Invariant Theory for Matrix Product States

Jacob Biamonte

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Talk based on: ***Invariant Theory for Matrix Product States***, with Ville Berghlom and Marco Lanzagorta, (2012) .

ISI Foundation

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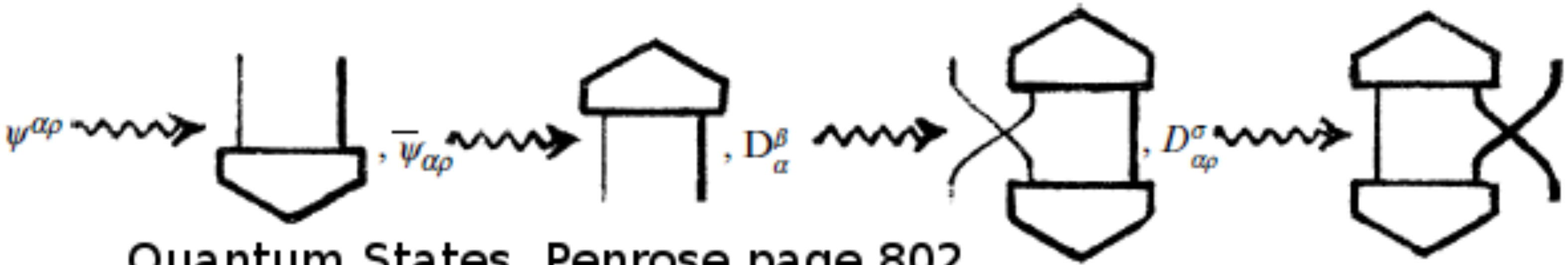
Overview

- History of the graphical notation (Penrose 1960's)
- Recent results
- Invariants

Seeing tensors and quantum theory



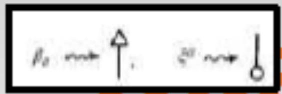
Quantum States, Penrose page 802

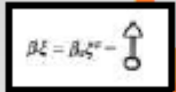
Fig. 29.5 Diagrammatic notation for density matrices constructed by 'summing over unknown states'. The normalized ket-vector $|\psi\rangle$ is expressed as $\psi^{\alpha\rho}$, where 'α' refers to 'here' (Earth) and 'σ' refers to 'there' (Titan). The Hermitian conjugate (bra-vector $\langle\psi|$) is $\bar{\psi}_{\alpha\rho}$ and the normalization is $\bar{\psi}_{\alpha\rho}\psi^{\alpha\rho} = 1$. The density matrix used 'here' is $D^{\beta}_{\alpha} = \bar{\psi}_{\alpha\rho}\psi^{\beta\rho}$, while that used 'there' is $\tilde{D}^{\sigma}_{\rho} = \bar{\psi}_{\alpha\rho}\psi^{\alpha\sigma}$.

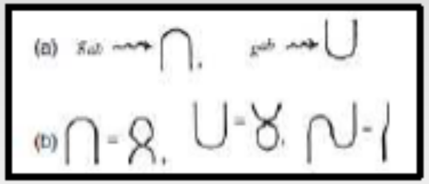
R. Penrose, Combinatorial quantum theory and quantized directions, in *Advances in Twistor Theory*, 1979, pp. 301-317.

Roger Penrose, Theory of quantized directions, 1967

Graphical Calculus for Quantum Theory [Penrose]







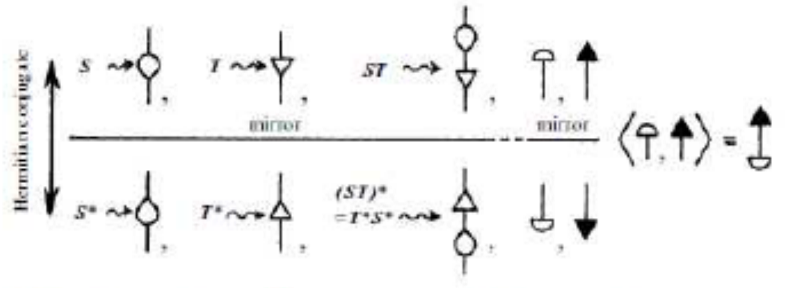


Fig. 13.19 The operation of Hermitian conjugation (*) conveniently depicted as reflection in a horizontal plane. This interchanges 'arms' with 'legs' and reverses the order of multiplication: $(ST)^* = T^*S^*$. The diagrammatic expression for the Hermitian scalar product $\langle v|w\rangle = v^*w$ is given (so that taking its complex conjugate would reflect the diagram on the far right upside-down).

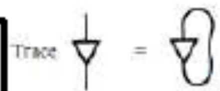
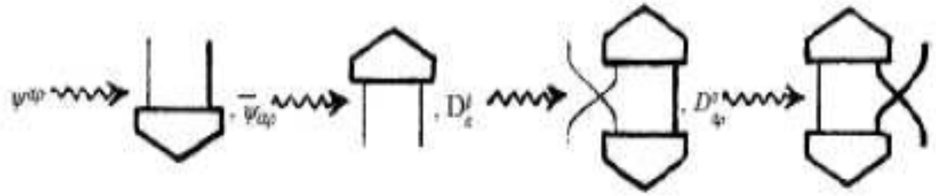


Fig. 13.9 Diagrammatic notation for trace $T(= T^a{}_a)$.



