Gapless Hamiltonians for non-injective Matrix Product States.

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Networking Tensor Networks, Benasque 2012.

Outline

- Non-robustness of parent Hamiltonian construction (under perturbation of the tensor description).
- Construction of uncle Hamiltonians (for non-injective MPSs).
- Properties of uncle Hamiltonians: ground space and spectrum.
- Injective MPS case.

Matrix Product State description

A **translationally invariant** with p.b.c. MPS can be written as

$$|M(A)\rangle = \sum_{i_1,\ldots,i_L=1}^{d} tr[A_{i_1}\cdots A_{i_L}]|i_1,\ldots,i_L\rangle,$$

with $A_{i_j} \in \mathcal{M}_D$.

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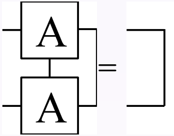
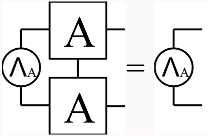
with $A_{i_j} \in \mathcal{M}_D$.
Canonical form: after blocking a finite number of spins, one can chose the matrices A_{i_j} with either $\mathrm{span}\{A_{i_j}\} = \mathcal{M}_D$ (called **injective**) or

$$\mathrm{span}\{A_{i_j}\} = \begin{pmatrix} \mathcal{M}_{D_1} & & & & & \\ & \mathcal{M}_{D_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \mathcal{M}_{D_k} \end{pmatrix},$$

$\sum D_i = D$ (called **block-injective**).

Normalization conditons:

1) $\sum_i A_i A_i^* = \mathbb{I}$: 

2) $\sum_i A_i^* \Lambda_A A_i = \Lambda_A$: 

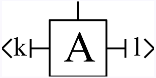
,

for certain diagonal full rank diagonal positive matrix Λ_A with $\mathrm{tr}(\Lambda_A) = 1$.

Parent Hamiltonian construction

The tensor A induces a map from the virtual to the physical level

$$\Phi_A(|k, l\rangle) =$$



$$= \sum_j \langle k | A_{i_j} | l \rangle | i_j \rangle$$

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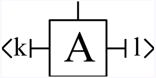


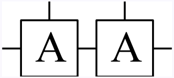
$$= \sum_j \langle k|A_{i_j}|l\rangle|i_j\rangle$$

A injective $\Rightarrow \Phi_A$ injective.

A block-injective $\Rightarrow \Phi_A$ is not injective.

$$A = \begin{pmatrix} A_{D_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{D_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{D_k} \end{pmatrix}$$

Parent Hamiltonian construction

We then consider the map induced by the contracted tensor at two (or more) sites, $A \overset{c}{-} A =$ , also injective or block-injective (as A is). One must then define the local Hamiltonian as the projector with kernel range$(\Phi_{A \overset{c}{-} A})$: $h_{\mathrm{loc}} = \mathbb{I} - \Pi_{\mathrm{range}(\Phi_{A \overset{c}{-} A})}$.
$H = \sum h_{\mathrm{loc}}$ is the parent Hamiltonian.

Ground space $= \ker(H) = \cap \ker(h_{\mathrm{loc}})$.

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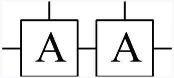
We then consider the map induced by the contracted tensor at two (or more) sites, $A \overset{c}{-} A =$ , also injective or block-injective (as A is). One must then define the local Hamiltonian as the projector with kernel range$(\Phi_{A \overset{c}{-} A})$: $h_{\mathrm{loc}} = \mathbb{I} - \Pi_{\text{range}(\Phi_{A \overset{c}{-} A})}$.
$H = \sum h_{\mathrm{loc}}$ is the parent Hamiltonian.

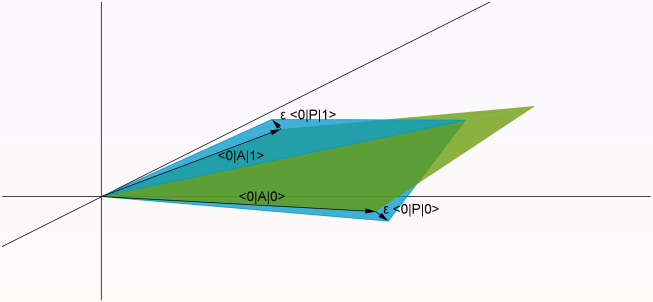
Ground space $=$ ker$(H) = \cap$ ker(h_{loc}).

Key properties: parent Hamiltonian is **local**, **frustration free**, **gapped** over the ground space, has as **unique ground state** the MPS (injective case) **or** is k-**fold degenerate** (k-block injective case, ground space spanned by the different blocks).

