Coarse Graining Tensor Renormalization by the Higher-Order Singular Value Decomposition

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References

H. C. Jiang et al, PRL 101, 090603 (2008)

Z. Y. Xie et al, PRL 103, 160601 (2009)

H.H. Zhao et al, PRB 81, 174411 (2010)

Q. N. Chen et al, PRL 107, 165701 (2011)

H. H. Zhao et al, PRB 85, 134416 (2012)

Z. Y. Xie et al, arXiv:1201.1144

Why should we study the renormalization of tensors?

Strong correlated systems

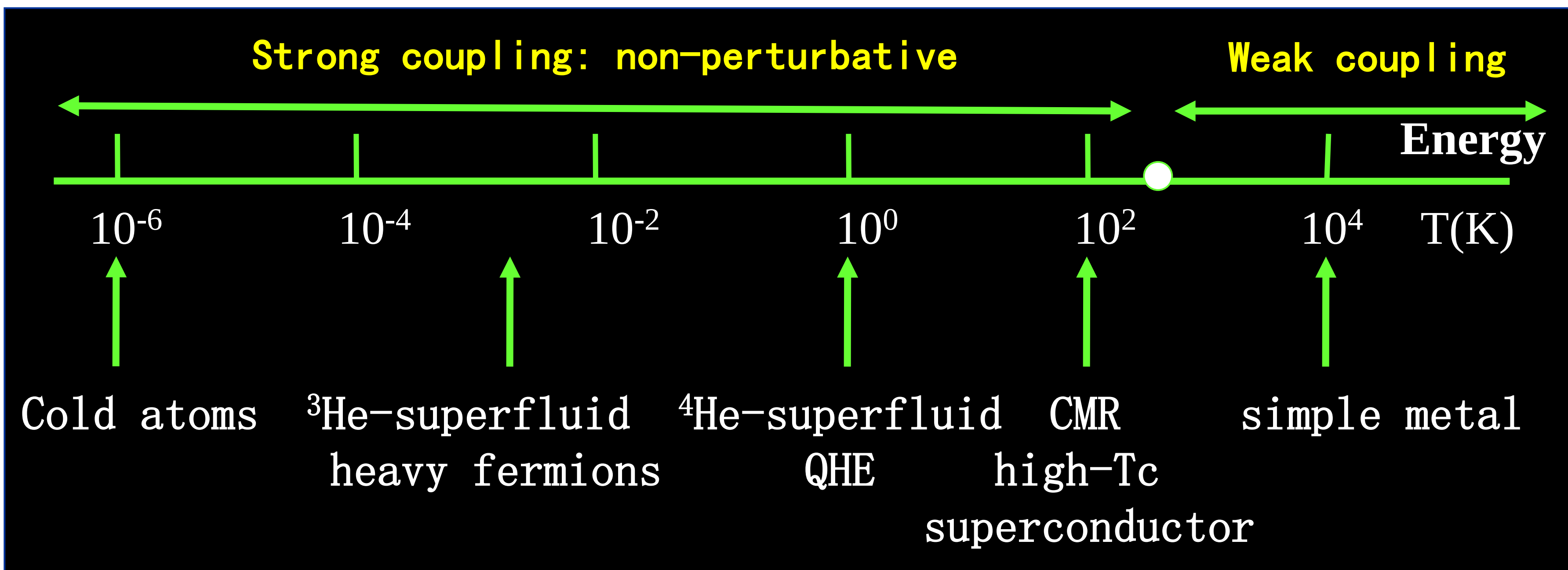
Strong many-body effect:

- ✓ Single particle approximation invalid
- ✓ No good analytic tool: particles are highly entangled

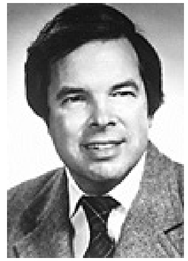
Weak coupling systems

Week interaction

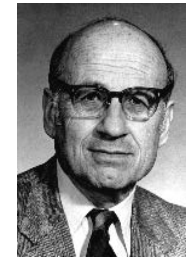
- ✓ Mean-field or single particle approximation works
- ✓ Particles are app. disentangled



Can we solve the problem numerically?



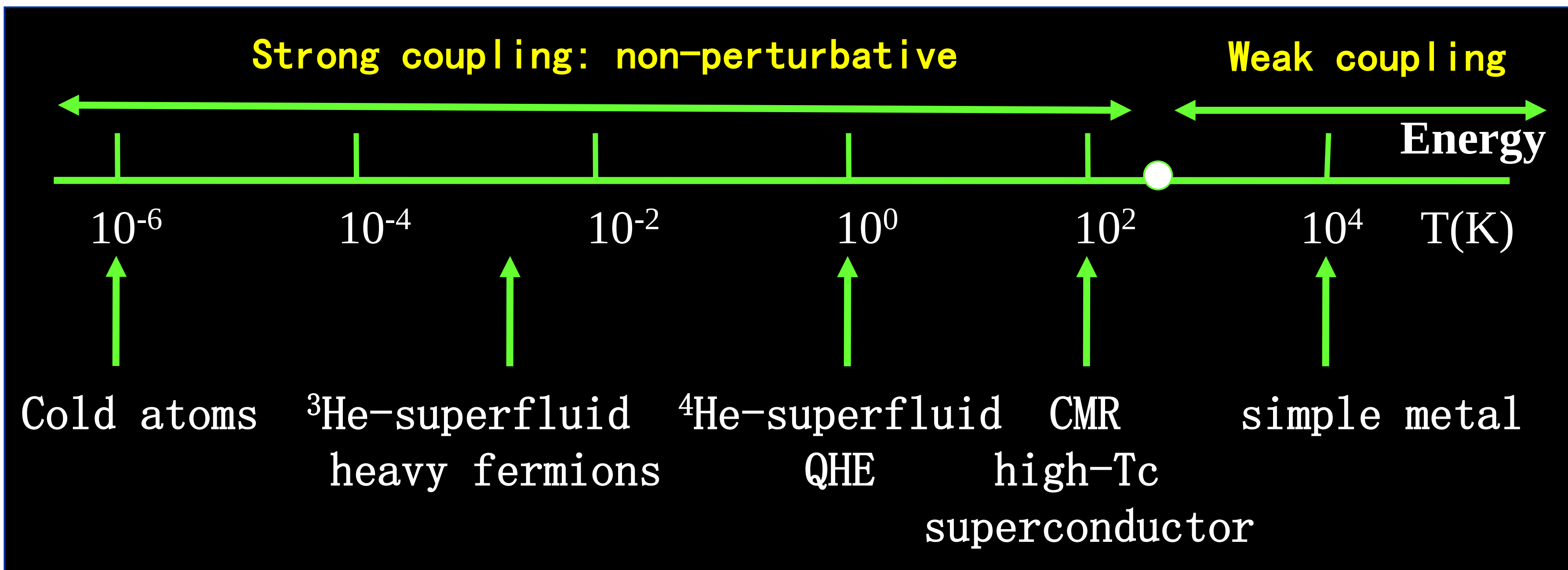
K. Wilson 1982



Walter Kohn 1997

Numerical Renormalization Group

Density Functional Theory



Three Stages of Numerical Renormalization Group Study

1. **Wilson NRG 1975 -**

   0 Dimensional problems (single impurity Kondo model)

2. **Density Matrix Renormalization Group (1D algorithm) 1992 -**

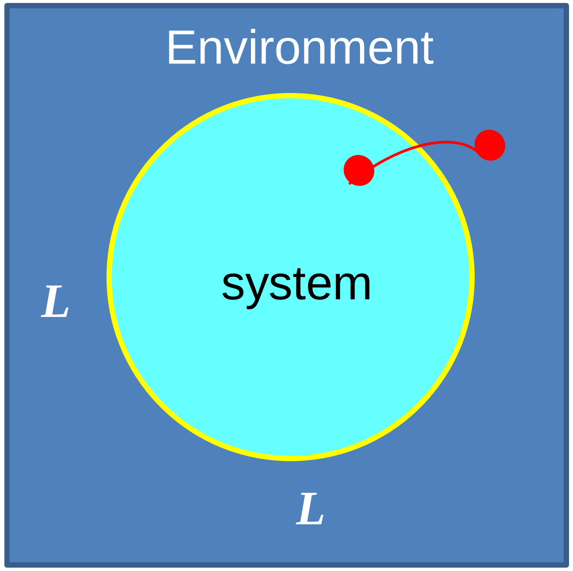
   1 Dimensional quantum lattice models

3. **Tensor Renormalization Group**

   2 or higher dimensional lattice models

Why is it more difficult to study higher-dimensional systems?