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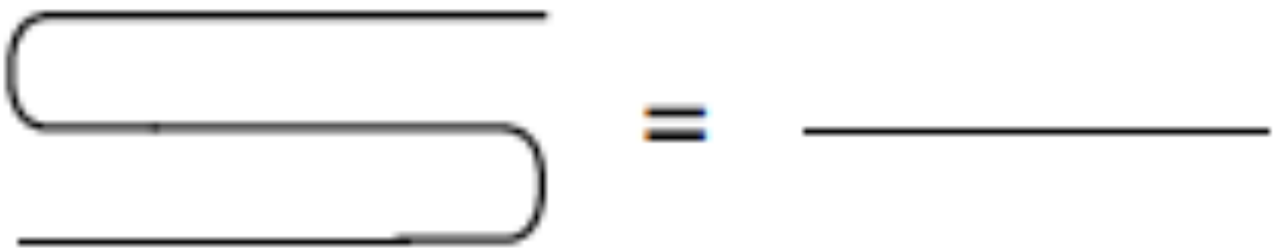
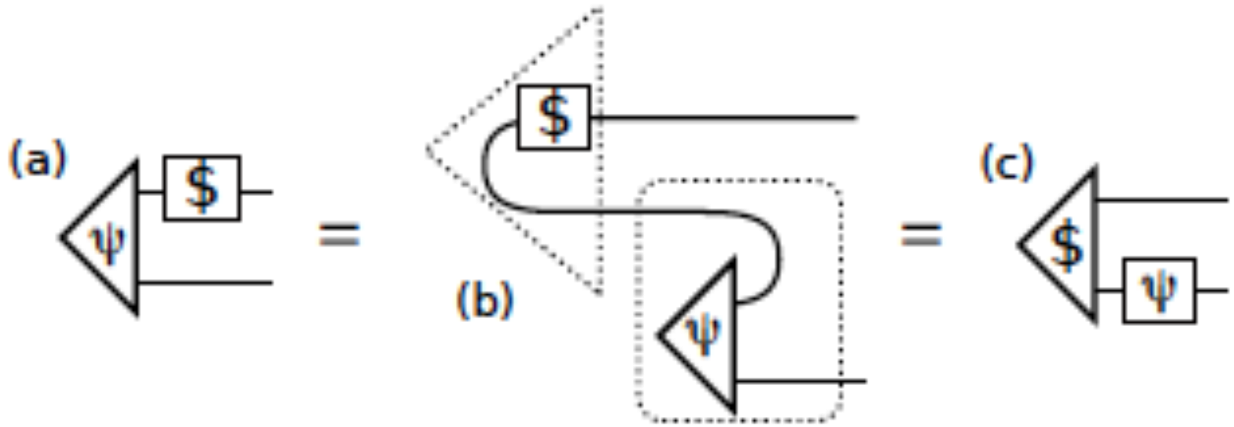
Applications of negative dimensional tensors, Roger Penrose in *Combinatorial Mathematics and its Applications*, Academic Press (1971).

Roger Penrose, *Theory of quantized directions*, 1967

Zig-zag equation



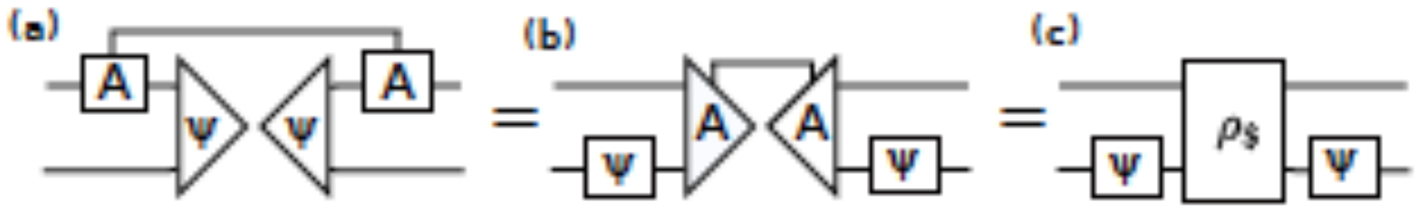
Version of map-state duality



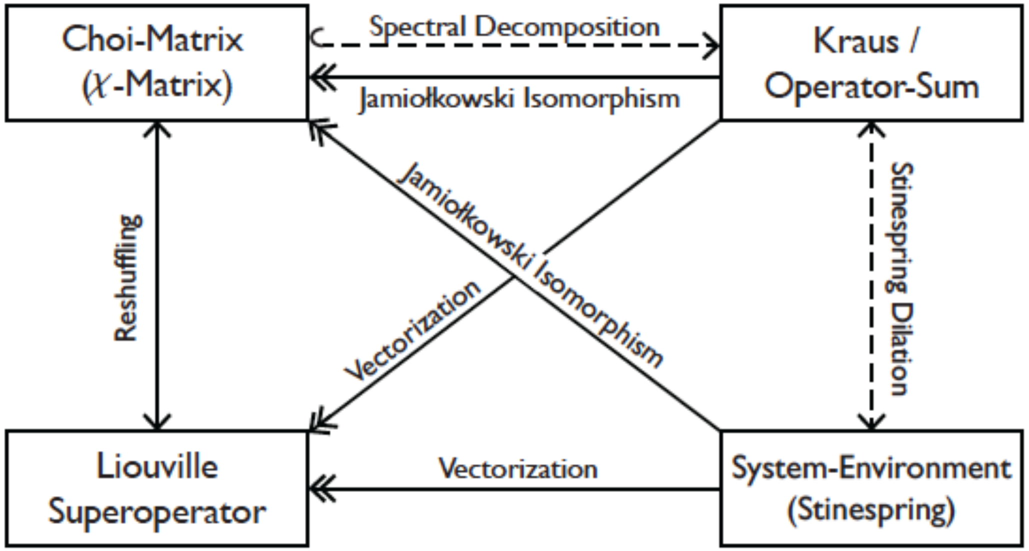
Tensor networks for entanglement evolution
Sebastian Meznaric and JDB
in review, (2012)
http://arxiv.org/abs/1204.3599

