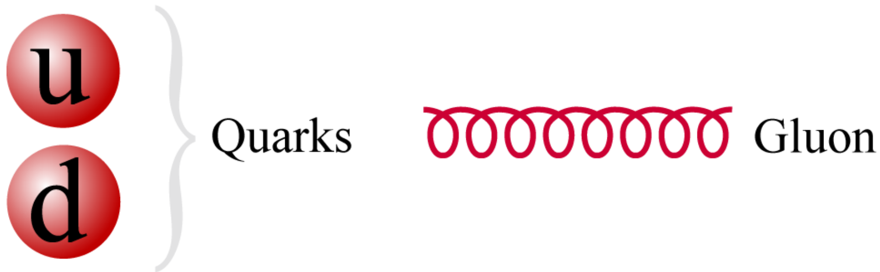
Lattice gauge theory problems for tensor networks

Karl Jansen

- **Introduction**
- **Studying strong interaction by a 4-dimensional lattice**
  - Breakthrough in Simulation Algorithm
  - Selected Results
- **A toy model challenge for tensor networks**
- **Summary**

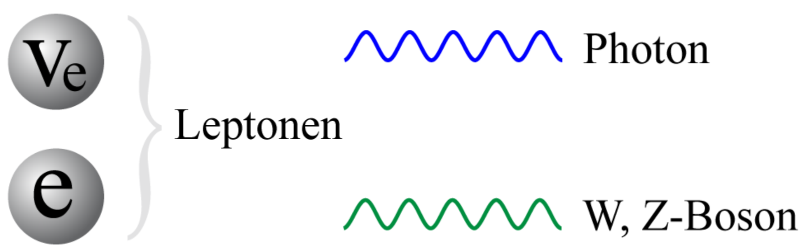
strong force

Particles **Interaction**

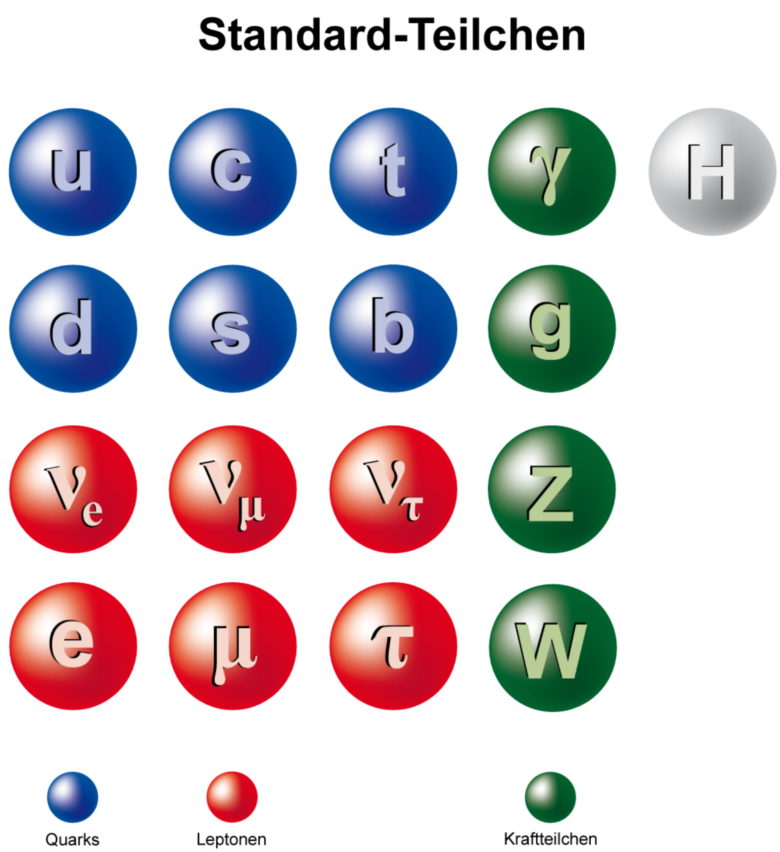


elektroweak force

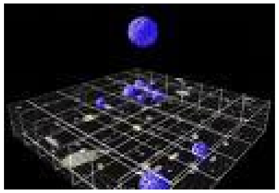
Particles **Interaction**



The complete Particle Spectrum



- there are three generations
- is there a Higgsboson?
- Gravitation, interaction particle $\rightarrow$ Graviton



Why we believe in Quantum Field Theories: Electromagnetic Interaction

Quantum Electrodynamics (QED)

coupling of the electromagnetic interaction is small
$\Rightarrow$ perturbation theory 4-loop calculation

anomalous magnetic moment of the electron

$$
\begin{aligned}
a_e(\text{theory}) &= 1159652201.1(2.1)(27.1)\cdot 10^{-12}\\
a_e(\text{experiment}) &= 1159652188.4(4.3)\cdot 10^{-12}
\end{aligned}
$$

anomalous magnetic moment of the muon

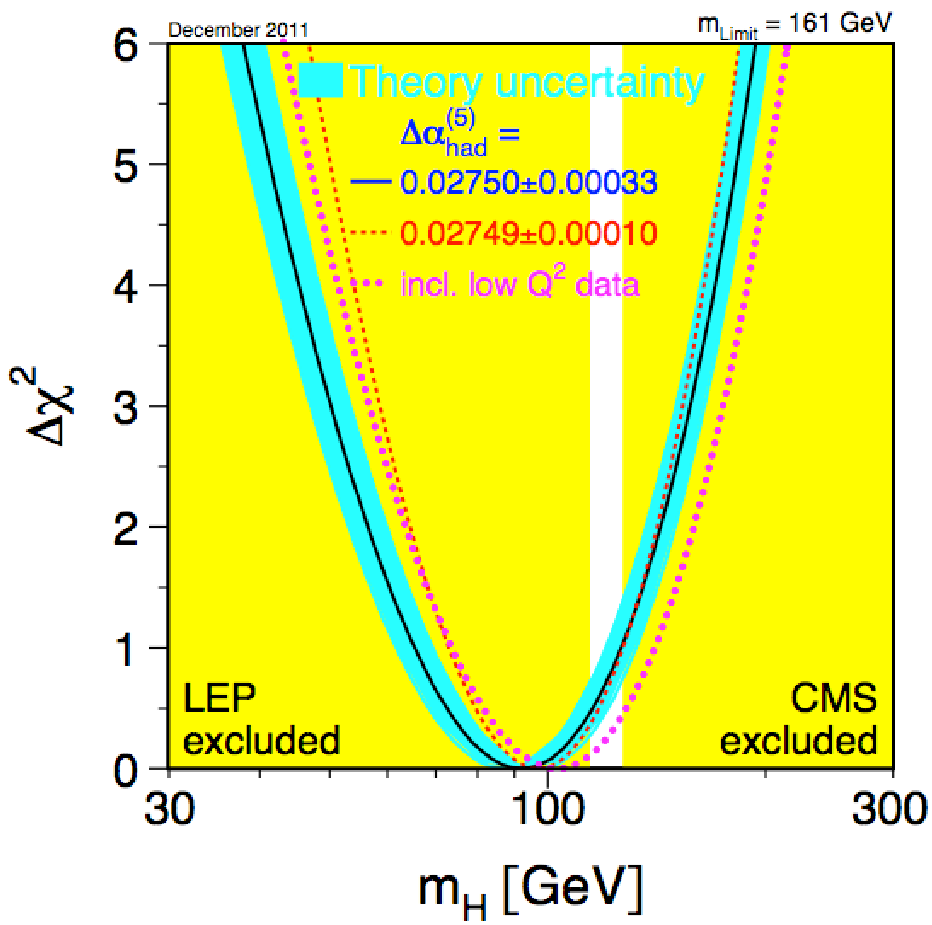
$$
\begin{aligned}
a_\mu(\text{theory}) &= 11659169.6(9.4)\cdot 10^{-10}\\
a_\mu(\text{experiment}) &= 11659203.0(8.0)\cdot 10^{-10}
\end{aligned}
$$

The Large Hadron Collider (LHC)

The search of the missing stone of the standard model of particle interaction:

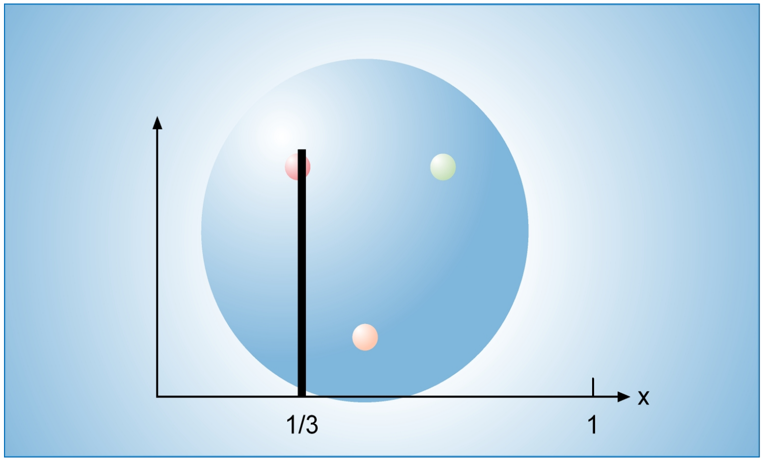
The Higgs Boson generates mass of all quarks and leptons

Experimental bounds on the Higgs boson mass



- only small window remains
- excess at $125 - 126\mathrm{GeV}$
- clarification with this years' run
  (most probably)

Quarks are the fundamental constituents of nuclear matter



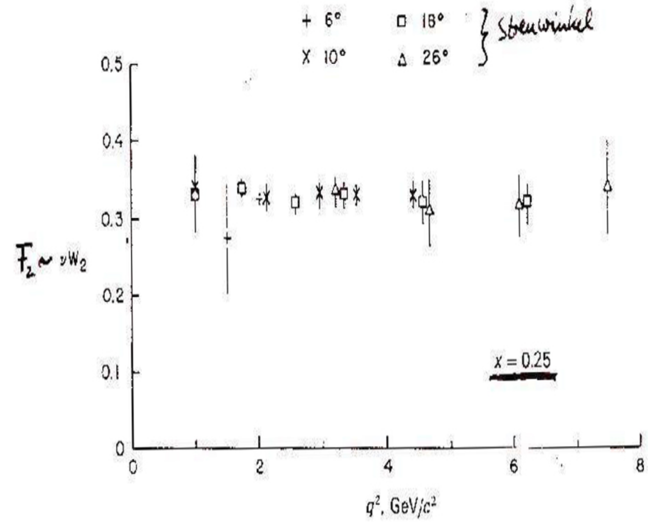


Fig. 7.17 νW_2 (or F_2) as a function of q^2 at $x = 0.25$. For this choice of x, there is practically no q^2-dependence, that is, exact "scaling". (After Friedman and Kendall 1972.)