Area Law of Entanglement Entropy



$$\boxed{\begin{array}{l} S \sim L^{d-1} \sim \ln D \\ D \sim \exp\left(L^{d-1}\right) \end{array}}$$

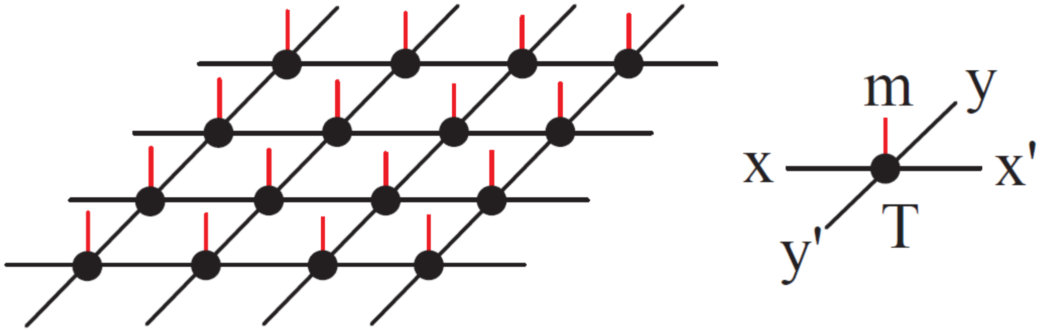
***D*: minimal number of basis states needed for describing the ground state**

***d*: spatial dimension**

What is the wavefunction that satisfies this area law?

The Answer: Tensor Network State

$$\boxed{|\Psi\rangle = Tr \prod T_{x_i x_i' y_i y_i'}[m_i]|m_i\rangle}$$



x = 1 … D

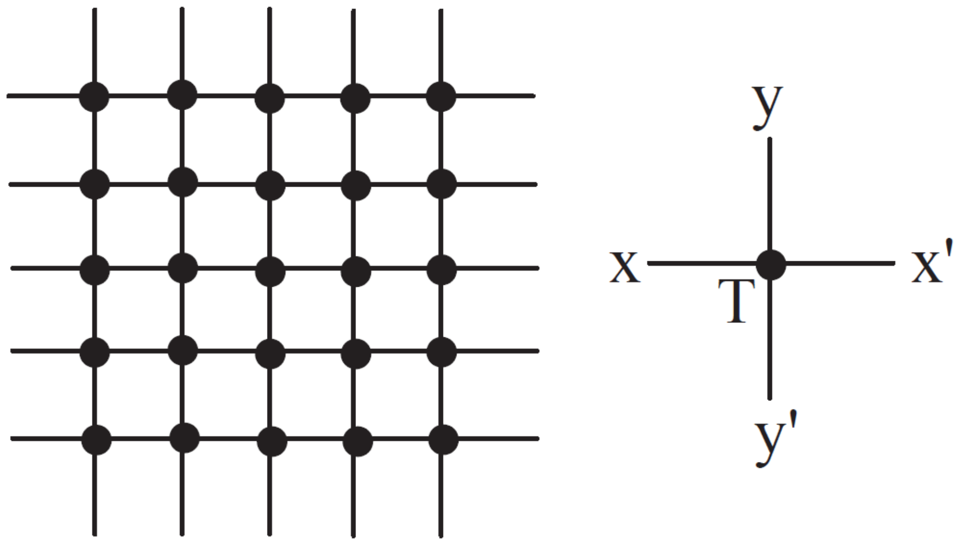
Tensor network state (projected entangled pair state) is a faithful representation of the ground state of a quantum lattice model

Verstraete, Cirac, arXiv:0407066

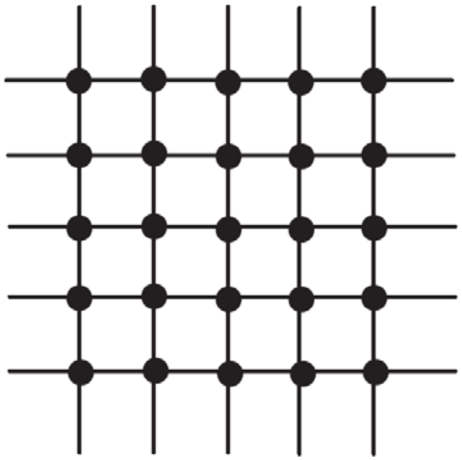
Classical Statistical Models = Tensor-network Model

$$\boxed{Z = Tr \prod_i T_{x_i x_i' y_i y_i'}}$$

The partitions for all statistical models with local interactions can be represented as tensor-network models



Tensor-Network Representation in the Original Lattice

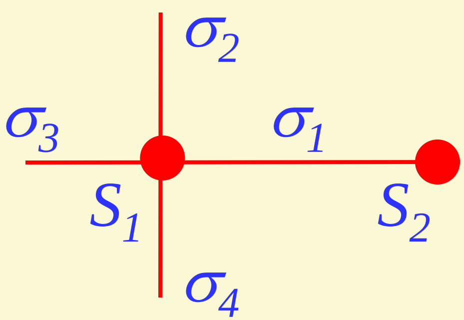


$$H = -J \sum_{\langle ij \rangle} S_i S_j$$

$$Z = Tr \prod_{\langle ij \rangle} \exp\left(-\beta H_{ij}\right) = Tr \prod_i T_{y_i x_i y'_i x'_i}$$

$$T_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \sum_{S_1} U_{S_1 \sigma_1} U_{S_1 \sigma_2} U_{S_1 \sigma_3} U_{S_1 \sigma_4} \sqrt{\Lambda_{\sigma_1} \Lambda_{\sigma_2} \Lambda_{\sigma_3} \Lambda_{\sigma_4}}$$

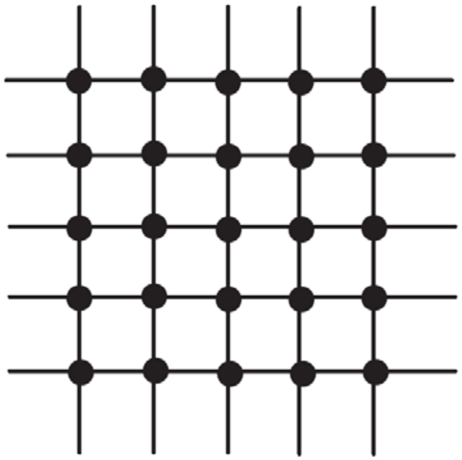
Singular Value Decomposition



$$M_{S_i S_j} = \exp\left(-\beta H_{ij}\right) = \exp\left(\beta J S_i S_j\right)$$

$$M_{S_1 S_2} = U_{S_1 \sigma_1} \Lambda_{\sigma_1} U_{S_2 \sigma_1}$$

Tensor-Network Representation in the Dual Lattice

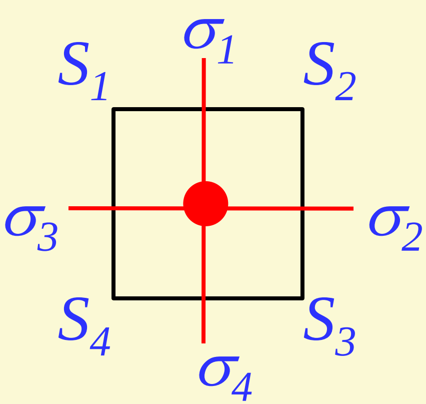


$$H = -J \sum_{\langle ij \rangle} S_i S_j$$

$$Z = Tr \prod_{\square} \exp\left(-\beta H_{\square}\right) = Tr \prod_{i} T_{y_i x_i y'_i x'_i}$$

$$T_{\sigma_1\sigma_2\sigma_3\sigma_4} = e^{-J\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)/2}\,\delta\left(\sigma_1\sigma_2\sigma_3\sigma_4 - 1\right)$$