Tensor networks and graphical calculus for open quantum systems
Christopher J. Wood, JDB and David G. Cory
in review (2011)
http://arxiv.org/abs/1111.6950

Version of map-state duality II



Pictures of open systems evolution



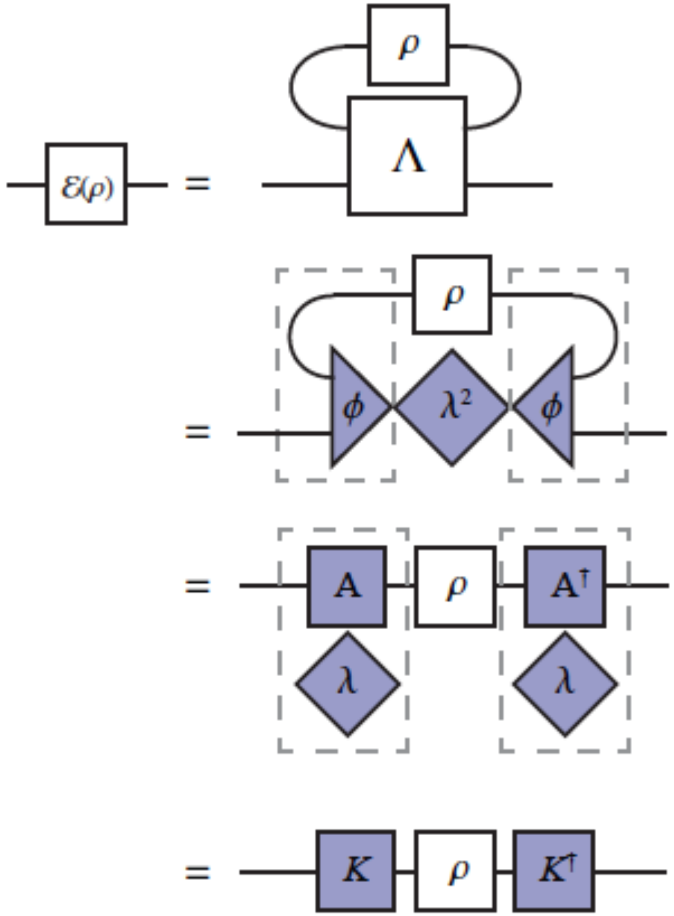


FIG. 42. Graphical proof of Fig. 41 for the construction of a Kraus representation $\{K_\alpha\}$ for a CPTP-map $\mathcal{E}$ from the Choi-matrix representation Λ.

Tensor networks and graphical calculus for open quantum systems
Christopher J. Wood, JDB and David G. Cory
in review (2011)
http://arxiv.org/abs/1111.6950

Boolean tensor network states

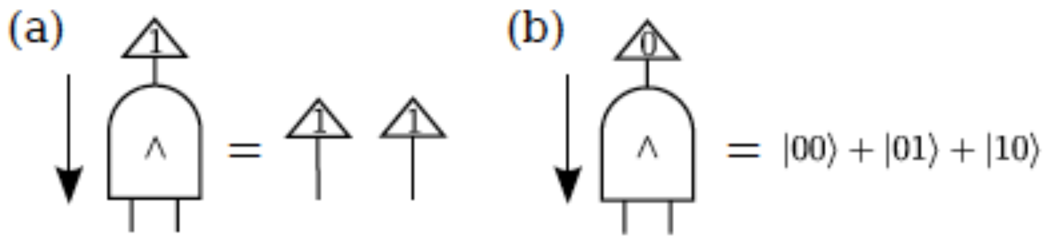
Theorem 3.3 (Boolean tensor network states). *A tensor network representing a Boolean quantum state is determined from the classical network description of the corresponding function.*

Theorem 3.3 was given in [2], where the quantum tensor networks are found by letting each classical gate act on a linear space and from changing the composition of functions, to the contraction of tensors.

Example 3.4 (AND-tensors). As an example of a Boolean logic tensor, consider the AND-tensor defined as

$$\mathrm{AND}^{i}_{jk} = |00\rangle\langle 0| + |01\rangle\langle 0| + |10\rangle\langle 0| + |11\rangle\langle 1|$$

We depict the contraction of the output of the AND-tensor with $|1\rangle$ and $|0\rangle$ as



In (a) the contraction results in creation of the product state $|11\rangle$ and in (b) the contraction yields $|00\rangle + |01\rangle + |10\rangle$.