Friedman and Kendall, 1972)

$f(x, Q^2)\big|_{x\approx 0.25, Q^2 > 10\mathrm{GeV}}$ independent of Q^2

(x momentum of quarks, Q^2 momentum transfer)

Interpretation (Feynman): scattering on single quarks in a hadron
$\rightarrow$ (Bjorken) scaling

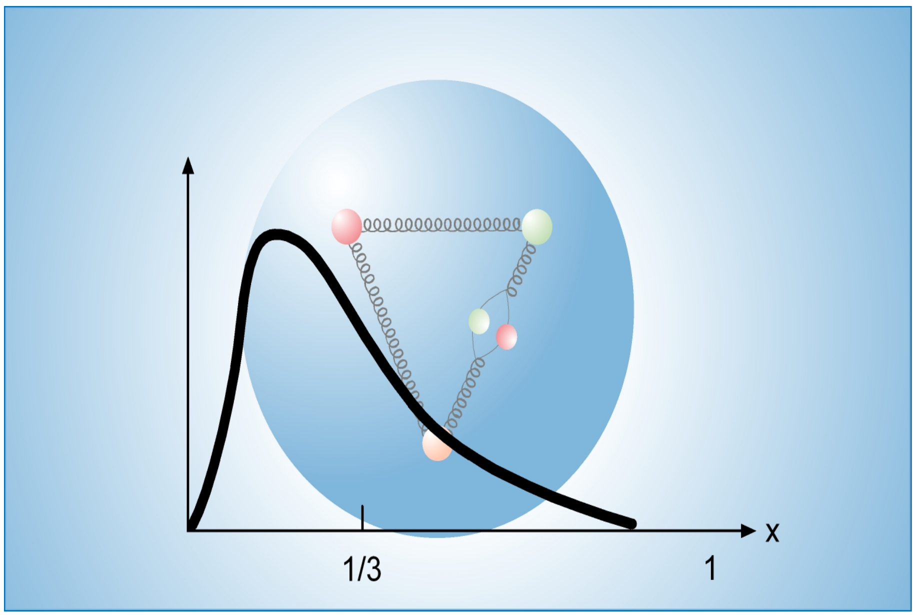
Quantum Fluctuations and the Quark Picture

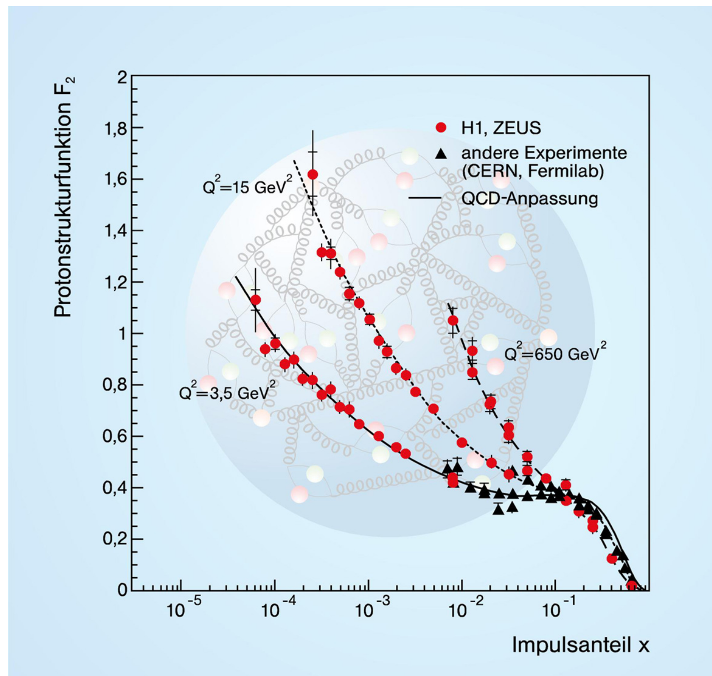
analysis in perturbation theory

$$\int_0^1 dx\, f(x,Q^2) = 3\left[1 - \frac{\alpha_s(Q^2)}{\pi} - a(n_f)\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 - b(n_f)\left(\frac{\alpha_s(Q^2)}{\pi}\right)^3\right]$$

– $a(n_f), b(n_f)$ calculable coefficients

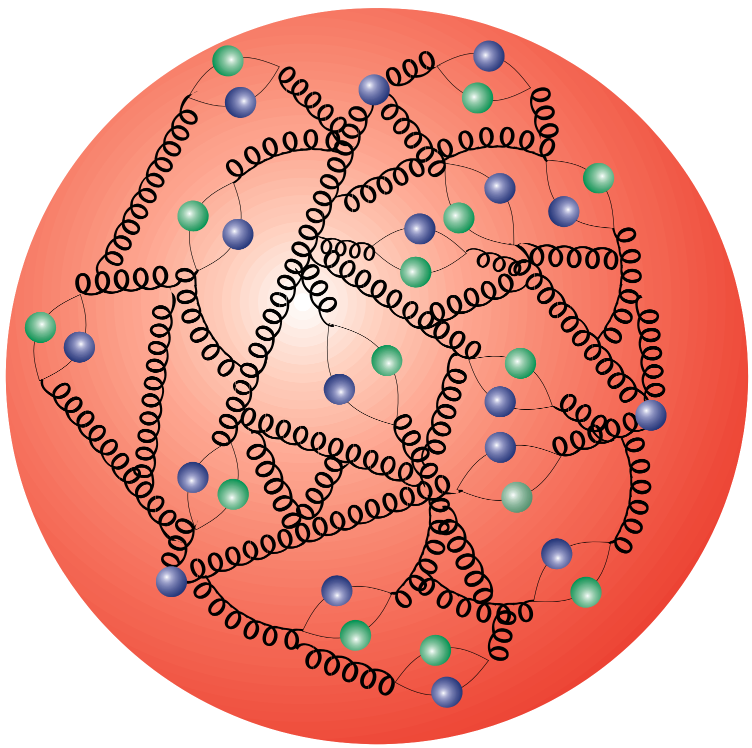
deviations from scaling $\rightarrow$ determination of strong coupling





Why Perturbation Theory fails for the Strong Interaction

- situation becomes incredibly complicated



- value of the coupling (expansion parameter) $\alpha_{\text{strong}}(1\text{fm}) \approx 1$

$\Rightarrow$ need different ("exact") method

$\Rightarrow$ has to be non-perturbative

- Wilson's Proposal: Lattice Quantum Chromodynamics

Lattice Gauge Theory had to be invented

$\rightarrow$ QuantumChromoDynamics

asymptotic freedom		confinement
distances $\ll 1\mathrm{fm}$		distances $\gtrsim 1\mathrm{fm}$
world of quarks and gluons		world of hadrons and glue balls
perturbative description		non-perturbative methods

Unfortunately, it is not known yet whether the quarks in quantum chromodynamics actually form the required bound states. To establish whether these bound states exist one must solve a strong coupling problem and present methods for solving field theories don't work for strong coupling.