Duality transformation



$$\sigma_1 = S_1 S_2$$

$$\sigma_2 = S_2 S_3$$

$$\sigma_3 = S_3 S_4$$

$$\sigma_4 = S_4 S_1$$

$$H_{\square} = -J\left(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4\right)/2$$

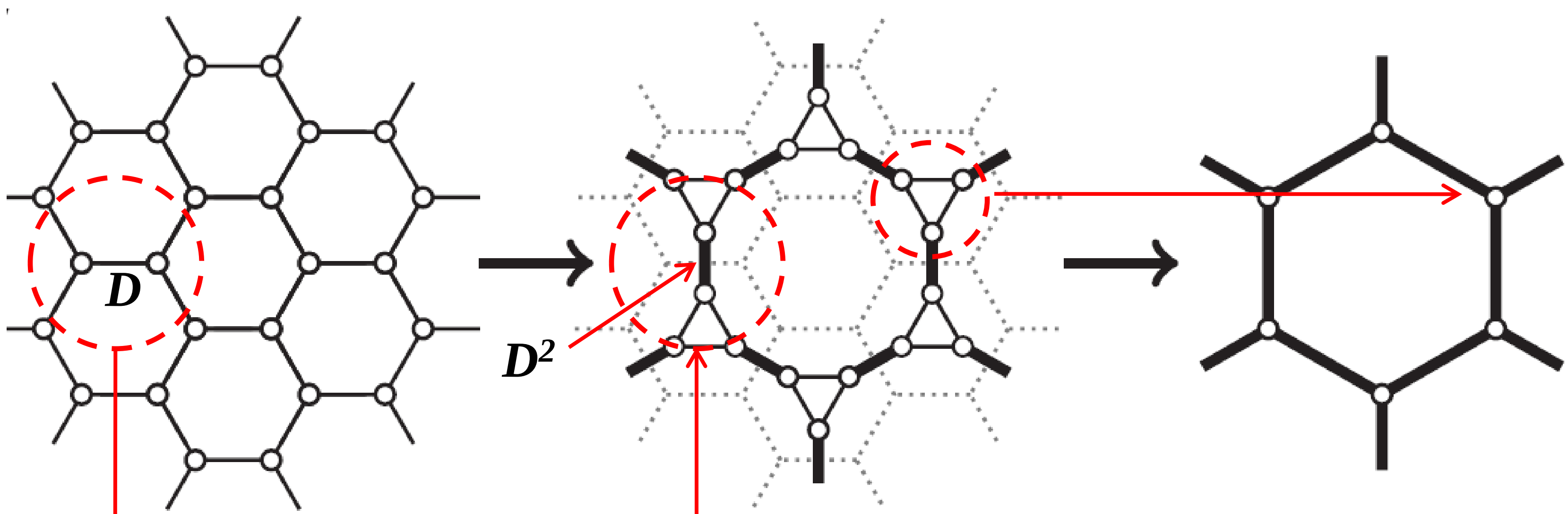
$$\sigma_1\sigma_2\sigma_3\sigma_4 = S_1 S_2 S_2 S_3 S_3 S_4 S_4 S_1 = 1$$

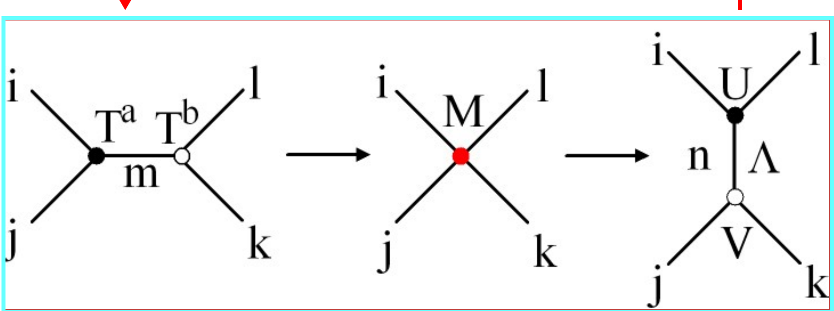
Coarse Grain Tensor Renormalization Group (TRG)

Levin, Nave, PRL 99 (2007) 120601

Step I: Rewiring

Step II: decimation



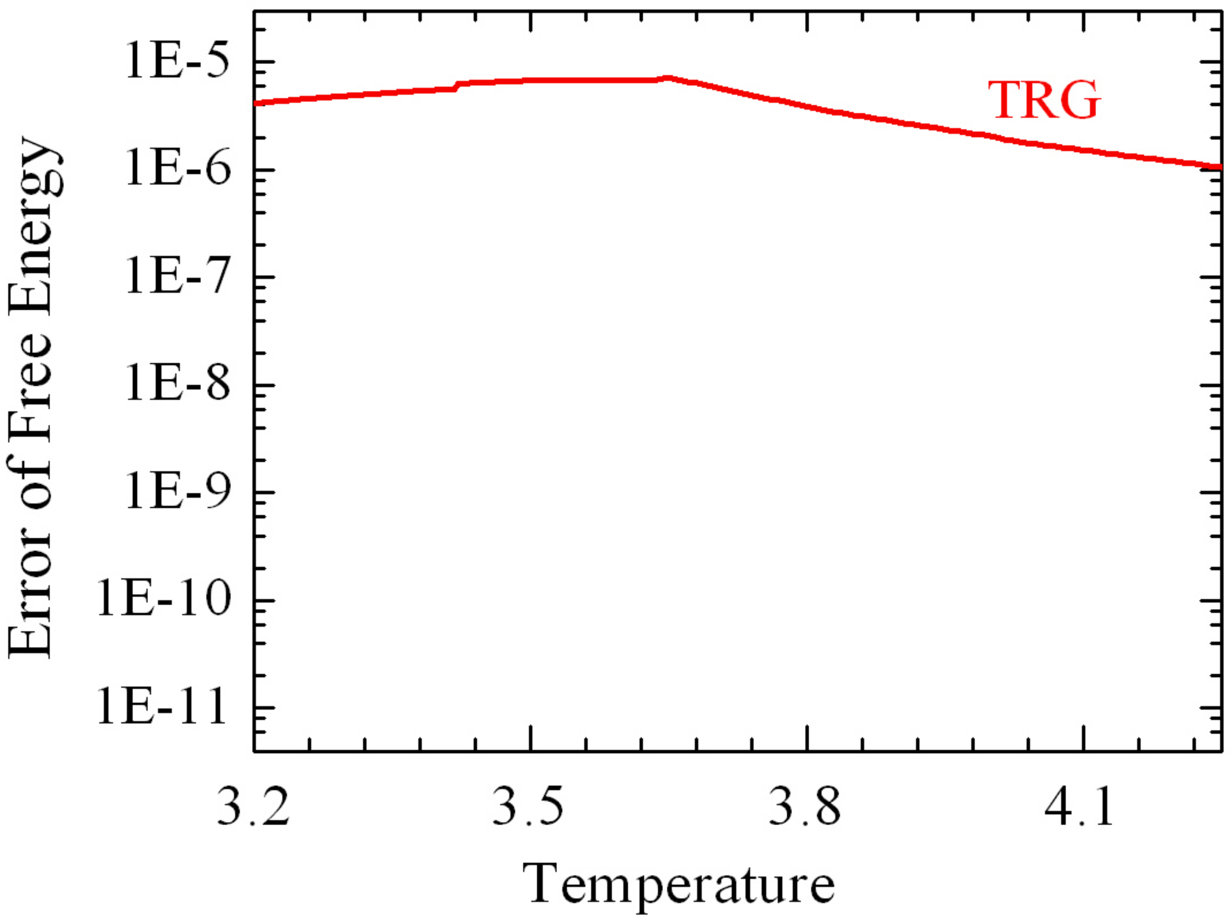


$$
\begin{aligned}
M_{kj,il} &= \sum_{m} T_{mji} T_{mlk} \\
&= \sum_{n=1}^{D^2 \to D} U_{kj,n} \Lambda_n V_{il,n}
\end{aligned}
$$

