantom universe with linear equation of state $P = \omega\rho$ with $\omega < 0$, scalar perturbations are amplified and the ratio of tensor to scalar perturbations is of the order $r \sim (1+\omega)^2$. Then, choosing an appropriate value of $\omega$ one will achieve the bound.

# Comparison with holonomy corrected LQC 1

The idea of introduccing holonomy corrections is very simple: starting from the perturbed classical Hamiltonian the introduction of holonomy corrections, like in isotropic models, is based in the replacement $\bar{c} \to \frac{\sin(n\bar{\mu}\bar{c})}{n\bar{\mu}}$, where $\bar{c} = \gamma\dot{a}$ is the Asthekar connection and $n \in \mathbb{N} \setminus \{0\}$. (Bojowald, Hossain PRD (2008), arXiv:0709.2365)

PROBLEM: the algebra of constrains ceases to be preserved, i.e., in the Poisson brackets of the constrains it appears additional terms called anomalies.

SOLUTION: anomalies could be removed and the algebra of constrains restored inserting some counter-terms. However some of them must contain the Asthekar connection which does not have a quantum analogue. (Cailleteau, Mielczarek, Barrau,Grain, Class.Quant.Grav (2012), arXiv:1111.3535)

# Comparison with holonomy corrected LQC 2

The evolution equation for the newtonian potencial in holonomy corrected LQC, is the same as in teleparallel LQC (eq. (8)), but with a square of the velocity of sound iqual to $c_s^2 = \Omega = 1 - \frac{2\rho}{\rho_c}$.

Note that in holonomy corrected LQC, $c_s^2 > 0$ when $\rho < \rho_c/2$ however when $\rho > \rho_c/2$ one has $c_s^2 < 0$, what means that in the super-inflationary phase, eq. (8) becomes elliptic. This behavior never happens in our formulation of LQC where $\tilde{c}_s^2$ is always positive, and thus, the equation is always hyperbolic.

The power spectrum for scalar perturbations in holonomy corrected LQC, is very similar to the one obtained in teleparallel LQC (eq. (25): (Wilson-Ewing, JCAP (2013), arXiv:1211.6269)

$$\mathcal{P}_\zeta(k) = \frac{\rho_c}{576}. \tag{30}$$

# Comparison with holonomy corrected LQC 3

The Mukhanov-Sasaki equation for tensor perturbations in holonomy corrected LQC is (Cailleteau, Barrau, Vidotto PRD (2012), arXiv:1206.6736)

$$v_t'' - c_s^2 \Delta v_t - \frac{z_t''}{z_t} v_t = 0, \tag{31}$$

with $z_t \equiv \frac{a}{\sqrt{\Omega}}$ and $v_t \equiv h z_t$.

It's important to realize that equation (31) has two singular points, at the beginning and at the end of the super-inflationary phase, what means that there is not any objective criterium of continuity to define the solution at these points.
Thus, there are infinite ways to match solutions at these points, and consequently infinite mode functions could be used to calculate the power spectrum of tensor perturbations.

# Comparison with holonomy corrected LQC 4

For example, when holonomy corrections are taken into account, for the modes we are considering, $z_t = \frac{a}{\sqrt{\Omega}}$ is a solution, but $\tilde{z}_t = \frac{a}{\sqrt{|\Omega|}}$ is another one. In fact, one can build infinite solutions, because we cannot impose its continuity at the singular points.

Taking as a mode solution $z_t$ one obtains $r \cong 0$. However using $\tilde{z}_t$ we have obtained $r \cong \frac{27}{\pi^2}$ which is of the order 1.

Finally, note that in teleparallel LQC, the corresponding Mukhanov-Sasaki equation for tensor perturbations (eq. (28)) has no singular points, what means that the solution $z_t = a\left|\frac{dF_\pm(T)}{dT}\right|^{-1/2}$ is unambiguously defined.

# Conclusions

1. Teleparallel LQC in the matter-bounce scenario could be an alternative to standard slow-roll inflation. **A scalar field that drive a extremely huge expansion of the universe in a very brief period of time, after that release its energy to created all the matter of the universe (religious people believe that the seventh day it took a nap), is not needed.**
2. The big bang singularity is avoided and provides an scale invariant spectrum for a matter-dominated universe.
3. The theory provides a very simple non-singular bounce (LQC) and has the advantages of $F(T)$ gravity (velocity of sound always positive, non-singular points in the M-S equation for tensor perturbations,...).

THANKS !!!!!