Wilson, Cargese Lecture notes 1976

Schwinger model: 2-dimensional Quantum Electrodynamics

Schwinger 1962

(typical system explored in DESY summerstudent programme; www.desy.de/summerstudents/)

Quantization via Feynman path integral

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-S_{\mathrm{gauge}} - S_{\mathrm{ferm}}}$$

Fermion action

$$S_{\mathrm{ferm}} = \int d^2x \bar{\Psi}(x) \left[D_\mu + m \right] \Psi(x)$$

gauge covriant derivative

$$D_\mu \Psi(x) \equiv (\partial_\mu - i g_0 A_\mu(x))\Psi(x)$$

with A_μ gauge potential, g_0 bare coupling

$$S_{\mathrm{gauge}} = \int d^2x F_{\mu\nu} F_{\mu\nu} \ , \ F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

equations of motion: obtain classical Maxwell equations

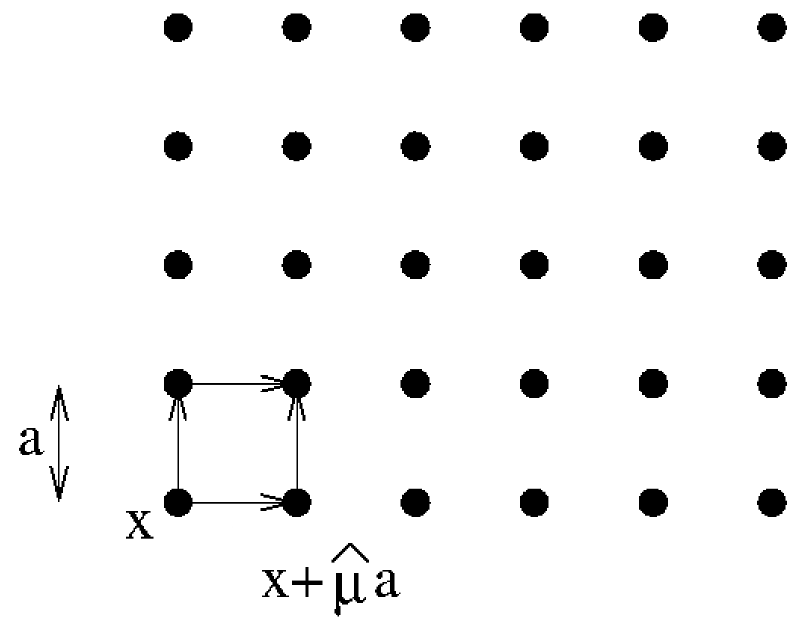
Properties of the Schwinger Model

- existence of bound states (mass gap)
- asymptotic free ($g_0 \to 0$ for distance between charges going to zero)
- exactly solvable for zero fermion mass (Coleman)
- super-renormalizable

$\Rightarrow$ valuable test laboratory for QCD
(simulations can be done on your desk top)

Lattice Schwinger model

introduce a 2-dimensional lattice with lattice spacing a



fields $\Psi(x)$, $\bar{\Psi}(x)$ on the lattice sites

$x = (t, \mathbf{x})$ integers

discretized fermion action

$S \rightarrow a^2 \sum_x \bar{\Psi} \left[\gamma_\mu \partial_\mu + m\right] \Psi(x)$

$\partial_\mu = \frac{1}{2}\left[\nabla_\mu^* + \nabla_\mu\right]$

$\nabla_\mu \Psi(x) = \frac{1}{a}\left[\Psi(x + a\hat{\mu}) - \Psi(x)\right]$, $\quad \nabla_\mu^* \Psi(x) = \frac{1}{a}\left[\Psi(x) - \Psi(x - a\hat{\mu})\right]$

fermion propagator

$$\tilde{G} = \frac{i}{a}\left[p_\mu \gamma_\mu + m \frac{1}{a}\left(i\gamma_\mu \sin p_\mu a\right) + m\right]^{-1}$$

$\rightarrow$ has poles for $p \approx 0$ AND $p \approx \pi/a$

$\Rightarrow$ doubling of spectrum

The Wilson-Dirac Operator

Wilson's suggestion: add a second derivative term

$$S \rightarrow a^2 \sum_x \bar{\Psi} \left[\gamma_\mu \partial_\mu - r \underbrace{\partial_\mu^2}_{\nabla_\mu^* \nabla_\mu} + m\right] \Psi(x)$$

$$\rightarrow \tilde{G} = \left[\frac{1}{a}\left(i\gamma_\mu \sin p_\mu a\right) + \frac{r}{a} \sum_\mu (1 - \cos p_\mu a) + m\right]^{-1}$$

pole for $p \approx 0$

at $p \approx \pi/a$: $\rightarrow \tilde{G}^{-1} \approx \frac{r}{a}$

$\Rightarrow$ unwanted fermion doubler decouples in continuum limit $(a \rightarrow 0)$

Problem: Wilson term acts as a mass term

$\Rightarrow$ breaking of chiral symmetry: $D\gamma_5 + \gamma_5 D = 0$ for fermion mass $m = 0$

clash between *chiral symmetry* and *fermion proliferation*

$\rightarrow$ Nielsen-Ninomiya theorem:

For any lattice Dirac operator D the conditions

- D is local (bounded by $Ce^{-\gamma/a|x|}$)
- $\tilde{D}(p) = i\gamma_\mu p_\mu + \mathrm{O}(ap^2)$ for $p \ll \pi/a$
- $\tilde{D}(p)$ is invertible for all $p \neq 0$
- $\gamma_5 D + D\gamma_5 = 0$

can not be simultaneously fulfilled

The theorem simply states the fact that the Chern number is a cobordism invariant (Friedan)

Implementing gauge invariance

Wilson's fundamental observation: introduce Paralleltransporter connecting the points x and $y = x + a\hat{\mu}$:

$$U(x,\mu) = e^{iaA_\mu(x)} \in U(1)$$

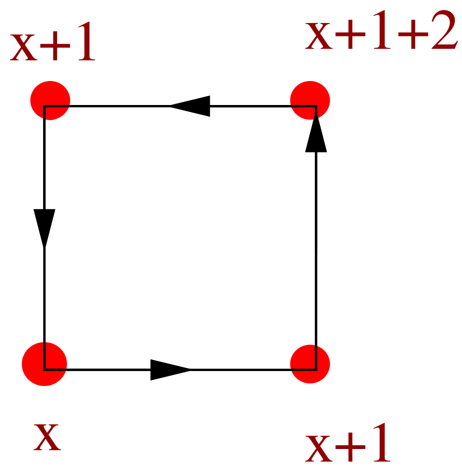
$\Rightarrow$ lattice derivatives
$$\begin{aligned}\nabla_\mu \Psi(x) &= \frac{1}{a}\left[U(x,\mu)\Psi(x+\mu) - \Psi(x)\right] \\ \nabla^*_\mu \Psi(x) &= \frac{1}{a}\left[\Psi(x) - U^{-1}(x-\mu,\mu)\Psi(x-\mu)\right]\end{aligned}$$

action gauge invariant under
$$\Psi(x) \rightarrow g(x)\Psi(x), \;\; \bar{\Psi}(x) \rightarrow \bar{\Psi}(x)g^*(x),$$

$$U(x,\mu) \rightarrow g(x)U(x,\mu)g^*(x+\mu)$$

Self-interaction of gauge fields

basic object: plaquette:



$$U_p = U(x,\mu)U(x+\mu,\nu)U^{-1}(x+\nu,\mu)U^{-1}(x,\nu) \rightarrow F_{\mu\nu}F^{\mu\nu}(x) \quad \text{for} \quad a \rightarrow 0$$

action

$$S_{\text{gauge}} = a^2 \sum_x \left\{ \beta \left[1 - \mathrm{Re}(U_{(x,p)}) \right] \right\}$$

Quantization of Theory

$$\mathcal{Z} = \int_{\mathrm{fields}} e^{-S}$$

Partition functions (pathintegral) with Boltzmann weight (action) S

$$S = a^2 \sum_x \left\{ \beta \left[1 - \mathrm{Re}(U_{(x,p)}) \right] + \bar{\psi} \left[m_0 + \tfrac{1}{2}\{\gamma_\mu(\partial_\mu + \partial_\mu^\star) - a\partial_\mu^\star \partial_\mu\} \right] \psi \right\}$$

Physical Observables

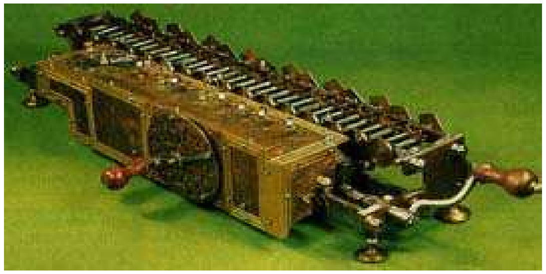
expectation value of physical observables $\mathcal{O}$

$$\underbrace{\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int_{\text{fields}} \mathcal{O} e^{-S}}$$

$\downarrow$ lattice discretization

$$0101110001100011110011$$

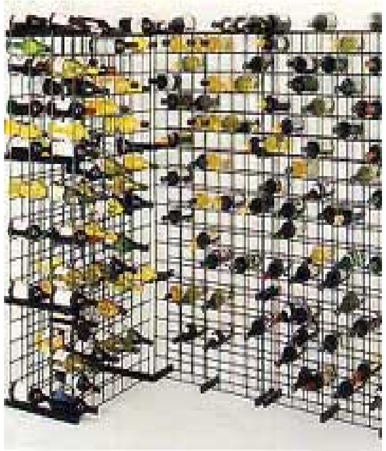
$$\downarrow$$



Lattice QCD

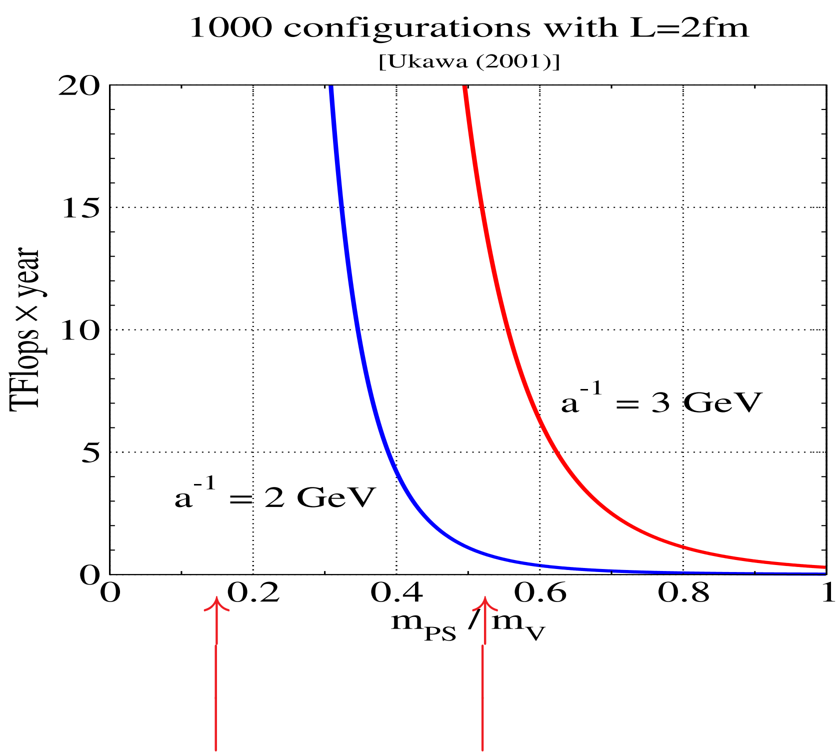
change:

- 2-d $\rightarrow$ 4-d
- gauge field $U(x,\mu) \in U(1) \rightarrow U(x,\mu) \in SU(3)$
- Pauli matrices $\sigma_\mu \rightarrow$ Dirac-matrices γ_μ
- spinors become 12-component complex vectors
- theory needs renormalization



Costs of dynamical fermions simulations, the "Berlin Wall"

see panel discussion in Lattice2001, Berlin, 2001



physical point

contact to χPT (?)

formula $C \propto \left(\frac{m_\pi}{m_\rho}\right)^{-z_\pi} (L)^{z_L} (a)^{-z_a}$

$$z_\pi = 6, \;\; z_L = 5, \;\; z_a = 7$$

"both a 10^8 increase in computing power AND spectacular algorithmic advances before a useful interaction with experiments starts taking place." (Wilson, 1989)

$\Rightarrow$ need of **Exaflops Computers**

Why are fermions so expensive?

- need to evaluate

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\bar{\psi}\{D_{\text{lattice}}^{\text{Dirac}}\}\psi} \propto \det[D_{\text{lattice}}^{\text{Dirac}}]$$

- bosonic representation of determinant

$$\det[D_{\text{lattice}}^{\text{Dirac}}] \propto \int \mathcal{D}\Phi^\dagger\mathcal{D}\Phi e^{-\Phi^\dagger\{D_{\text{lattice}}^{-1}\}\Phi}$$

- need vector $X = D_{\text{lattice}}^{-1}\Phi$

- solve linear equation $D_{\text{lattice}}X = \Phi$

  D_{lattice} matrix of dimension $100\text{million} \otimes 100\text{million}$ (however, sparse)

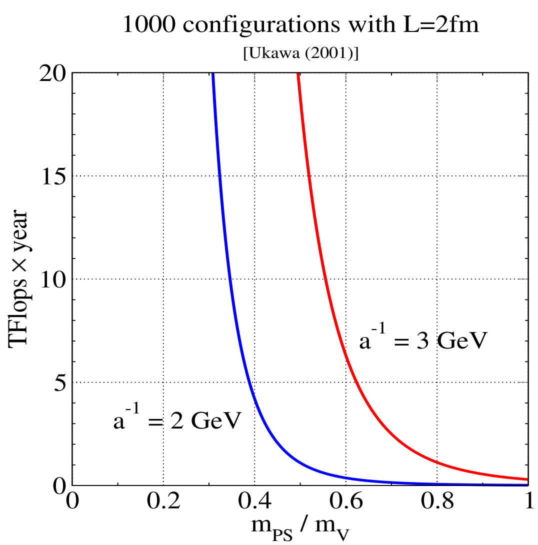
- number of such "inversions": $O(100)$ for one field configuration

- want: $O(1000 - 10000)$ such field configurations

A generic improvement for Wilson type fermions

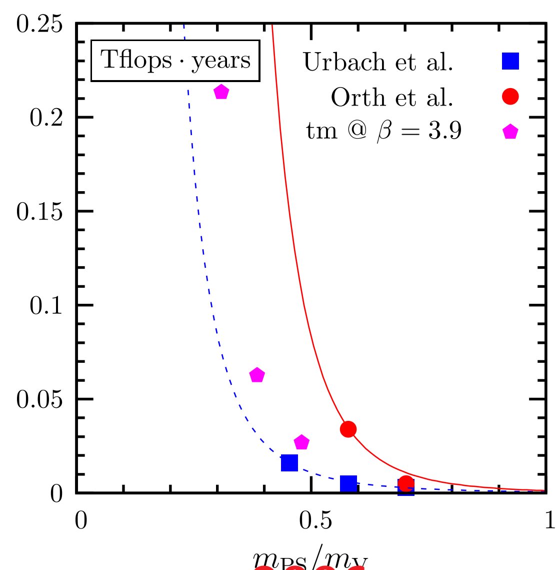
New variant of HMC algorithm (Urbach, Shindler, Wenger, K.J.)
(see also SAP (Lüscher) and RHMC (Clark and Kennedy) algorithms)

- even/odd preconditioning
- (twisted) mass-shift (Hasenbusch trick)
- multiple time steps



2001

$\longrightarrow$



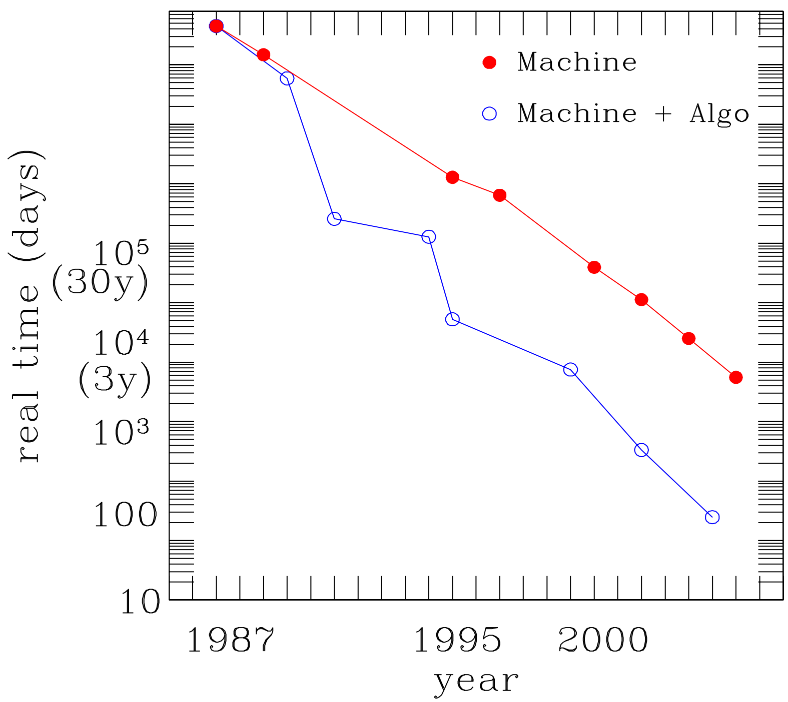
2006

– comparable to staggered

– reach small pseudo scalar masses $\approx 300\text{MeV}$

Computer and algorithm development over the years

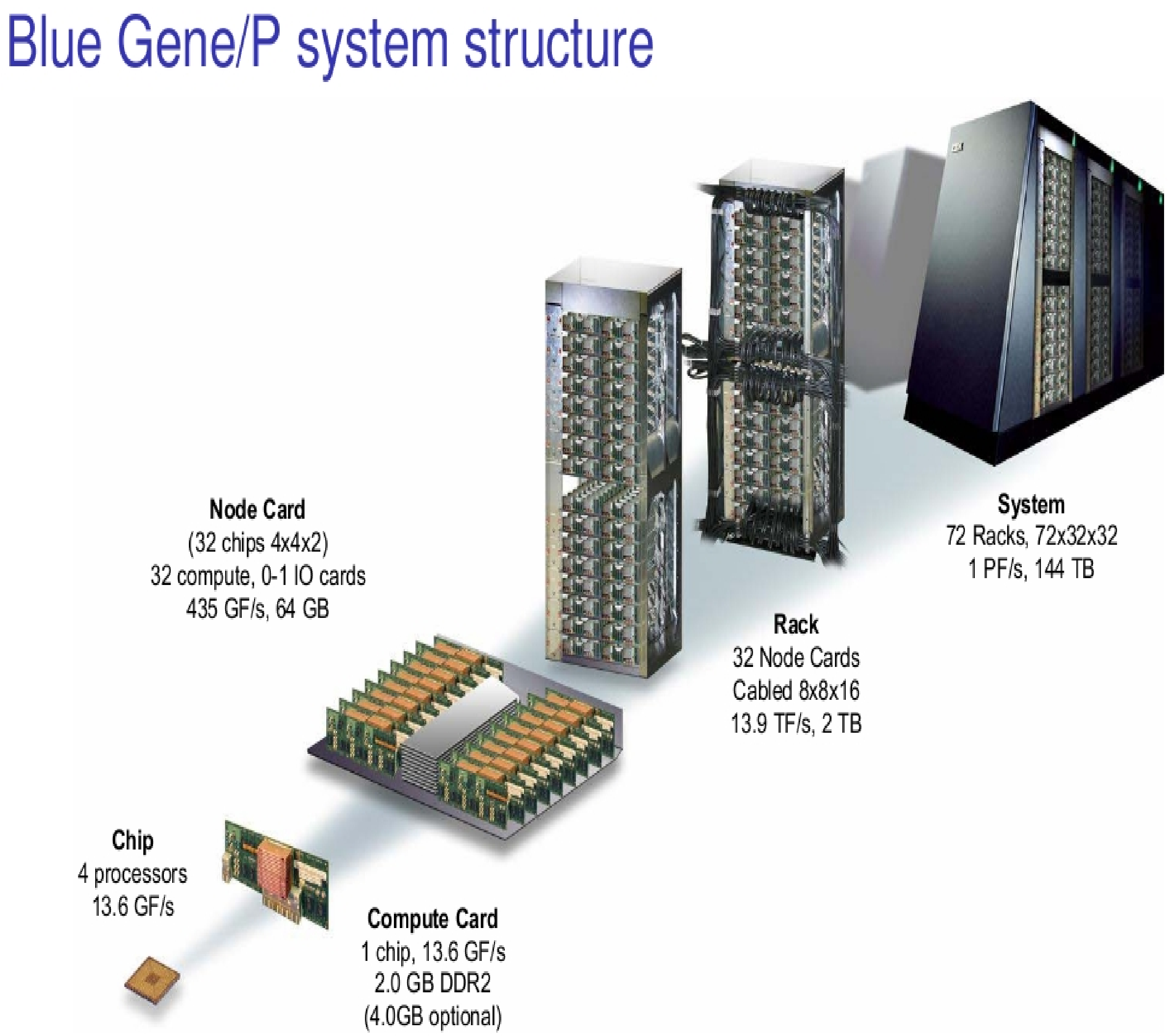
time estimates for simulating $32^3 \cdot 64$ lattice, 5000 configurations