Singular value decomposition: SVD
best scheme for truncating a matrix

Accuracy of TRG

TRG is a good method, but it can be further improved

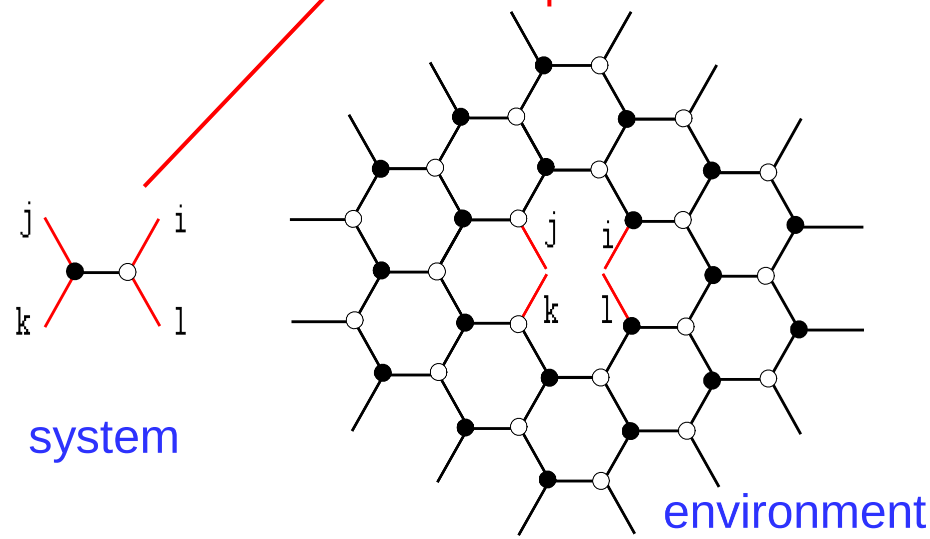


D = 24

Ising model on a triangular lattice

Second renormalization of tensor-network state (SRG)

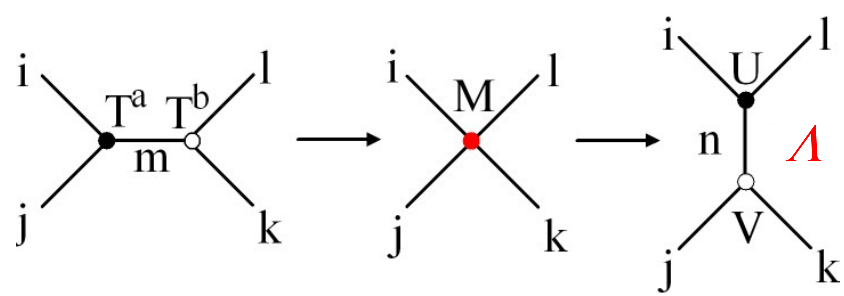
$$Z = Tr\left(M M^{env} \right)$$



- TRG:
  truncation error of M is minimized by the singular value decomposition

But, what really needs to be minimized is the error of Z!

- SRG:
  The renormalization effect of M^{env} to M is considered



Xie et al, PRL **103**, 160601 (2009)
Zhao, et al, PRB **81**, 174411 (2010)

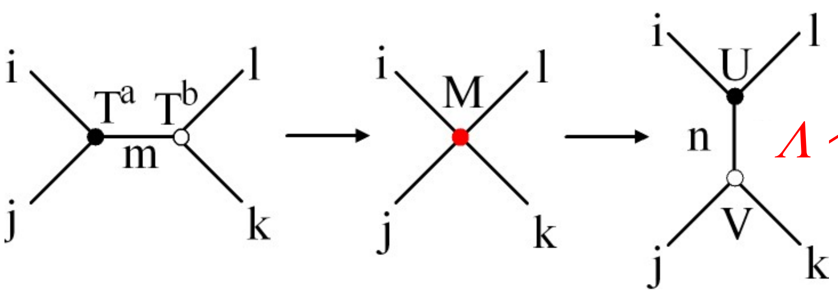
I. Poor Man's SRG: entanglement mean-field approach

$$Z = Tr\left(MM^{env}\right)$$

$$M^{env}_{kl,ij} \approx \Lambda_k^{1/2}\Lambda_l^{1/2}\Lambda_i^{1/2}\Lambda_j^{1/2}$$

Mean field (or cavity) approximation

$$M_{kj,il} = \sum_{n=1\ldots D^4} U_{kj,n}\Lambda_n V_{il,n}$$



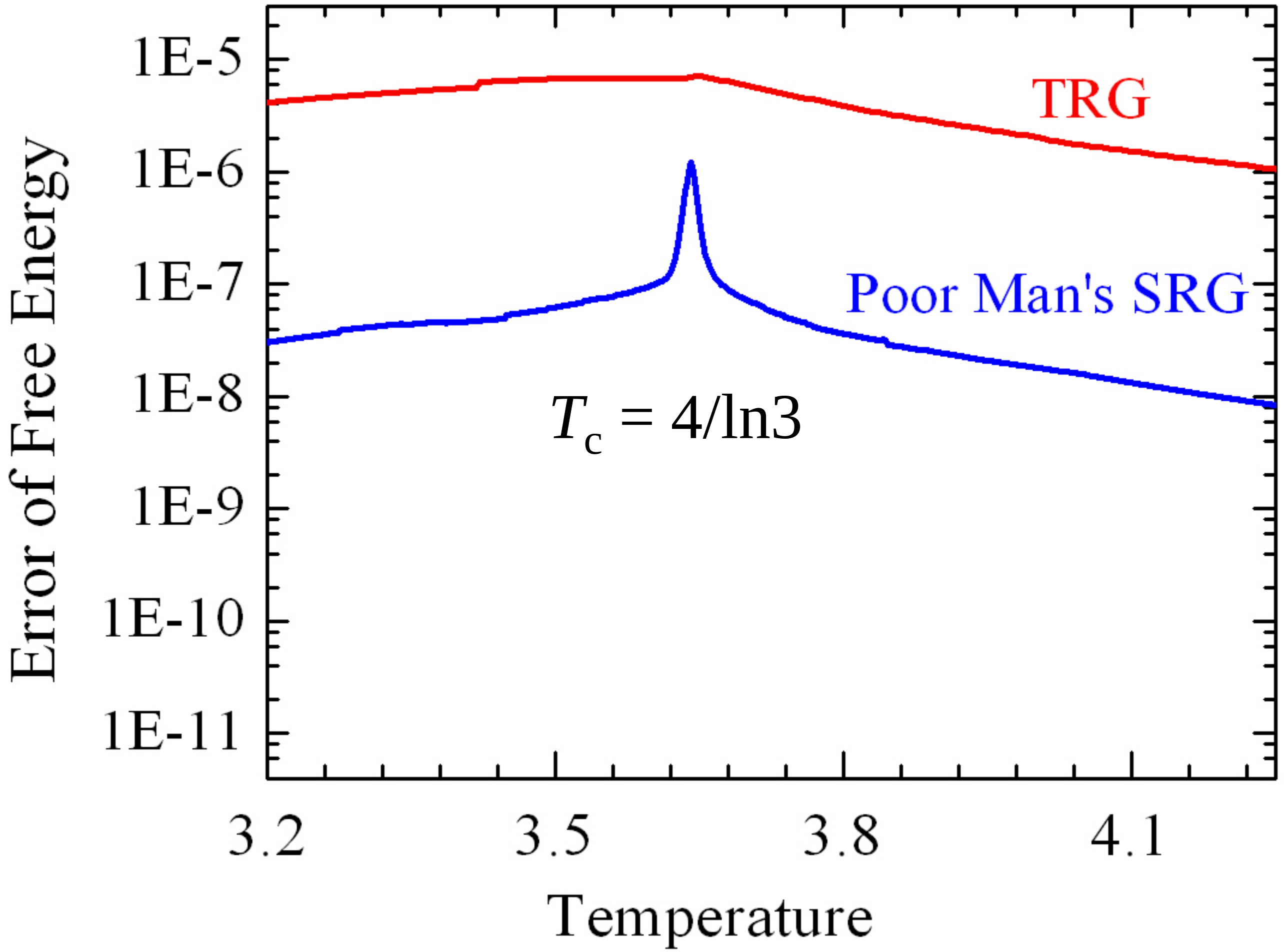
Bond field – measures the entanglement between *U* and *V*