Robustness of the parent Hamiltonian construction

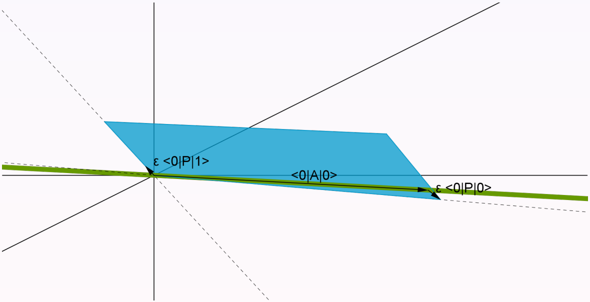
Perturbing A means perturbing h_{loc}.
Let P be the perturbation tensor and $A + \varepsilon P$ the perturbed tensor, which turns to be generally injective.

A injective $\Rightarrow \dim(\mathrm{range}(\phi_A)) = \dim(\mathrm{range}(\phi_{A+\varepsilon P})) = D^2$. Good way back when $\varepsilon \to 0$.

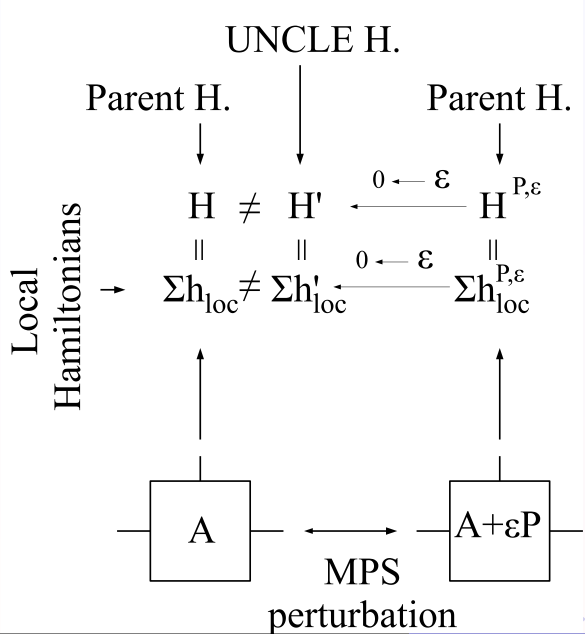


Robustness of the parent Hamiltonian construction

A non-injective $\Rightarrow \dim(\text{range}(\phi_A)) < \dim(\text{range}(\phi_{A+\varepsilon P})) = D^2$. New directions appear when $\varepsilon \to 0$. Parent Hamiltonian not robust in this case.



Uncle Hamiltonian



GHZ parent Hamiltonian 'perturbed'

GHZ unnormalized state: $|\mathrm{GHZ}\rangle = |00\cdots 0\rangle + |11\cdots 1\rangle$.

$|\mathrm{GHZ}\rangle = \sum_{i_1,\ldots,i_L=1}^{d} tr[A_{i_1}\cdots A_{i_L}]|i_1,\ldots,i_L\rangle$,
with $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Parent Hamiltonian: $h_{\mathrm{loc}} = \mathbb{I} - \Pi_{\mathrm{range}(\Phi_{\boldsymbol{A} \overset{\boldsymbol{c}}{\_} \boldsymbol{A} \overset{\boldsymbol{c}}{\_} \boldsymbol{A}})}$.
$\ker h_{\mathrm{loc}} = \mathrm{span}\{|000\rangle, |111\rangle\}$.

GHZ parent Hamiltonian 'perturbed'

Under a generic perturbation $P_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ and $P_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, we get a new injective tensor $A + \varepsilon P$.

The new parent Hamiltonians have as local Hamiltonian $h^{\varepsilon}_{\mathrm{loc}}$ the projection with kernel spanned by

$$|000\rangle + O(\varepsilon), (b_0 + b_1)(|001\rangle + |011\rangle) + O(\varepsilon),$$

$$(c_0 + c_1)(|100\rangle + |110\rangle) + O(\varepsilon), |111\rangle + O(\varepsilon).$$

If $b_0 + b_1 \neq 0$ and $c_0 + c_1 \neq 0$, the limit as $\varepsilon \to 0$ is h'_{loc}, with kernel spanned by
$|000\rangle, |001\rangle + |011\rangle, |100\rangle + |110\rangle, |111\rangle$.