$\rightarrow$ O(few months) nowadays with a typical collaboration supercomputer contingent

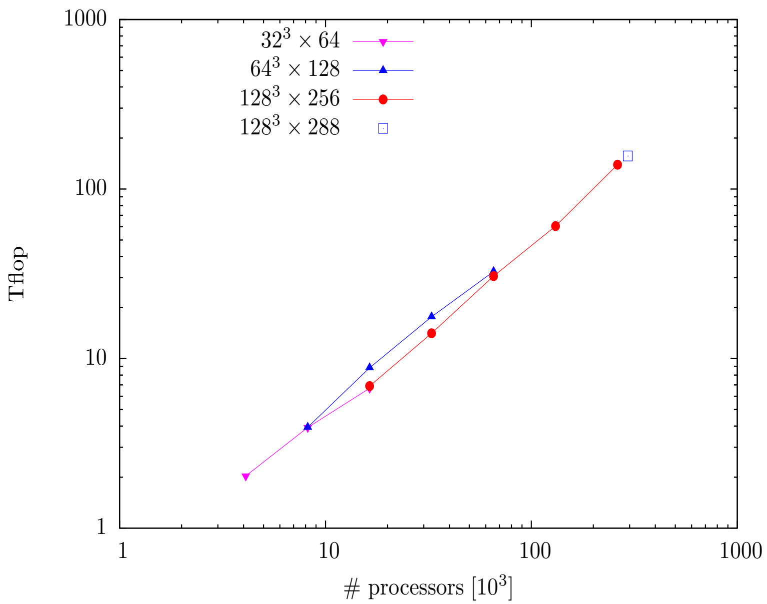
State of the art

- **BG/P**



Strong Scaling

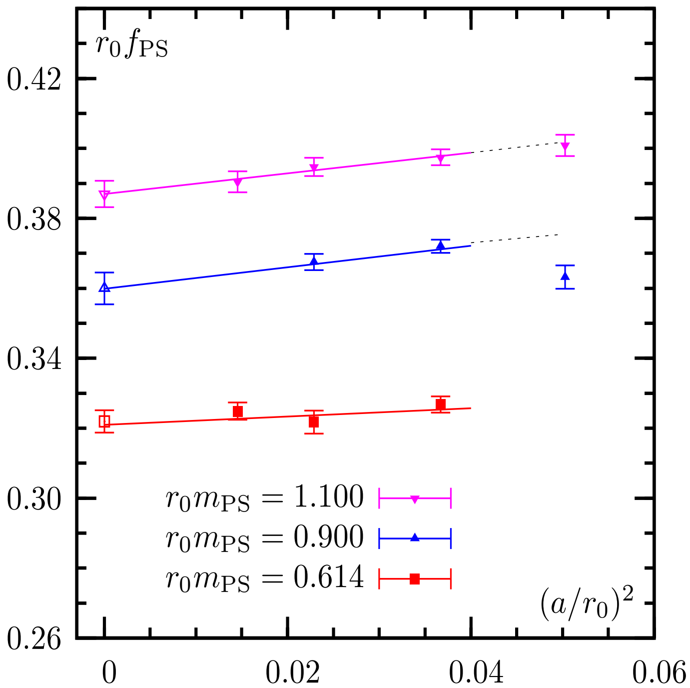
- Test on 72 racks BG/P installation at supercomputer center Jülich
- using tmHMC code



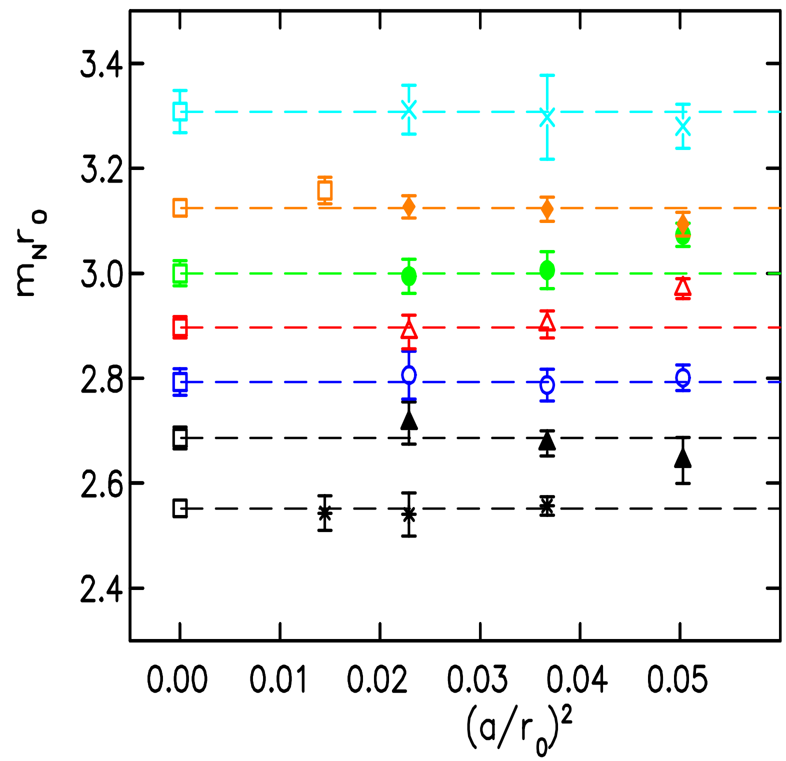
- **Cyprus (Nicosia)**
  C. Alexandrou, M. Constantinou, T. Korzec, G. Koutsou
- **France (Orsay, Grenoble)**
  R. Baron, B. Bloissier, Ph. Boucaud, M. Brinet, J. Carbonell, V. Drach, P. Guichon, P.A. Harraud, Z. Liu, O. Pène
- **Italy (Rome I,II,III, Trento)**
  P. Dimopoulos, R. Frezzotti, V. Lubicz, G. Martinelli, G.C. Rossi, L. Scorzato, S. Simula, C. Tarantino
- **Netherlands (Groningen)**
  A. Deuzeman, E. Pallante, S. Reker
- **Poland (Poznan)**
  K. Cichy, A. Kujawa
- **Spain (Valencia)**
  V. Gimenez, D. Palao
- **Switzerland (Bern)**
  U. Wenger
- **United Kingdom (Glasgow, Liverpool)**
  G. McNeile, C. Michael, A. Shindler
- **Germany (Berlin/Zeuthen, Hamburg, Münster)**
  F. Farchioni, X. Feng, J. González López, G. Herdoiza, K. Jansen, I. Montvay, G. Münster, M. Petschlies, D. Renner, T. Sudmann, C. Urbach, M. Wagner

Lattice spacing scaling

Pion decay constant



Nucleon mass

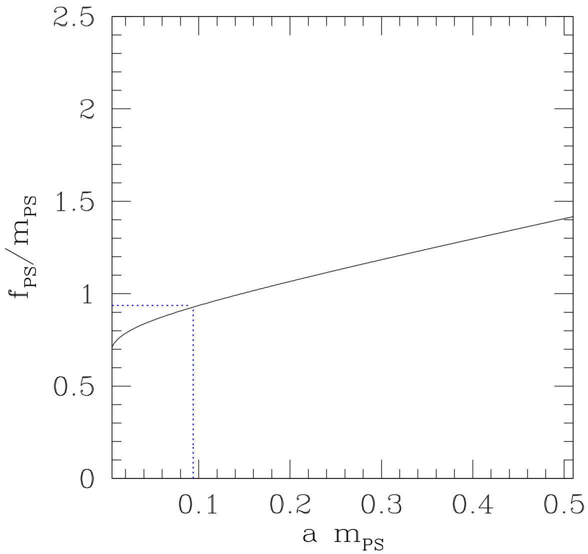


$\rightarrow$ observe (flat) $O(a^2)$ scaling

Setting the scale

$$m_{\rm PS}^{\rm latt} = a m_{\rm PS}^{\rm phys}\ , \quad f_{\rm PS}^{\rm latt} = a f_{\rm PS}^{\rm phys}$$

$$\frac{f_{\rm PS}^{\rm phys}}{m_{\rm PS}^{\rm phys}} = \frac{f_{\rm PS}^{\rm latt}}{m_{\rm PS}^{\rm latt}} + \mathrm{O}(a^2)$$