Categorical Tensor Network States
Jacob D. Biamonte, Stephen R. Clark and Dieter Jaksch
AIP Advances **1(4)**, 042172 (2011). http://arxiv.org/abs/1012.0531

Boolean tensor network states

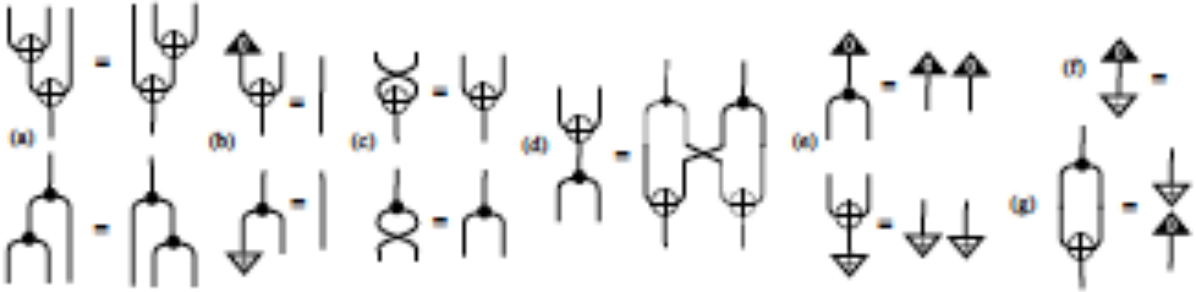


FIG. 3. Read top to bottom. A presentation of the linear fragment of the Boolean calculus. The plus ($\oplus$) dots are XOR and the black ($\bullet$) dots represent COPY. The details of (a)-(g) will be given in Sections III and IV. For instance, (d) represents the bialgebra law and (g) the Hopf-law (in the case of qubits $x \oplus x = 0$, in higher dimensions the units $\langle +|$ becomes $\langle 0| + \langle 1| + \cdots + \langle d-1|$).

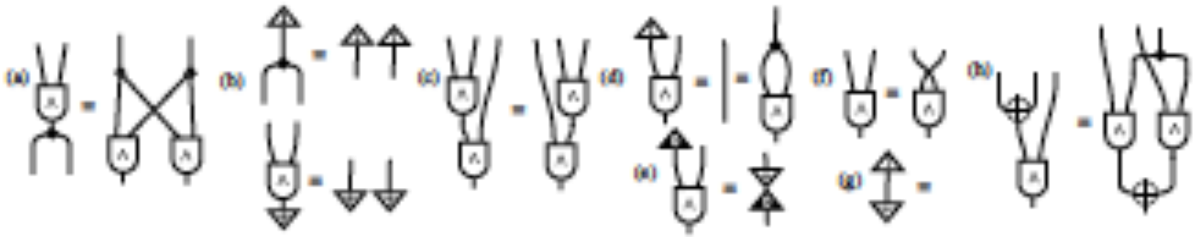
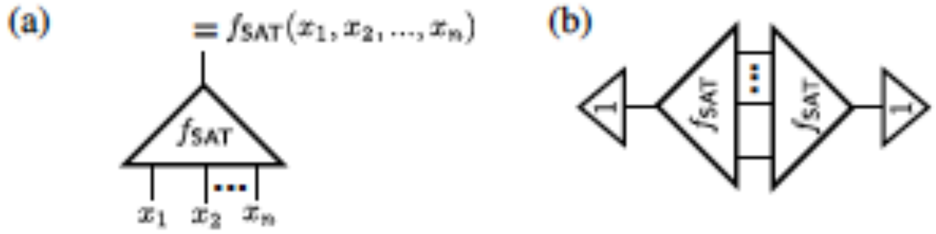


FIG. 4. Diagrams read top to bottom. A presentation of the Boolean-calculus with Figure 3. The details of (a)-(g) will be given in Sections III and IV. For instance, (h) represents distributivity of AND($\wedge$) over XOR ($\oplus$), and (d) shows that $x \wedge x = x$.

Contracting tensor networks to solve problems

Remark 3.1 (Graphical depiction) *The algorithm is depicted below. (a) gives a network realization of the function and determining if the network in (b) contracts to a value greater than zero solves a SAT instance.*



Related also to work in progress, **Problem solving by strongly simulating quantum circuits**
T. H. Johnson J. D. Biamonte, S. R. Clark, and D. Jaksch and also related work with Jason Morton

When a tensor network represents a physical state

Theorem 3.2 (Penrose, 1967) *The norm of a tensor network vanishes iff the physical situation it represents is forbidden by the rules of quantum mechanics.*

Deciding if tensors are physical?

Example 3.3 (Examples of Penrose's theorem) *For example, consider a Bell state $\Phi^+ = |00\rangle + |11\rangle$. The amplitude of the first party measuring $|0\rangle$ followed by the second party measuring $|1\rangle$ is zero. This vanishing tensor network contraction is given by $\langle 01, \Phi^+\rangle$. A second example is found by considering the norm of a state $|\psi\rangle$ formed by a network of connected tensors, by taking an inner product with a conjugated copy of itself $\langle \psi, \psi\rangle$. If this inner product vanishes, the network necessarily represents a non-physical quantum state, by Penrose's theorem.*

Corollary 3.4 *All physical Boolean states are satisfiable.*

Quantum measurement occurrence is undecidable,
J. Eisert, M. P. Mueller, C. Gogolin,
To appear in PRL
http://arxiv.org/abs/1111.3965

Problem solving by strongly simulating quantum circuits
T. H. Johnson J. D. Biamonte, S. R. Clark, and D. Jaksch and also related work with Jason Morton