General construction of uncle Hamiltonians

MPS tensor $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, perturbation tensor $P = \varepsilon \begin{pmatrix} P^A & R \\ L & P^B \end{pmatrix}$

Two-sites perturbed tensor: $(A + \varepsilon P) \stackrel{c}{-} (A + \varepsilon P) =$

$$\begin{pmatrix} A \stackrel{c}{-} A + O(\varepsilon) & \varepsilon(A \stackrel{c}{-} R + R \stackrel{c}{-} B) + O(\varepsilon^2) \\ \varepsilon(B \stackrel{c}{-} L + L \stackrel{c}{-} A) + O(\varepsilon^2) & B \stackrel{c}{-} B + O(\varepsilon) \end{pmatrix}.$$

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In the limit, the tensor determining the local Hamiltonian projector $h'_{\mathrm{loc},P}$ is

$$\begin{pmatrix} A \stackrel{c}{-} A & A \stackrel{c}{-} R + R \stackrel{c}{-} B \\ B \stackrel{c}{-} L + L \stackrel{c}{-} A & B \stackrel{c}{-} B \end{pmatrix},$$

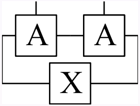
as long as this last tensor is injective (which happens almost surely).

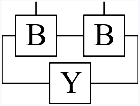
General construction of uncle Hamiltonians

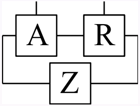
For the projector $h'_{\mathrm{loc},P}$ induced by the tensor

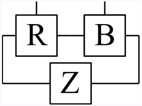
$$\begin{pmatrix} A \overset{c}{-} A & A \overset{c}{-} R + R \overset{c}{-} B \\ B \overset{c}{-} L + L \overset{c}{-} A & B \overset{c}{-} B \end{pmatrix},$$

the local kernel is

$\Big\{$ 

, 

, 

$+$ 

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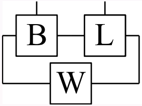
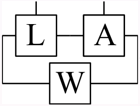


$+$ 

$, X, Y, Z, W \Big\}$.

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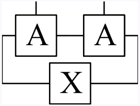
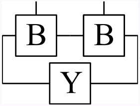
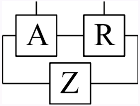
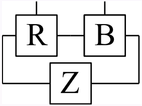
$\Bigg\{$ 

, 

, 

$+$ 

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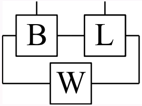
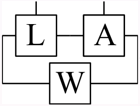


$+$ 

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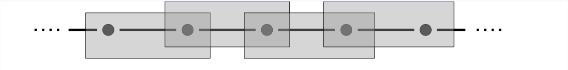
The uncle Hamiltonian is then $H'_P = \sum h'_{\mathrm{loc},P}$.

Uncle Hamiltonian ground space

$H'_P = \sum h'_{\mathrm{loc},P} \Rightarrow \ker(H'_P) = \cap \ker(h'_{\mathrm{loc},P})$
Many consecutive kernels intersected (not all!):



GHZ: $\cap \ker(h'_{\mathrm{loc}}) =$
$\mathrm{span}\left\{ \otimes^m |0\rangle, \otimes^m|1\rangle, \sum_i |0\cdots 01^i 1\cdots 1\rangle, \sum_i |1\cdots 11^i 0\cdots 0\rangle \right\}$

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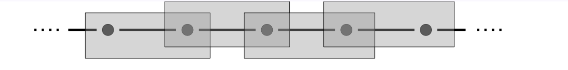

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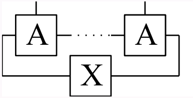
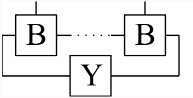
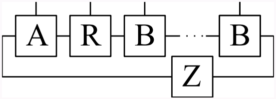
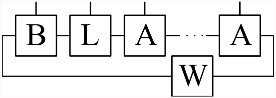
Non-injective MPSs: $\cap\, \mathrm{ker}(h'_{\mathrm{loc},P}) =$

$\Bigg\{$ 

, 

, $\sum_{\mathrm{posR}}$ 

$\sum_{\mathrm{posL}}$ 

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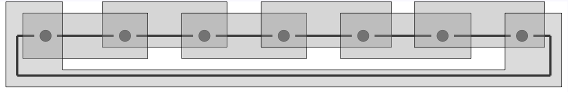
Closing the loop: 

GHZ case:
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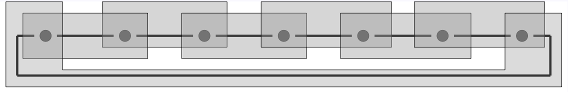
Closing the loop: 

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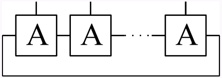
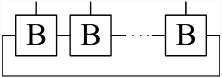
For non-injective MPSs:

$\mathrm{ker}(H'_P) = \mathrm{span}\Bigg\{$  ,  $\Bigg\}$.