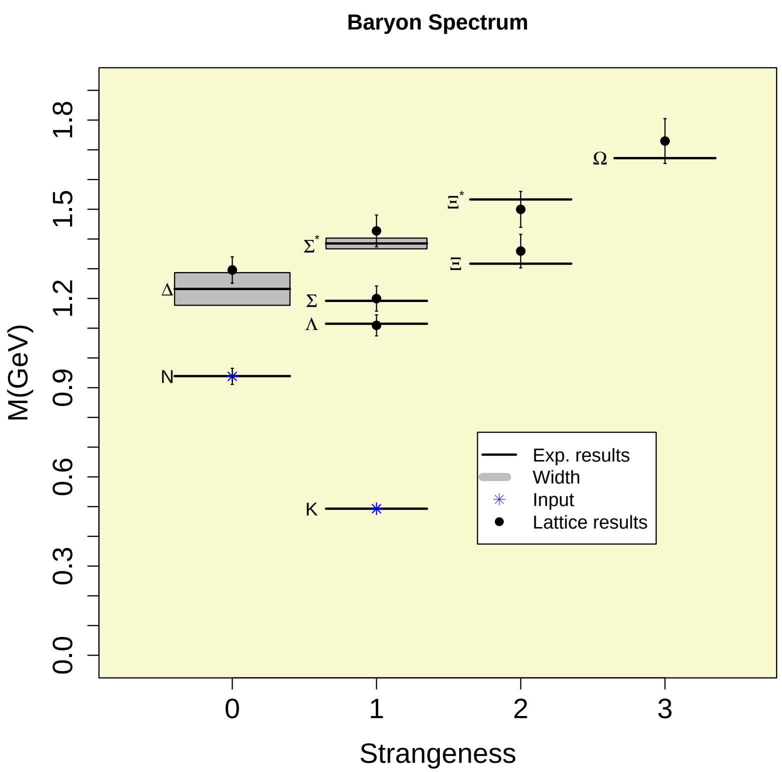
$\rightarrow$ setting $\frac{f_{\rm PS}^{\rm latt}}{m_{\rm PS}^{\rm latt}} = 130.7/139.6$

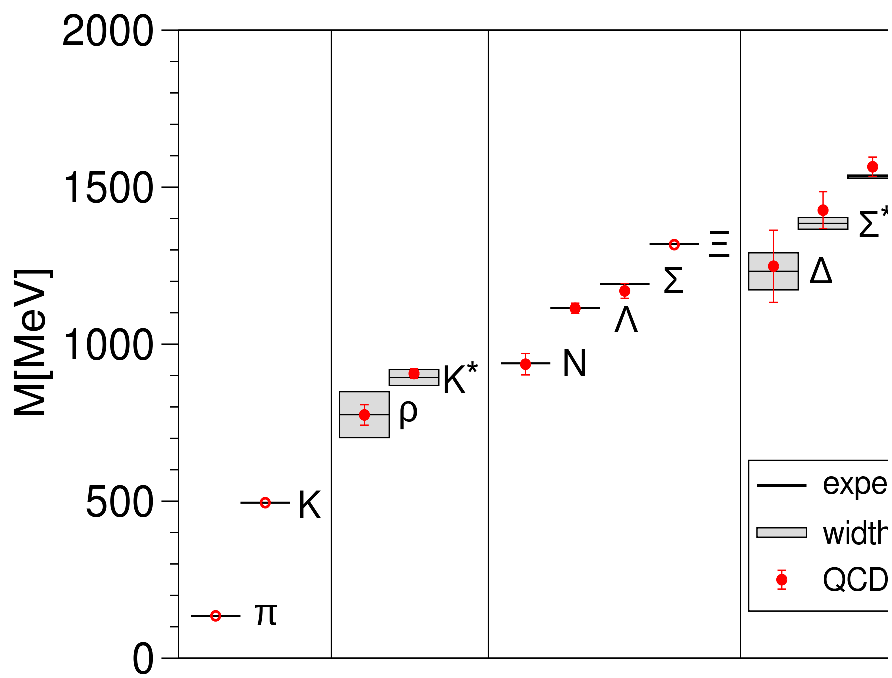
$\rightarrow$ obtain $m_{\rm PS}^{\rm latt} = a139.6$[Mev]

$\rightarrow$ value for lattice spacing a

The lattice QCD benchmark calculation: the spectrum

ETMC ($N_f = 2$), BMW ($N_f = 2+1$)

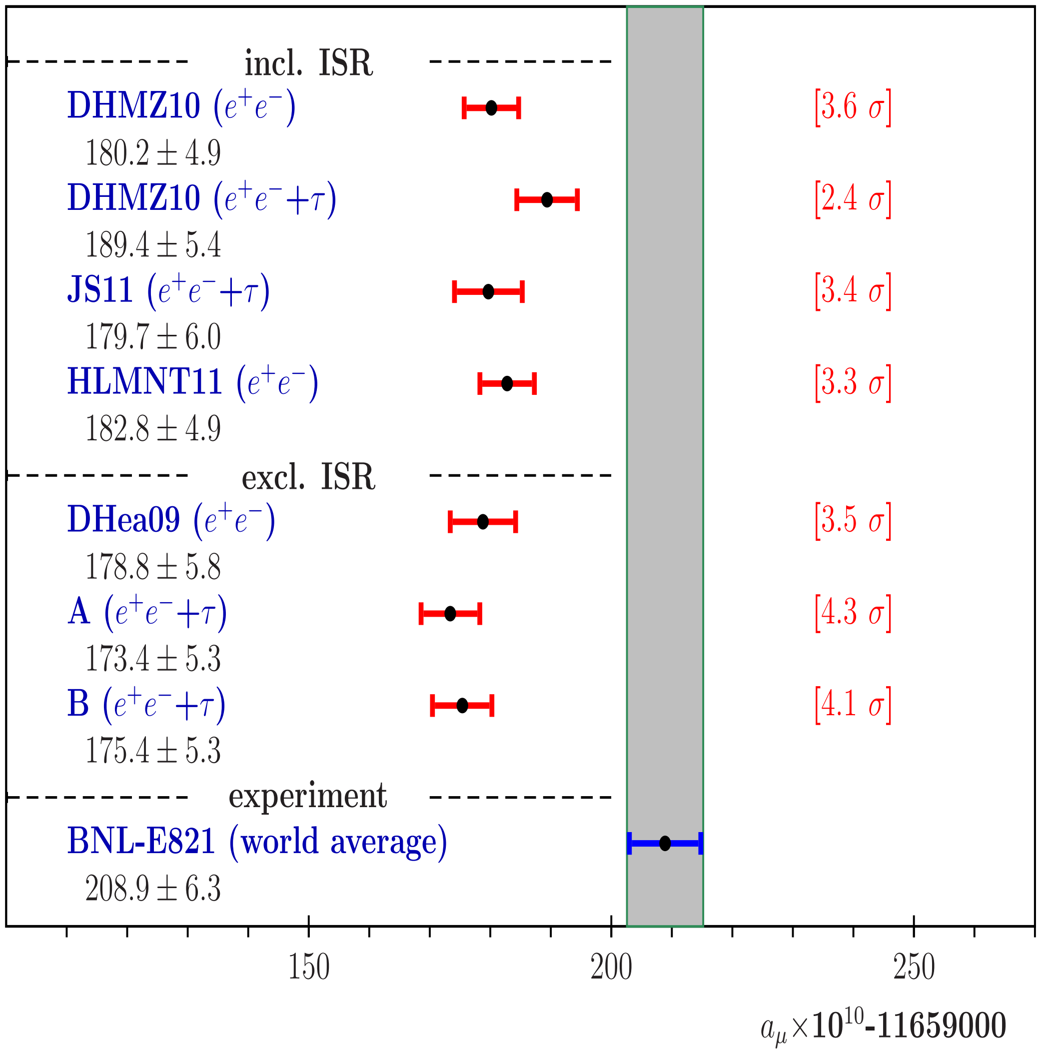




$N_f = 2$ $\qquad$ $N_f = 2+1$

Muon magnetic moment: a tension between theory and experiment

$\rightarrow$ signs of new physics?



latest analysis

(Benayoun, David, DelBuono, Jegerlehner)

larger than $\underline{4\sigma}$ discrepancy

largest error: non-perturbative QCD contribution

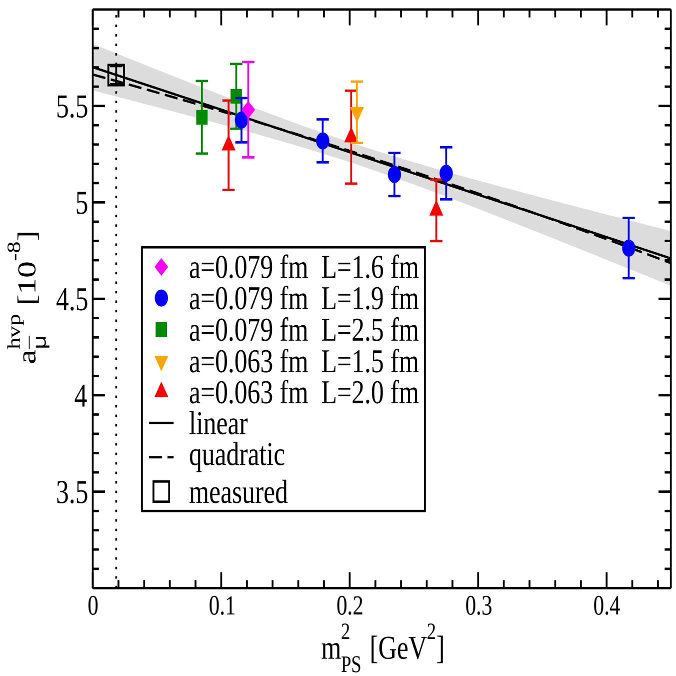
Some numbers

- experimental value: $a_{\mu,N_f=2}^{\mathrm{hvp,exp}} = 5.66(05)10^{-8}$
- from our old analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,old}} = 2.95(45)10^{-8}$