$\Lambda = \Lambda^{1/2}\ \Lambda^{1/2}$

From environment

From system

Accuracy of Poor Man's SRG

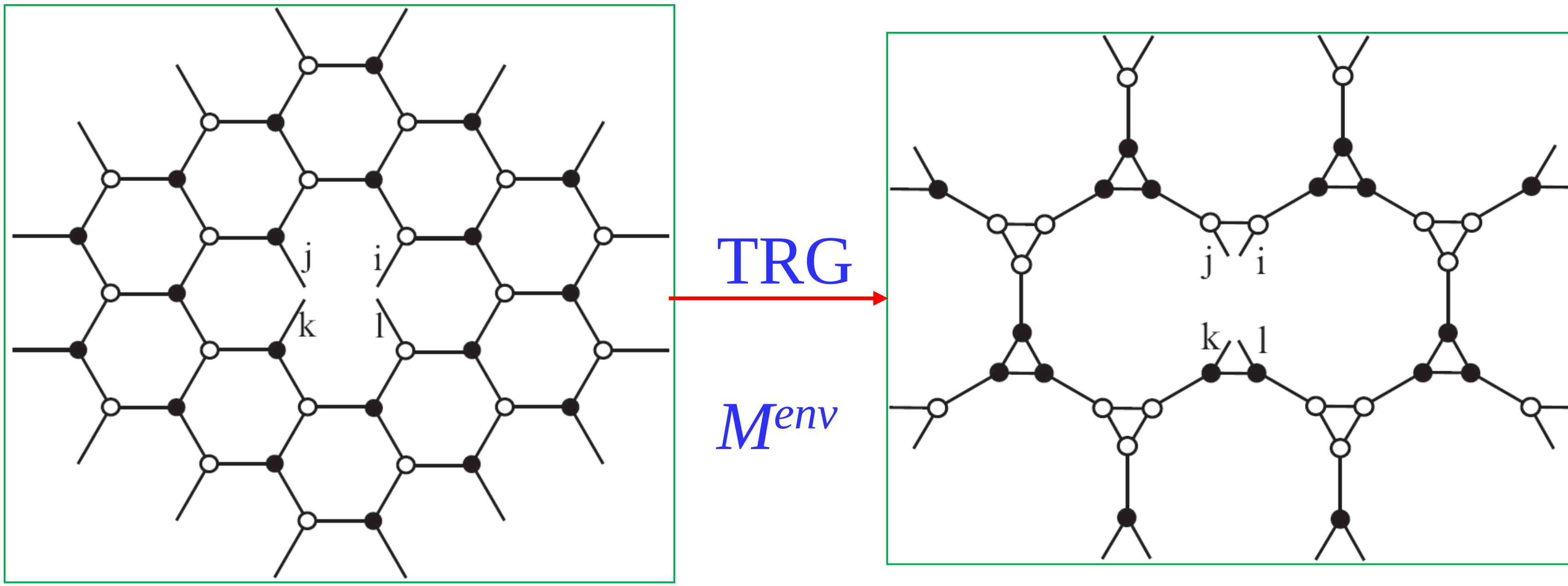


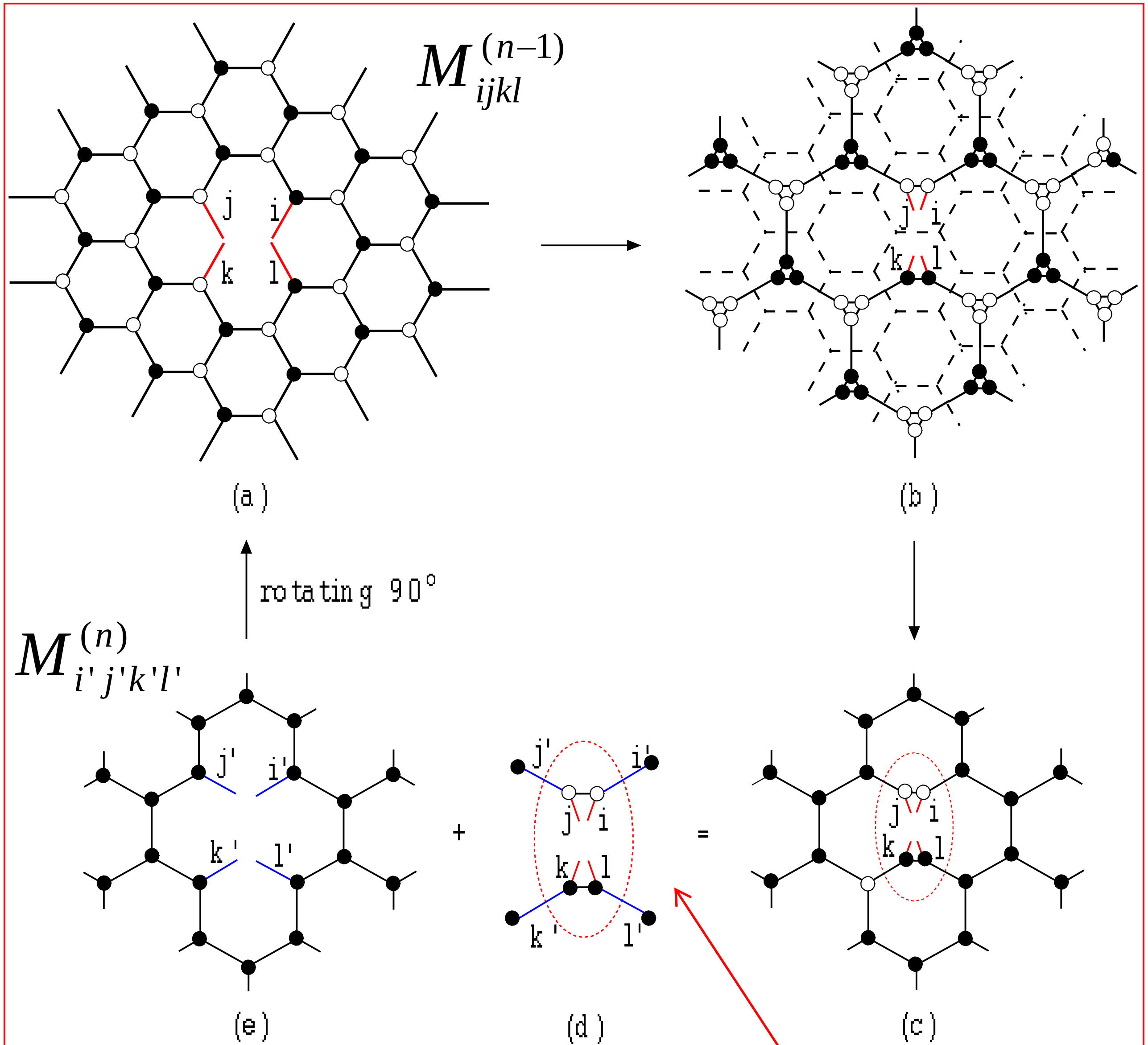
D = 24

Ising model on a triangular lattice

II. More accurate treatment of SRG

Evaluate the environment contribution M^{env} using TRG





1. Forward iteration

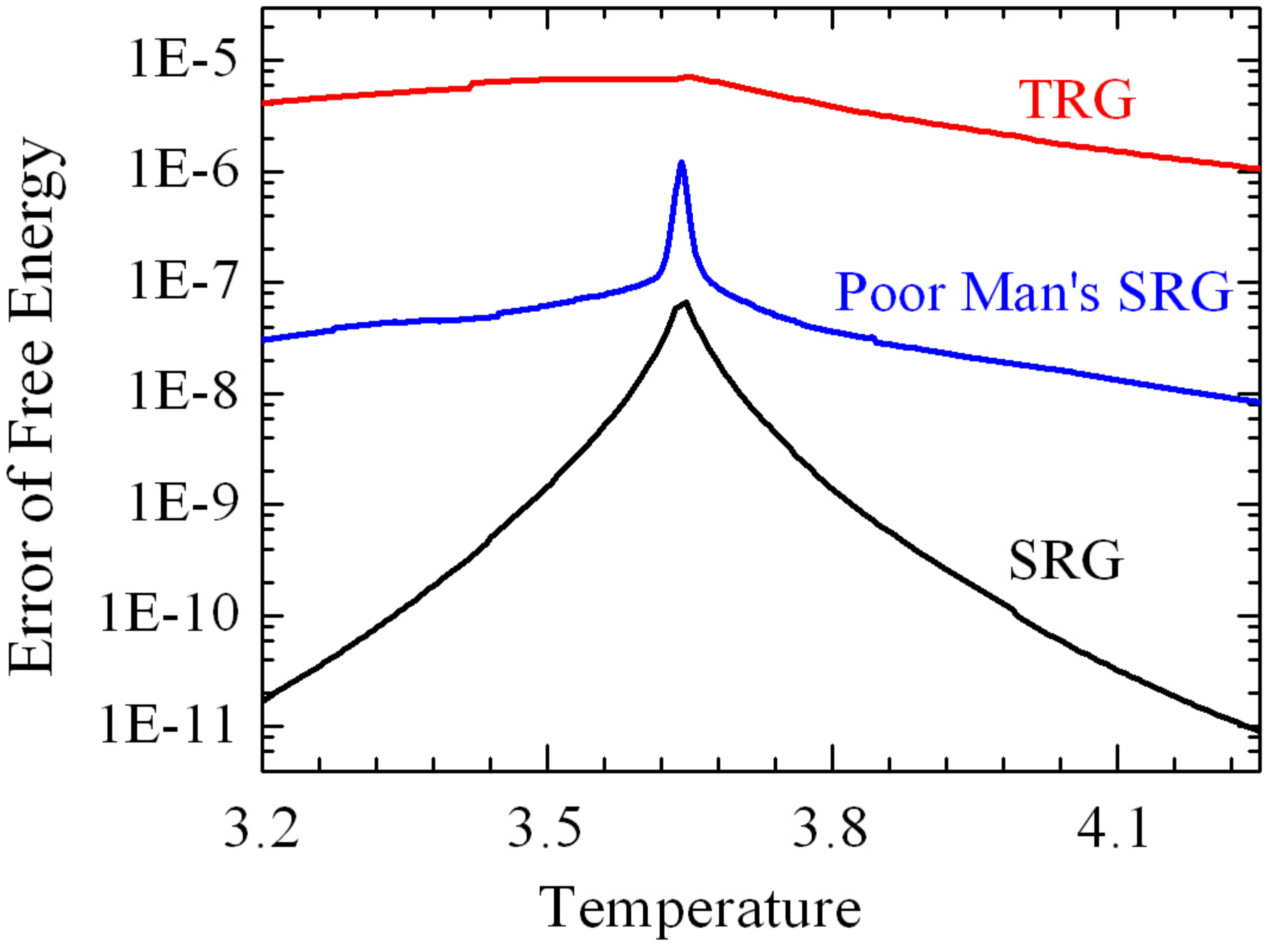
$$M^{(0)} \rightarrow M^{(1)} \rightarrow \cdots \rightarrow M^{(N)}$$

2. Backward iteration

$$M^{(N)} \rightarrow M^{(N-1)} \rightarrow \cdots \rightarrow M^{(0)} = M^{env}$$