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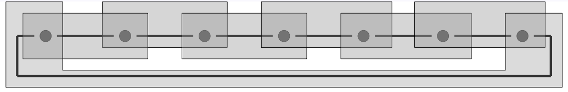
Closing the loop: 

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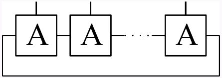
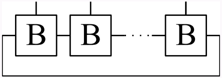
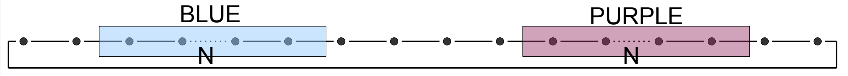
$\ker(H'_P) = \text{span}\Bigg\{$  ,  $\Bigg\}$.

$\ker(H'_P) = \ker(H)$, for almost every perturbation.

Uncle Hamiltonians are gapless

GHZ Uncle:



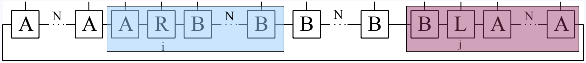
$$|\varphi_N\rangle = \sum_{\substack{i\in\text{BLUE}\\ j\in\text{PURPLE}}} |00\cdots0^i11\cdots1^j0\cdots0\rangle$$

$\langle\varphi_N|\varphi_N\rangle = \Theta(N^2)$ and $\langle\varphi_N|H'|\varphi_N\rangle = \Theta(N)$

The states $|\varphi_N\rangle$ are orthogonal to the ground space, and have energy $O(1/N)$.

Uncle Hamiltonians are gapless

Non-injective MPSs Uncle:

$|\varphi_N\rangle = \sum_{\substack{i \in \mathrm{BLUE} \\ j \in \mathrm{PURPLE}}}$ 

$\langle \varphi_N | \varphi_N \rangle = \Theta(N^2)$ and $\langle \varphi_N | H'_P | \varphi_N \rangle = \Theta(N)$

The states $|\varphi_N\rangle$ tend to be orthogonal to the ground space (exponentially), and have energy $O(1/N)$.

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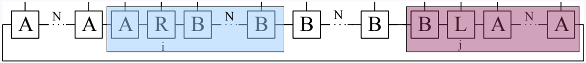
$|\varphi_N\rangle = \sum_{\substack{i \in \mathrm{BLUE} \\ j \in \mathrm{PURPLE}}}$ 

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The states $|\varphi_N\rangle$ tend to be orthogonal to the ground space (exponentially), and have energy $O(1/N)$.

The uncle Hamiltonians are gapless, for almost every perturbation.

$\sigma(H') = \mathbb{R}^+$

H' acting on the closure of $S = \cup_{i<j} |\cdots 00^i\rangle \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes |0^j 00 \cdots\rangle$.

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Low energy vectors:
$|\varphi_N\rangle = \sum_{\substack{-N<i<-1 \\ 1<j<N}} |\cdots 00\rangle \otimes |0^{-N}0\cdots 0^i 11\cdots 1^j 0\cdots 0^N\rangle \otimes |00\cdots\rangle$

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H' acting on the closure of $S = \cup_{i<j} | \cdots 00^i\rangle \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes |0^j 00 \cdots\rangle$.

Low energy vectors:
$|\varphi_N\rangle = \sum_{\substack{-N<i<-1 \\ 1<j<N}} | \cdots 00\rangle \otimes |0^{-N}0 \cdots 0^i 11 \cdots 1^j 0 \cdots 0^N\rangle \otimes |00 \cdots\rangle$

$\exists \lambda_j \in \sigma(H'),\ \lambda_j \to 0$, with 'approximated eigenvectors' (Weyl sequences) in S such that $\|(H' - \lambda_j I)|\varphi_{\lambda_j,k}\rangle\| < 1/k$.

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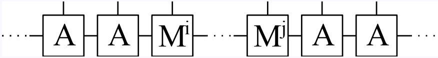
$$H'(|\cdots 000\varphi_{\lambda_j,k}00000\varphi_{\lambda_l,k}000\cdots\rangle) =$$
$$= |\cdots 0H'(00\varphi_{\lambda_j,k}00)000\varphi_{\lambda_l,k}00\cdots\rangle + |\cdots 00\varphi_{\lambda_j,k}000H'(00\varphi_{\lambda_l,k}00)0\cdots\rangle \sim$$
$$\sim (\lambda_j + \lambda_l)(|\cdots 000\varphi_{\lambda_j,k}00000\varphi_{\lambda_l,k}000\cdots\rangle)$$

And $\lambda_j + \lambda_l \in \sigma(H')$. The same procedure for any finite sum.