$\rightarrow$ misses the experimental value
$\rightarrow$ order of magnitude larger error

- from our new analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,new}} = 5.66(11)10^{-8}$

$\rightarrow$ error (including systematics) almost matching experiment



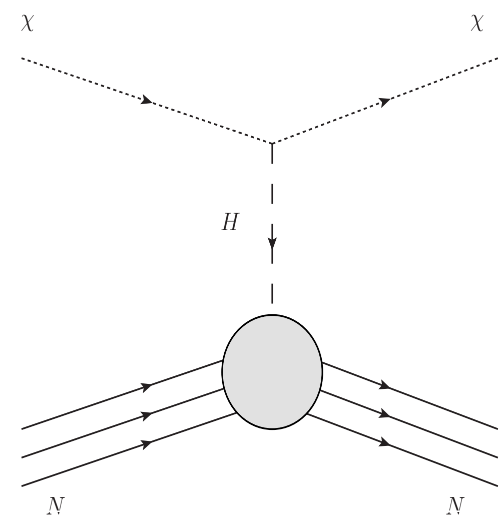
– different volumes

– different values of lattice spacing

– included dis-connected contributions

Dark matter detection

- size of quark content of nucleon important to detect dark matter candidates
- scattering dark matter particle with Higgs boson
  $\rightarrow$ cross-section changes an order of magnitude with small changes of quark content



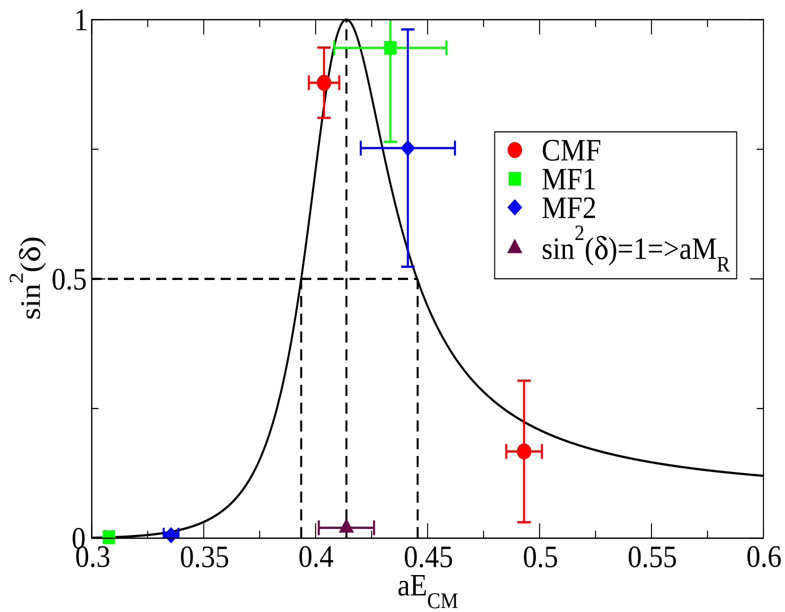
diagrammatic scattering process

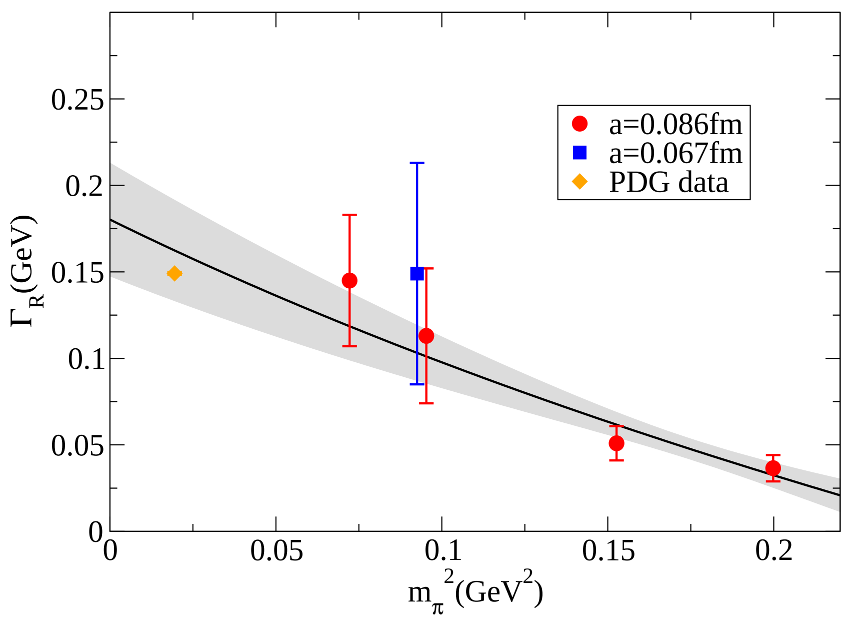
$$y_N \equiv \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u+\bar{d}d|N\rangle} = 0.082(16)(2)$$

- value much smaller than earlier thought
  $\rightarrow$ severe consequences for experiments