Graphical SVD

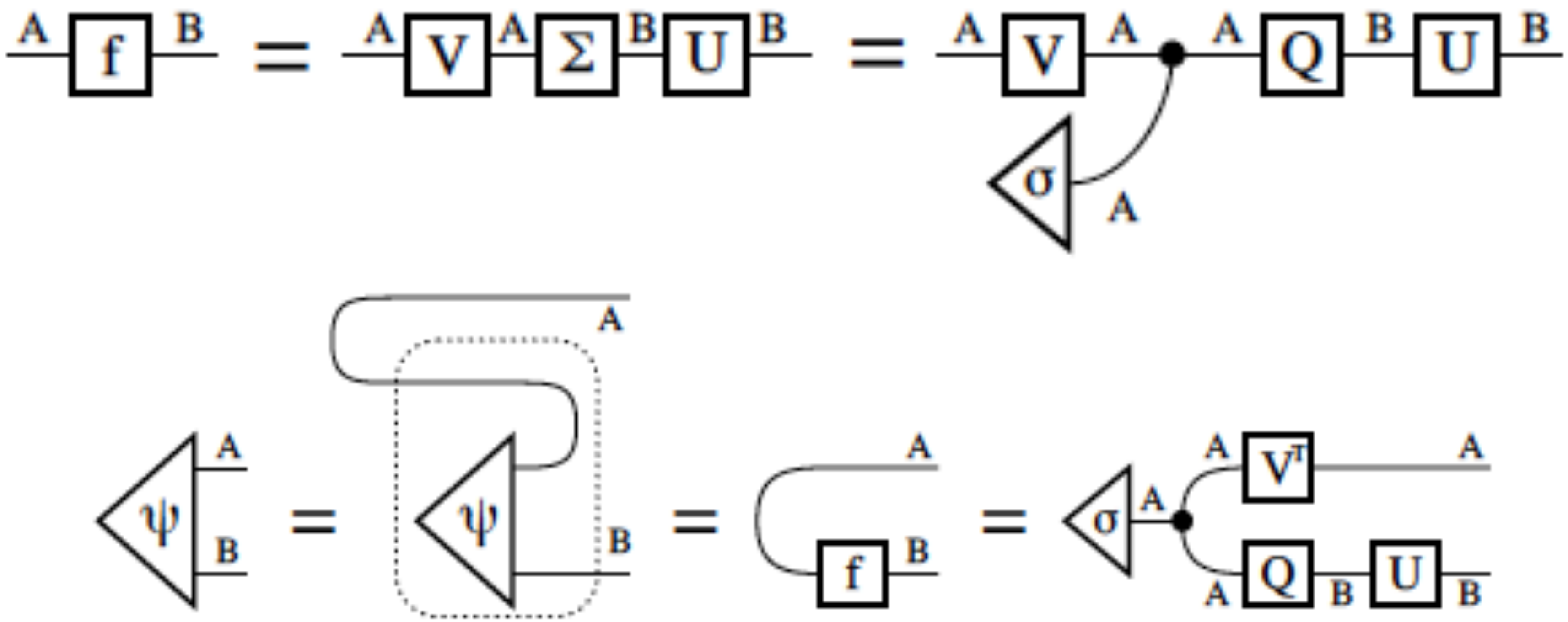
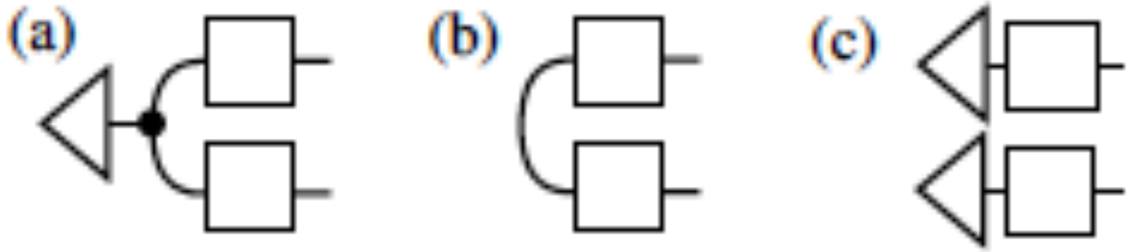


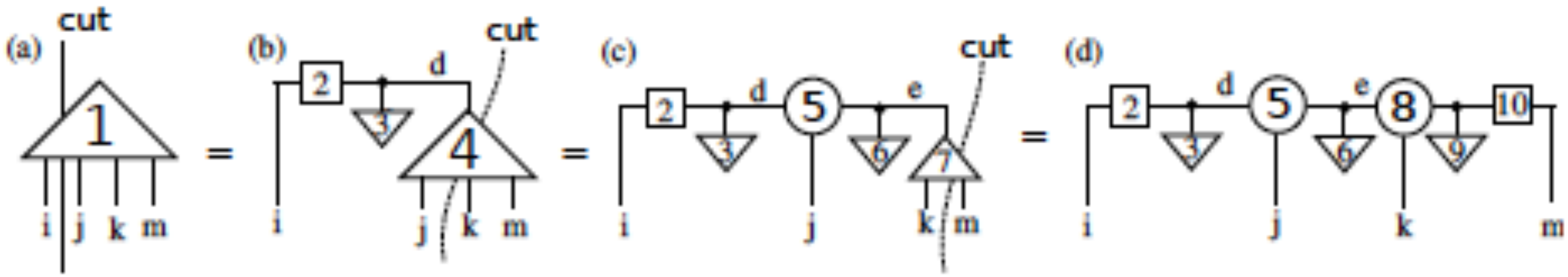
FIG. 3. Diagrammatic Schmidt decomposition for $|\psi\rangle \in A \otimes B$.