$$M_{ijkl}^{(n-1)} = \sum_{i'j'k'l'} M_{i'j'k'l'}^{(n)} \sum_{pq} \underline{S_{k'jp} S_{j'pi} S_{i'lq} S_{l'qk}}$$

Accuracy of SRG

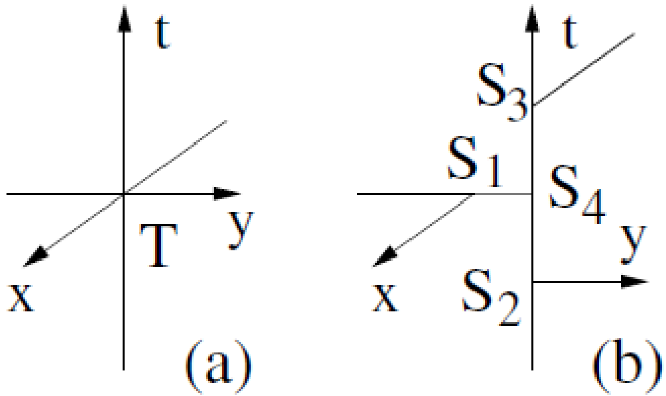


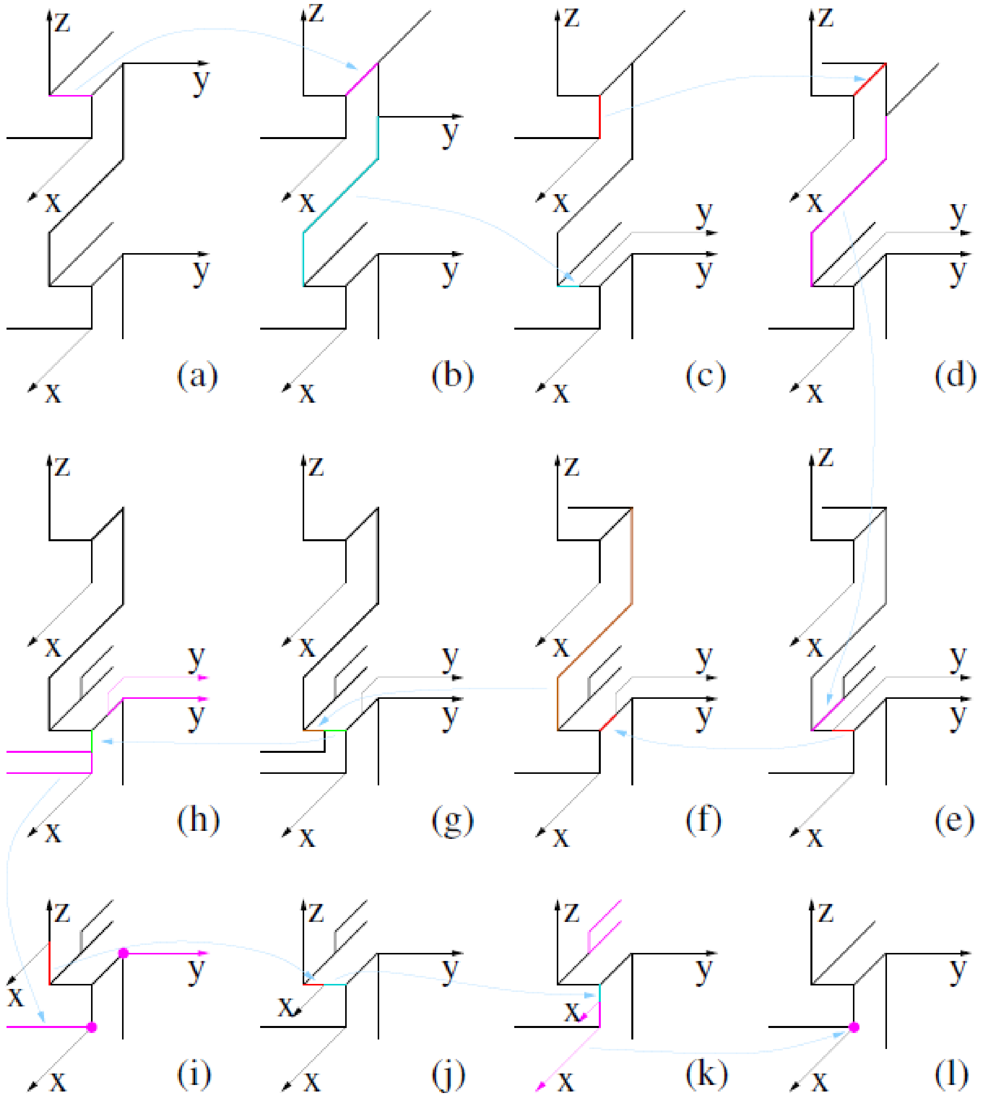
D = 24

Ising model on a triangular lattice

Extension to 3D is difficult

Z. C. Gu, M. Levin, and X. G. Wen, unpublished





- **Decompose a tensor into 3 parts**
- **11 RG moves**
- **Error ~ 1 %**

Efforts made by Nishino and collaborators

Corner Transfer Tensor Renormalization Group method

Nishino & Okunishi, JPSJ 67, 3066 (1998)