The ρ-meson resonance: dynamical quarks at work

- usage of three Lorentz frames



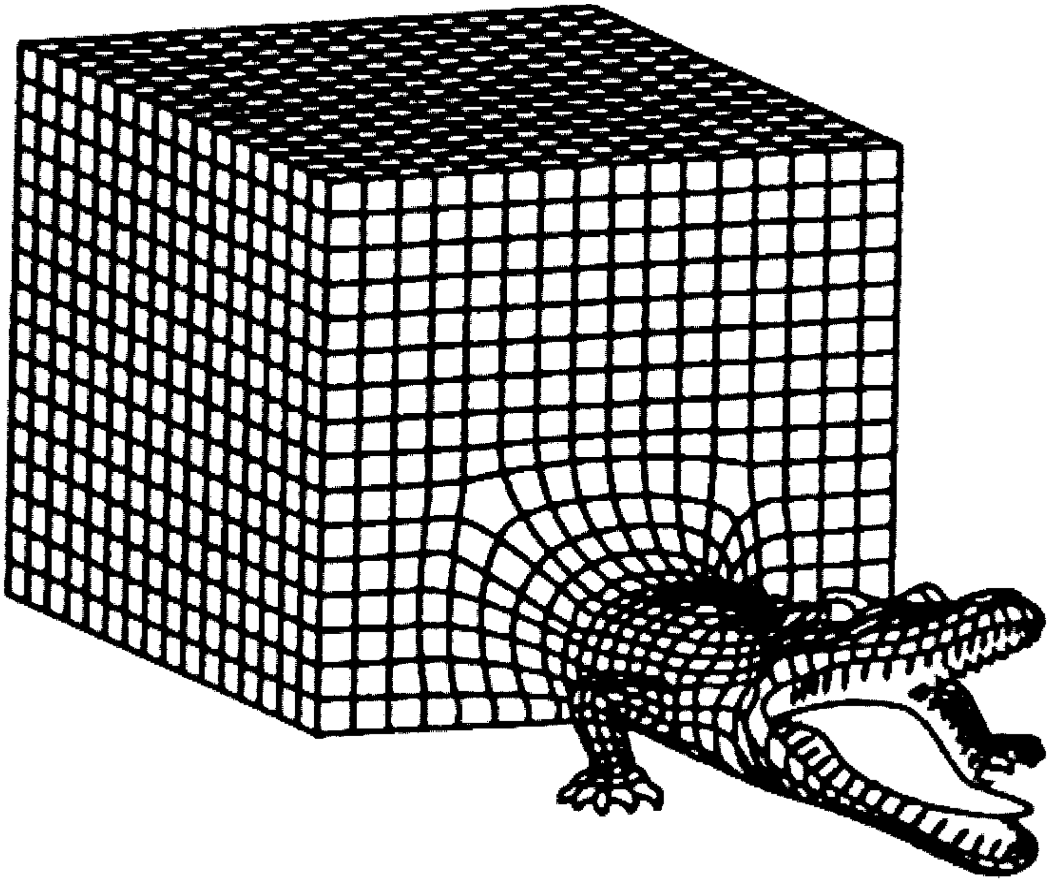


$m_{\pi^+} = 330$ MeV, $a = 0.079$ fm, $L/a = 32$

fitting $z = (M_\rho + i\frac{1}{2}\Gamma_\rho)^2$

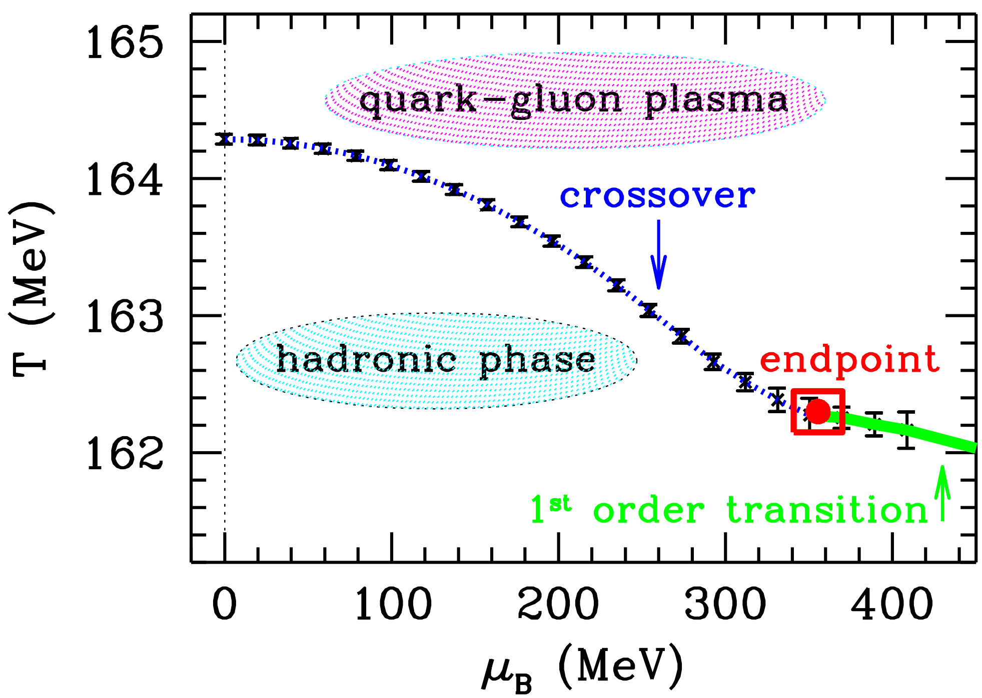
$m_\rho = 1033(31)$ MeV, $\Gamma_\rho = 123(43)$ MeV

There are dangerous lattice animals



Non-zero temperature and density

suggested phase diagram



Lattice Action with chemical potential

$$\mathcal{Z} = \int_{\text{fields}} e^{-S}$$

Partition functions (pathintegral) with Boltzmann weight (action) S

$$S = a^2 \sum_x \left\{ \beta \left[1 - \mathrm{Re}(U_{(x,p)}) \right] + \bar{\psi} \left[m_0 + \tfrac{1}{2} \{ \gamma_\mu (\partial_\mu + \partial_\mu^\star) - a \partial_\mu^\star \partial_\mu \} \right] \psi \right\}$$

- adding a chemical potential
  $S \rightarrow S + i\tau_3 \bar{\Psi}\Psi$
- action becomes complex, MC methods not possible
- partial solution: reweighting

The Schwinger model Hamiltonian

(Irving and Thomas, Nucl.Phys.B215 [FS7] (1983) 23)

Recent paper:

L. Tagliacozzo, A. Celi, A. Zamora, M. Lewenstein, *Optical Abelian Lattice Gauge Theories*, arXiv:1205.0496

Free continuum Hamiltonian

$$H = \int dx \Psi^\dagger (i\alpha \cdot \partial/\partial x + m_f \beta)\Psi$$

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

fermion field: $\Psi = \begin{pmatrix} b \\ d^\dagger \end{pmatrix}$

with $b^\dagger$ $(d^\dagger)$ creates fermion (antifermion)

The Schwinger model Hamiltonian on the lattice

Wilson discretization, derivative $(x \to j \cdot a)$

$$\alpha \cdot \partial/\partial x \Psi \to \frac{1}{2a} [\alpha \cdot (\Psi_{j+1} - \Psi_{j-1} + i\beta \cdot (\Psi_{j+1} + \Psi_{j-1} - 2\Psi_j]$$

2-component fermion field $\Psi_j = \begin{pmatrix} \Psi_j^{(1)} \\ \Psi_j^{(2)} \end{pmatrix}$

"pseudofermion" description: $\Phi_j = \Psi_j^{(1)}$, $\bar{\Phi}_j = i\Psi_j^{\dagger(1)}$

$$\begin{aligned} H_{\text{Wilson}} &= \frac{1}{2a} \left\{ \sum_j \left(\bar{\Phi}^\dagger_{j+1} \Phi^\dagger_j - \Phi_{j+1} \bar{\Phi}^\dagger_j - \bar{\Phi}^\dagger_{j+1} \bar{\Phi}^\dagger_j + \Phi_{j+1} \Phi^\dagger_j \right) + \text{h.c.} \right. \\ &+ \left. 2(m_f a + 1)(\Phi^\dagger_j \Phi_j + \bar{\Phi}^\dagger_j \bar{\Phi}_j) \right\} \end{aligned}$$

vacuum: $|\, 0\ 0\ 0\ 0\ 0\ 0 \rangle$ $\qquad \Phi^\dagger_1 \bar{\Phi}^\dagger_2 |\, 0\ 0\ 0\ 0\ 0\ 0 \rangle = |\, 1\ 0\ 0\ 1\ 0\ 0 \rangle$

The Schwinger model Hamiltonian on the lattice

Staggered (Kogut-Susskind) discretization

$\rightarrow$ associate

upper components $\Psi_j^{(1)}$ with *even* lattice sites

lower components $\Psi_j^{(2)}$ with *odd* lattice sites

$\Rightarrow$ introduce single component field Φ_j with $\{\Phi_j^\dagger, \Phi_{j'}\} = \delta_{j,j'}$

can use naive derivative: $\partial/\partial x \Phi \rightarrow \frac{1}{2a}\left[\Phi_{j+1} - \Phi_{j-1}\right]$

special trick for staggered fermions: Jordan-Wigner transformation

$$\Phi(j) = \prod_{j<n}[i\sigma_3]\sigma^-(n)$$

$$\Phi^\dagger(j) = \prod_{j<n}[-i\sigma_3]\sigma^+(n)$$

$\rightarrow$ allows for a direct matrix representation of Hamiltonian

Adding gauge fields

$$H = \int dx \Psi^\dagger (i\alpha \cdot \partial/\partial x - gA_x + m_f\beta)\Psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

As in the pathintegral: introduce parallel transporters

$$U_x(j) = \exp\left\{ig\int_{ja}^{(j+1)a} A_x(x)dx\right\}$$

for Schwinger model: just a phase factor

$$U_x(j) = e^{i\theta(n)}$$

Implementing gauge fields

temporal gauge: $A_t = 0 \Rightarrow F_{t,x}(j) = \partial_t A_x(j)$

energy:

$$E(j) = F_{t,x}(j)\ , \quad \text{vacuum}: \ E(j)|0\rangle = 0$$

lattice gauge fields $U_x(j)$ introduce a ladder space:

$$U_x(j)^l|0\rangle = |j\rangle_j\ , \quad E(j)|l\rangle_j = l|l\rangle_l$$

Field strength tensor

$$\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}dx \to \tfrac{1}{2}g^2 a \textstyle\sum_j E^2(j)$$

Observables

- bound "positronium" states:
  - vector state:

$$|v\rangle = \frac{1}{\sqrt{N}} \sum_j^N \left[\Phi^\dagger(j) e^{i\theta(n)} \Phi(j+1) + \text{h.c.} \right] |0\rangle$$

  - scalar state:

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_j^N [\Phi^\dagger(j) e^{i\theta(n)} \Phi(j+1) - \text{h.c.}] |0\rangle$$

- average electric field:

$$\Gamma = \frac{1}{N} \sum_j E(j)$$

- condensates:

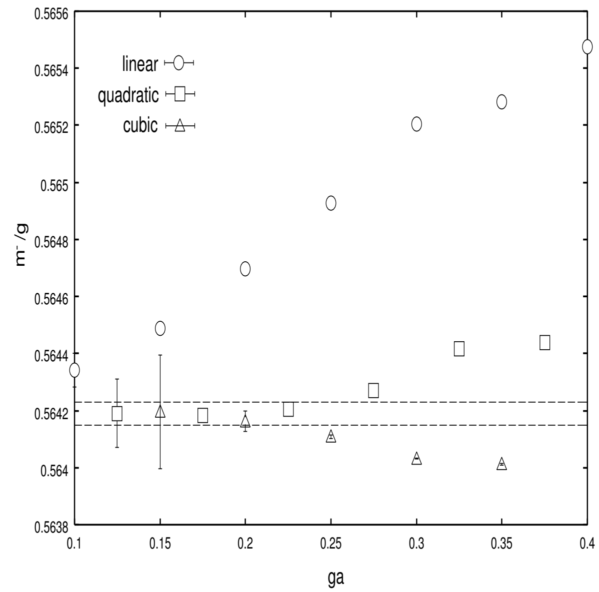
$$\Gamma^5 = \langle i\bar{\Psi}\sigma_3\Psi/g\rangle \propto \langle \sum_j \left[\Phi^\dagger(j)\Phi(j+1) - h.c. \right] \rangle_0$$

DMRG improvement

(Byrnes, Sriganesh, Bursill, Hamer; 2008)

plain Hamiltonian calculations: (Crewther and Hamer, 1980; Irving and Thomas, 1982)

m/g	DMRG 2008	plain H 1980
0	0.56419(4)	0.56(1)
0.125	0.53950(7)	0.54(1)
0.25	0.51918(5)	0.52(1)
0.5	0.48747(2)	0.50(1)
2	0.398(1)	0.413(5)
8	0.287(8)	0.299(5)
16	0.238(5)	0.245(5)
32	0.194(5)	0.197(5)



Summary

- Progress in solving QCD with lattice techniques
  - dramatic algorithm improvements
  - new supercomputer architectures
- offers possibility to
  - reach continuum limit and chiral limit

  → have computed the baryon spectrum

  → anomalous magnetic moment of muon
- challenges
  - cannot reliably simulate chemical potential
  - real time evolution not controlled
- 

from participants of this workshop

  (started game with M.C. Bañuls, K. Cichy)