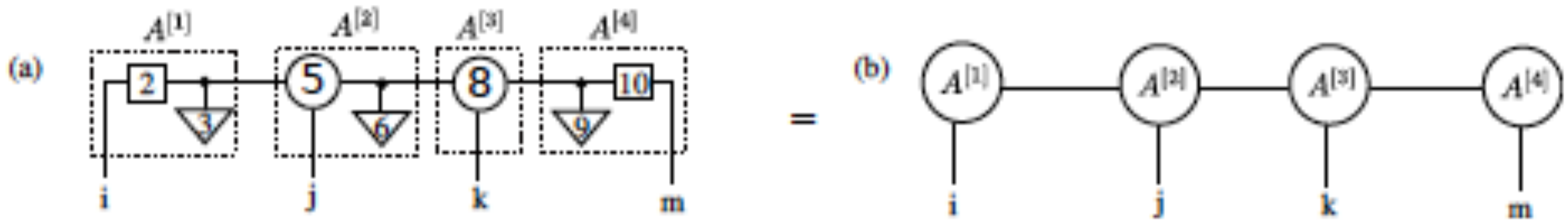
Invariant Theory for Matrix Product States
Jacob Biamonte, Ville Berghlom and Marco Lanzagorta
(2012)

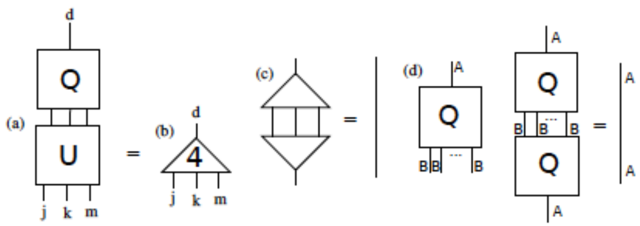
Entanglement topology



MPS





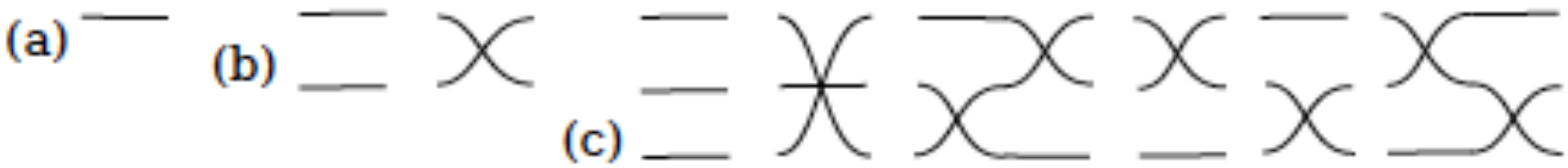


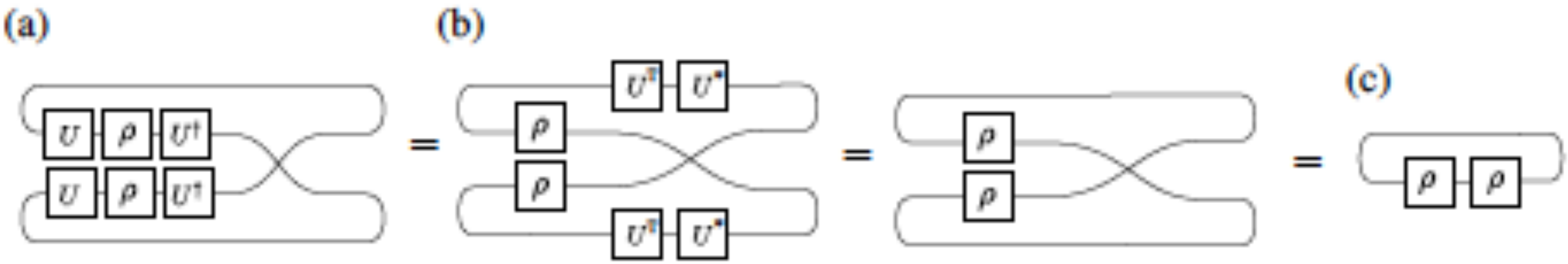
References for MPS in the paper

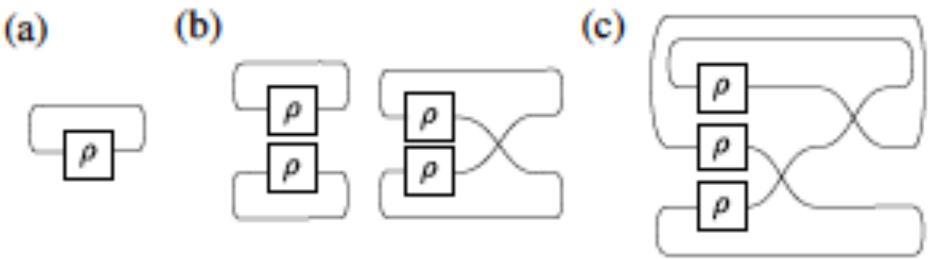
Polynomial invariants

- Written in terms of a coefficients of a state

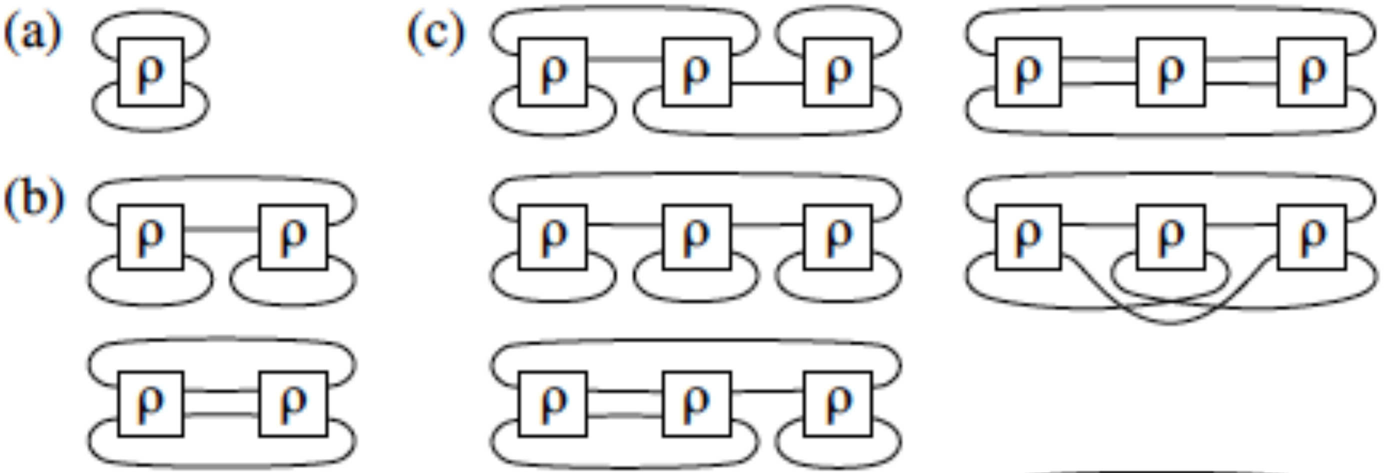
Contracting permutations

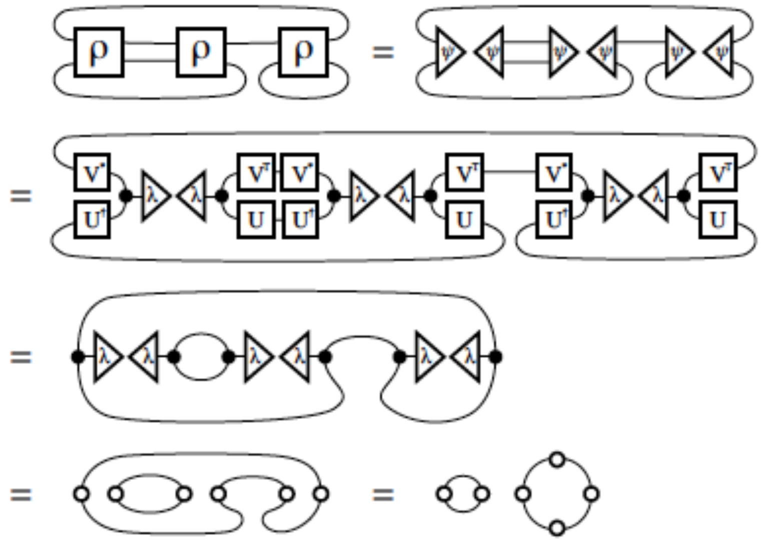




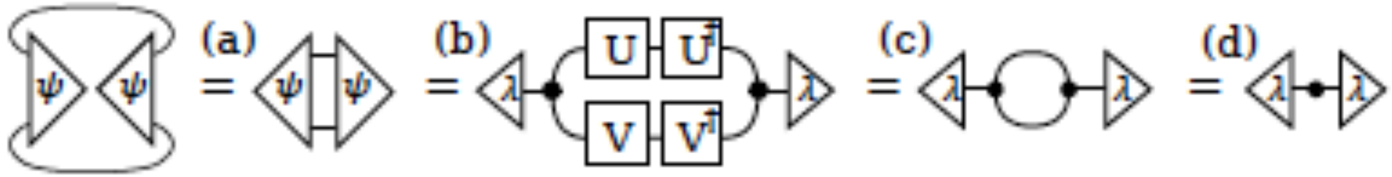


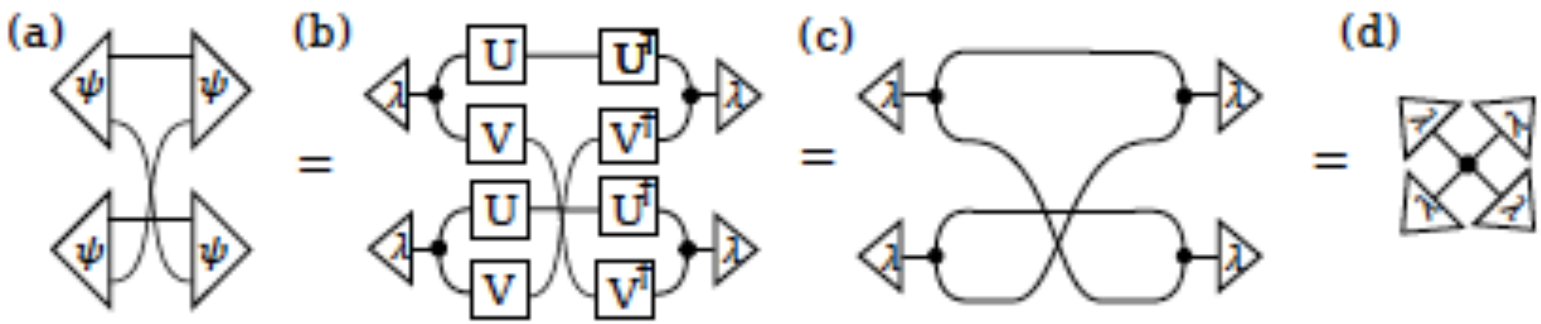
Alternatives to Tr ρ^n



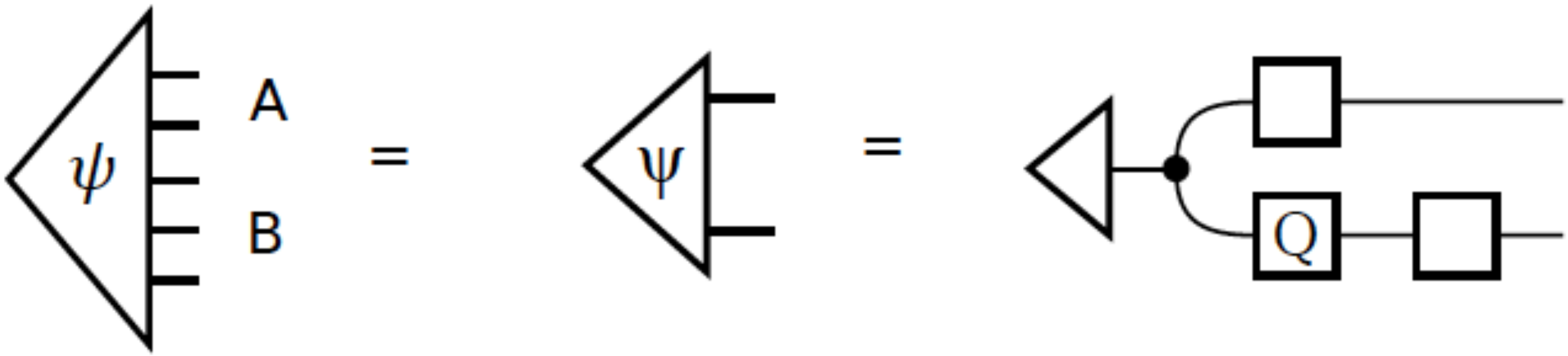