Variational wavefunction (transfer matrix)

Okunishi, Nishino, Prog. Theor. Phys, 103, 541 (2000);

Maeshima, Hieida, Akutsu, Nishino, Okunishi, PRE 64, 016705 (2001);

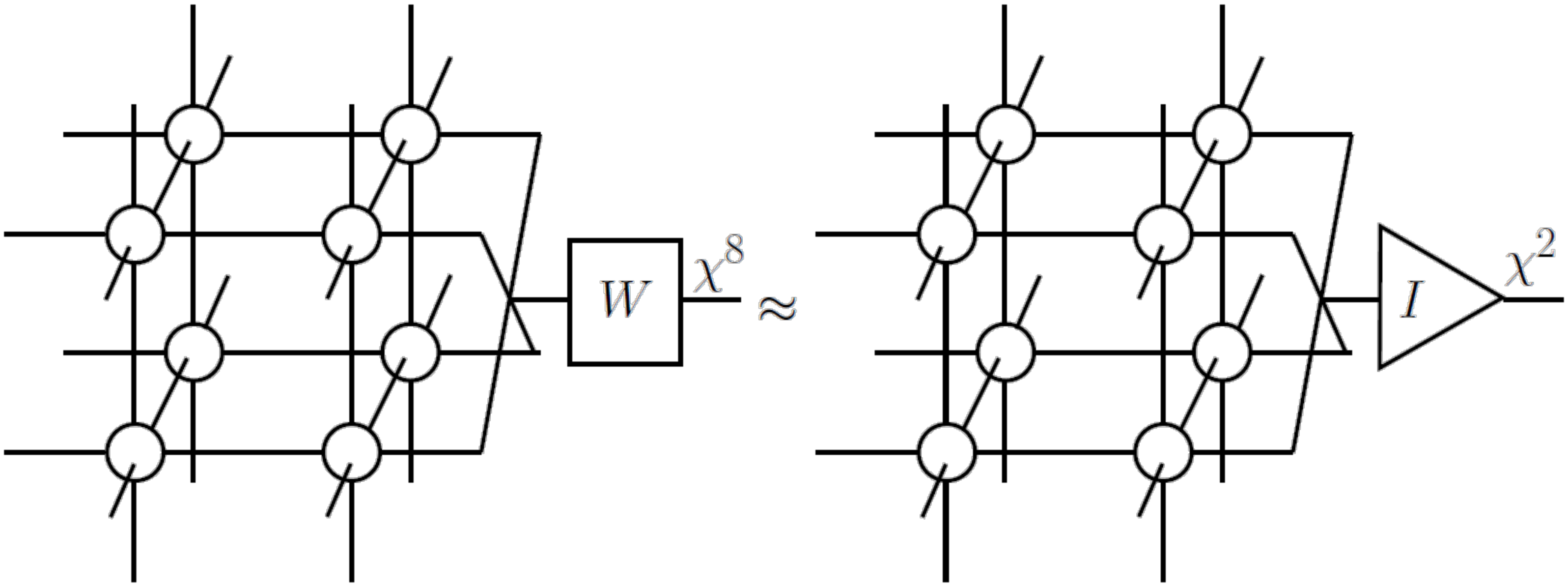
Nishino, Hieida, Okunishi, Maeshima, Akutsu, Gendiar, Prog. Theo. Phys. 105, 409 (2001).

Gendiar & Nishino, PRB 71, 024404 (2005)

Main problem: tensor dimension D = 2 ~ 5

error > 0.6 %

A. Garcia-Saez, and J. I. Latorre, arXiv:1112.1412

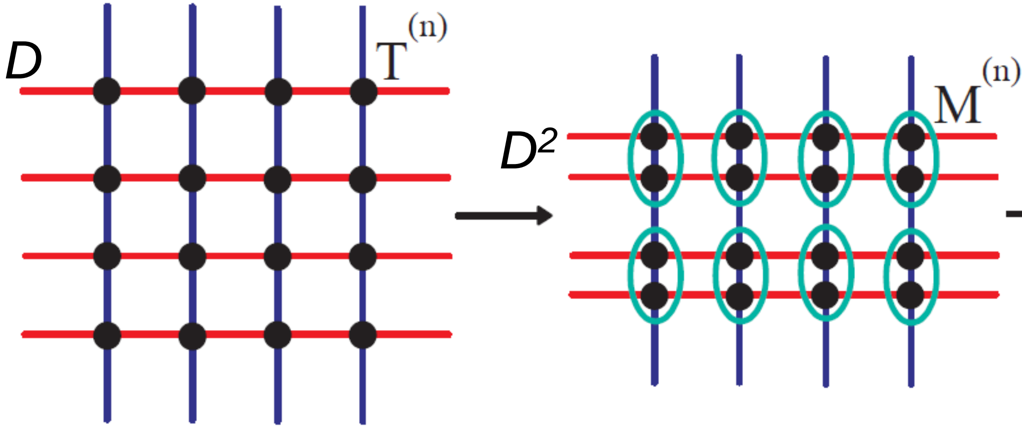


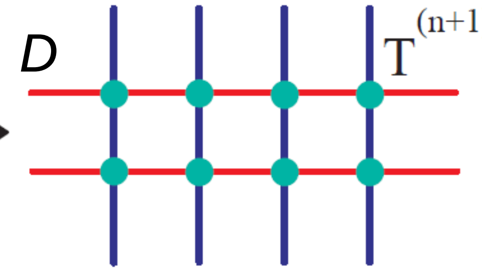
$D_{max} \sim 5$

error ~ 2.7 %

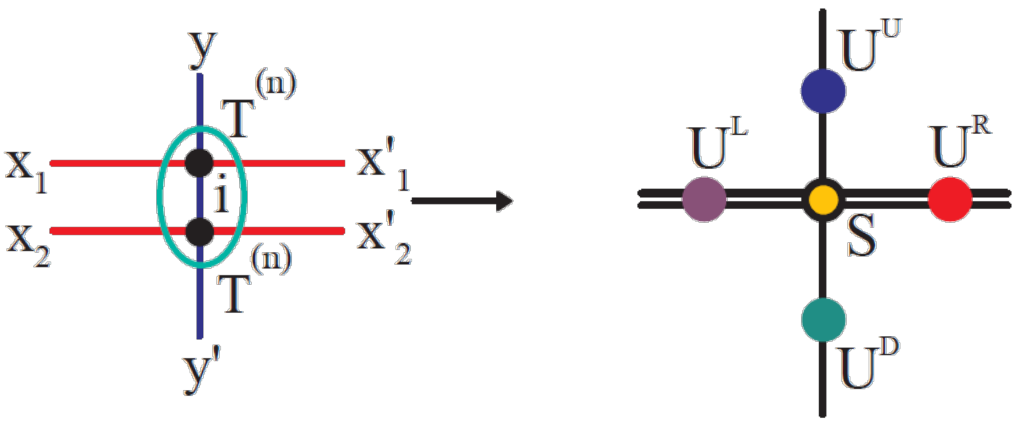
TRG with Higher-Order Singular Value Decomposition of Tensors

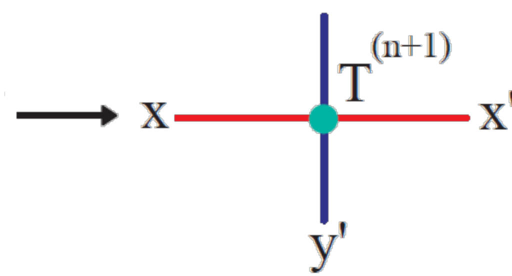
Z. Y. Xie et al, arXiv:1201.1144





$$M^{(n)}_{xx'yy'} = \sum_i T^{(n)}_{x_1 x'_1 y i} T^{(n)}_{x_2 x'_2 i y'}$$





Higher order singular value decompostion

$$\boxed{M^{(n)}_{xx'yy'} = \sum_{ijkl} S_{ijkl} U^L_{xi} U^R_{x'j} U^U_{yk} U^D_{y'l}}$$

Higher-Order Singular Value Decomposition(HOSVD)

$$M^{(n)}_{xx'yy'} = \sum_{ijkl} S_{ijkl} U^L_{xi} U^R_{x'j} U^U_{yk} U^D_{y'l}$$