H'_P acting on the closure of

$S = \Big\{$ 

$, i < j \Big\}.$

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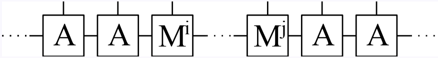
$S = \Big\{$  $, i < j \Big\}.$

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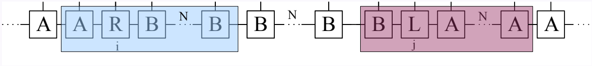
$|\varphi_N\rangle = \sum_{\substack{i \in \text{blue} \\ j \in \text{purple}}}$ 

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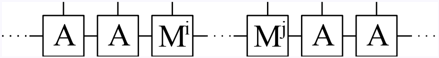
$S = \Big\{$ 

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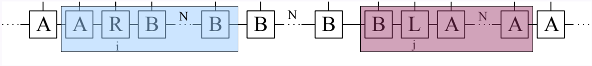
$|\varphi_N\rangle = \sum_{\substack{i \in \text{blue} \\ j \in \text{purple}}}$ 

Concatenation must me made separating the 'core' blocks enough.

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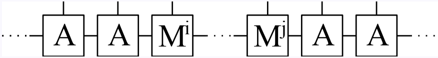
$S = \Big\{$  $, i < j \Big\}.$

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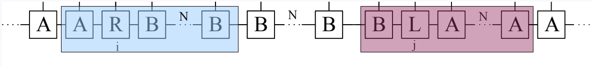
$|\varphi_N\rangle = \sum_{\substack{i \in \text{blue} \\ j \in \text{purple}}}$ 

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Finite sums of any sequence tending to 0 is dense in $\mathbb{R}^+$, and the spectrum is closed.

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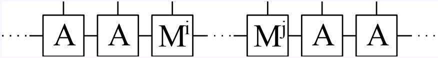
$S = \Big\{$ 

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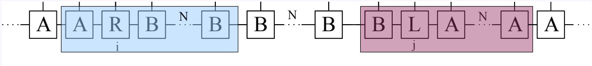
$|\varphi_N\rangle = \sum_{\substack{i \in \text{blue} \\ j \in \text{purple}}}$ 

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Finite sums of any sequence tending to 0 is dense in $\mathbb{R}^+$, and the spectrum is closed.

The spectra of the thermodynamic limit of the uncle Hamiltonians are equal to $\mathbb{R}^+$, for almost every perturbation.

$\sigma(H'_P) = \mathbb{R}^+$

Approximated eigenvectors for $\lambda_1, \ldots, \lambda_n$ embedded (or approximately embedded) in some finite size chain $\Rightarrow$ existence of eigenvalues close to $\lambda_1, \ldots, \lambda_n$.

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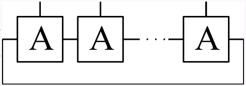
The spectra of the uncle Hamiltonians on finite size chains tend to be dense in $\mathbb{R}^+$, for almost every perturbation.

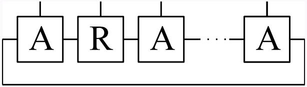
Injective MPSs

For an injective MPS with tensor description A, one can consider a non-injective tensor description:

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

In this case $ker(H') =$

span$\{$ 

$, \sum_{\mathrm{pos} R}$ 

$\}$.

Further work

Uncle Hamiltonians for G-injective PEPS.

For 'orthogonal' perturbations, example with the toric code at arXiv:1111.5817v2

Uncle Hamiltonian for toric code: toric code as ground space, local, frustration free, gapless, with spectrum equal to $\mathbb{R}^+$

Summary

- We have perturbed the MPS description of a state and considered the corresponding parent Hamiltonian.
- As the limit of these Hamiltonians we have gotten a new family: the **uncle Hamiltonians**.
- The uncle Hamiltonians are **local** and **frustration free** and have **the same ground space as the parent Hamiltonian** (for non-injective MPSs).
- The uncle Hamiltonians are generally **gapless**, and their spectra are generally equal to $\mathbb{R}^+$, in contrast with the gap that the parent Hamiltonian exhibits.

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Thank you very much for your attention!