Returning to the diagrammatic SVD

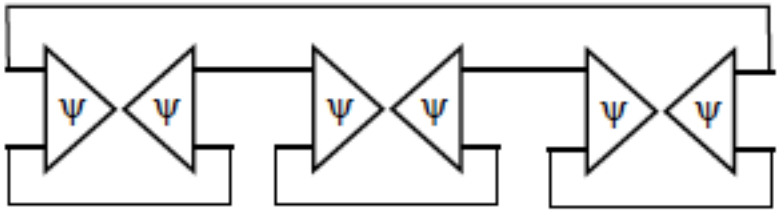




Example



Example part II



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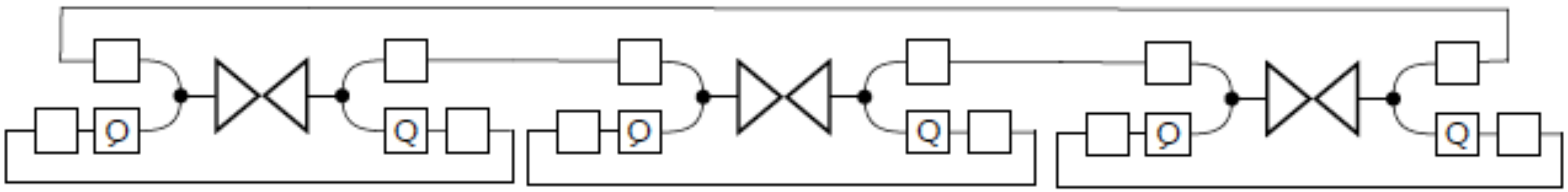

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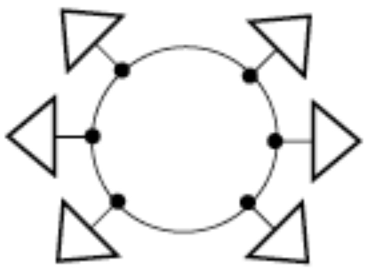

References

- **Categorical Tensor Network States,** Jacob D. Biamonte, Stephen R.Clark and Dieter Jaksch, AIP Advances **1(4)**, 042172 (2011). http://arxiv.org/abs/1012.0531
- **Categorical Quantum Circuits**, Ville Bergholm and Jacob Biamonte, In *Journal of Physics A: Mathematical and Theoretical*, Vol. 44, No. 17, pages 25304-25324, 2011, http://arxiv.org/abs/1010.4840
- **Algebraically contractible topological tensor network states**, S. J. Denny, J. D. Biamonte, D. Jaksch and S. R. Clark, J. Phys. A: Math. Theor. 45 015309, (2012), http://arxiv.org/abs/1108.0888
- **Tensor networks for entanglement evolution**, Sebastian Meznaric and Jacob Biamonte, in review, (2012)http://arxiv.org/abs/1204.3599
- **Tensor networks and graphical calculus for open quantum systems**, Christopher J. Wood, Jacob D. Biamonte and David G. Cory, *in review (2011)*, http://arxiv.org/abs/1111.6950
- **Quantum Techniques for Stochastic Mechanics ,** John Baez and Jacob Biamonte, Book in progress, (2012), http://math.ucr.edu/home/baez/quantum-mathematics.pdf
- **Invariant Theory for Matrix Product States,** Jacob Biamonte, Ville Berghlom and Marco Lanzagorta, (2012)