Core tensor

- **all-orthogonal:**

$$\langle S_{:,j,:,:} \mid S_{:,j',:,:} \rangle = 0, \qquad \text{if } j \neq j'$$

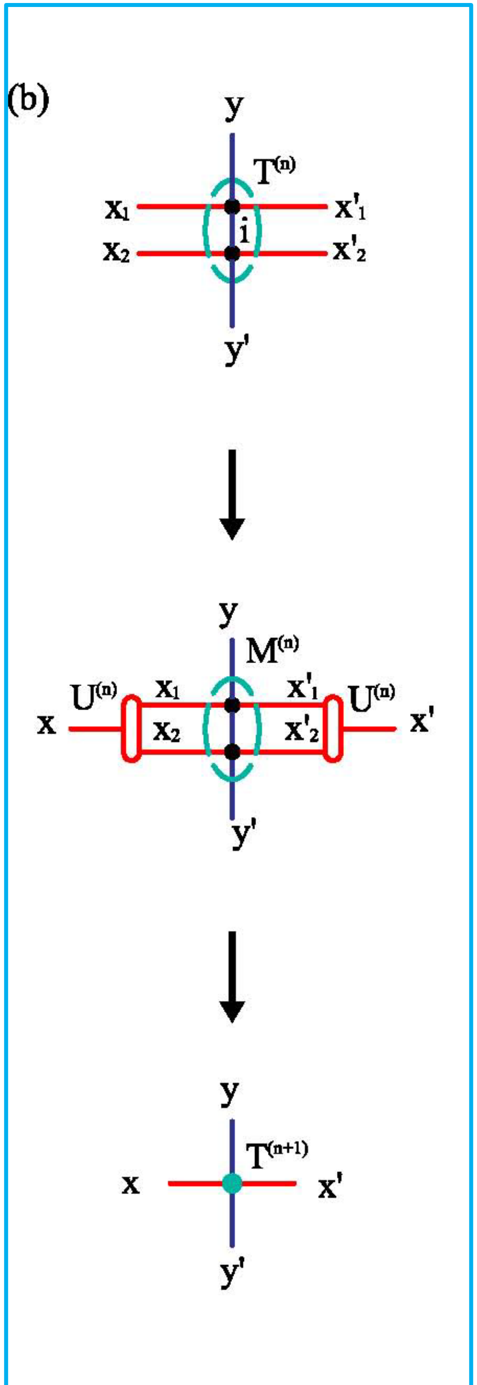
- **pseudo-diagonal / ordering:**

$$|S_{:,j,:,:}| \geq |S_{:,j',:,:}|, \qquad \text{if } j < j'$$

Nearly optimal low-rank approximation

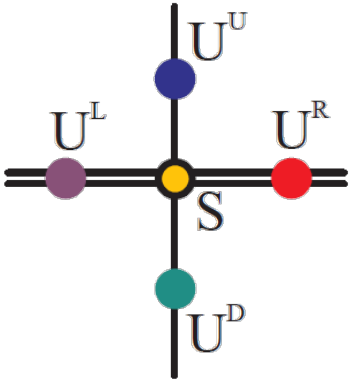
L. de Latheauwer, B. de Moor, and J. Vandewalle, SIAM, J. Matrix Anal. Appl, 21, 1253 (2000).

Unitary Transformation Matrix



$$\varepsilon_1 = \sum_{i>D} |S(i,:,:,:)|^2$$

$$\varepsilon_2 = \sum_{j>D} |S(:,j,:,:)|^2$$



Only horizontal bonds need to be cut

if $\varepsilon_1 < \varepsilon_2$, $U^{(n)} = U^L$

if $\varepsilon_1 > \varepsilon_2$, $U^{(n)} = U^R$

truncation error = min(ε_1, ε_2)

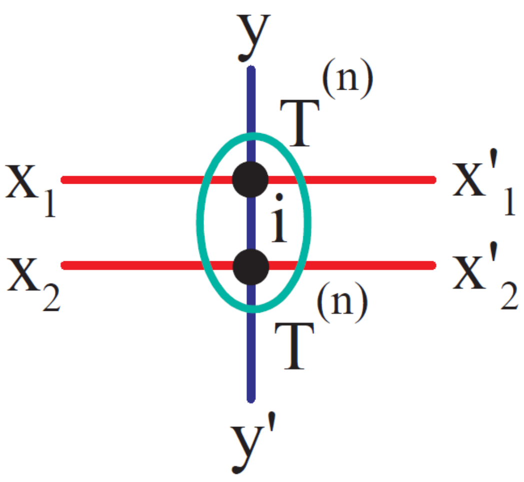
$$\boxed{T^{(n+1)}_{xx'yy'} = \sum_{ij} U^{(n)}_{ix} M^{(n)}_{ijyy'} U^{(n)}_{jx'}}$$

How to do the HOSVD

HOSVD can be achieved by successive SVD for each index of the tensor

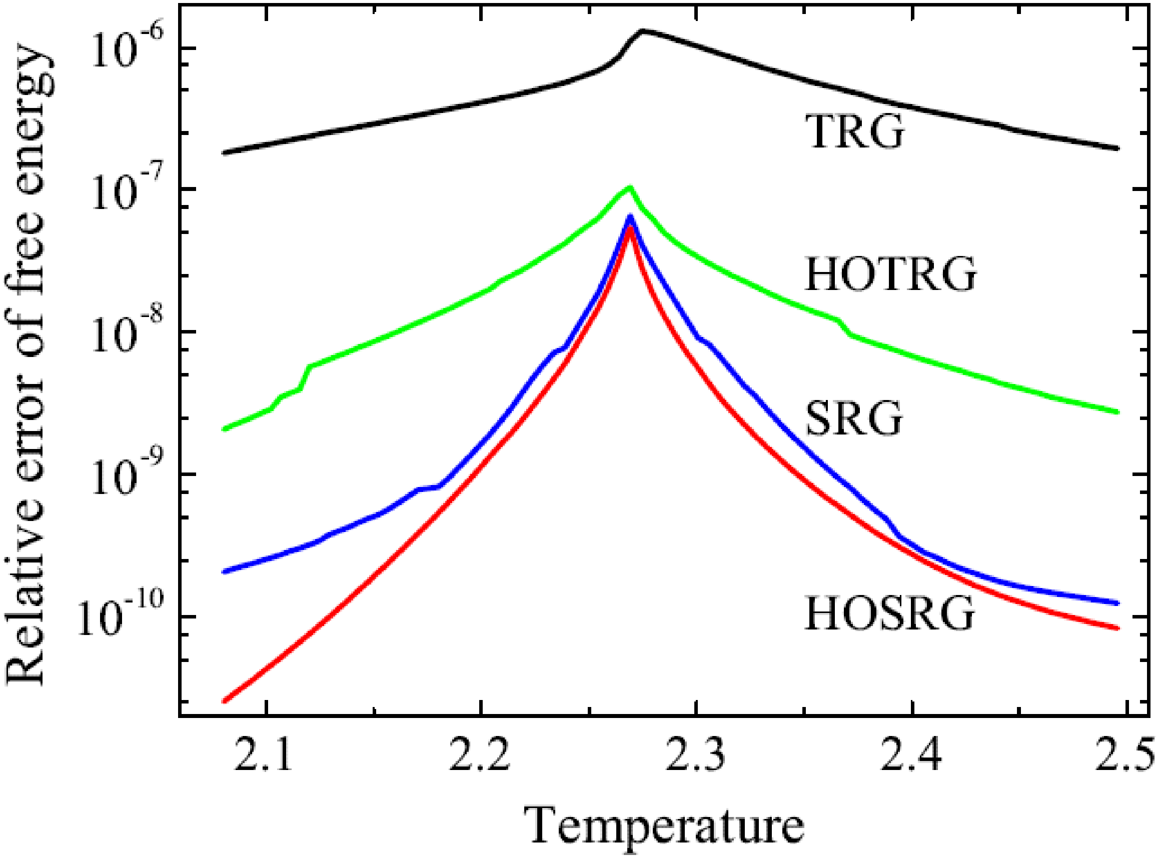
For example

$$M^{(n)}_{(x;x'yy')} = U^L \Lambda V^\dagger$$



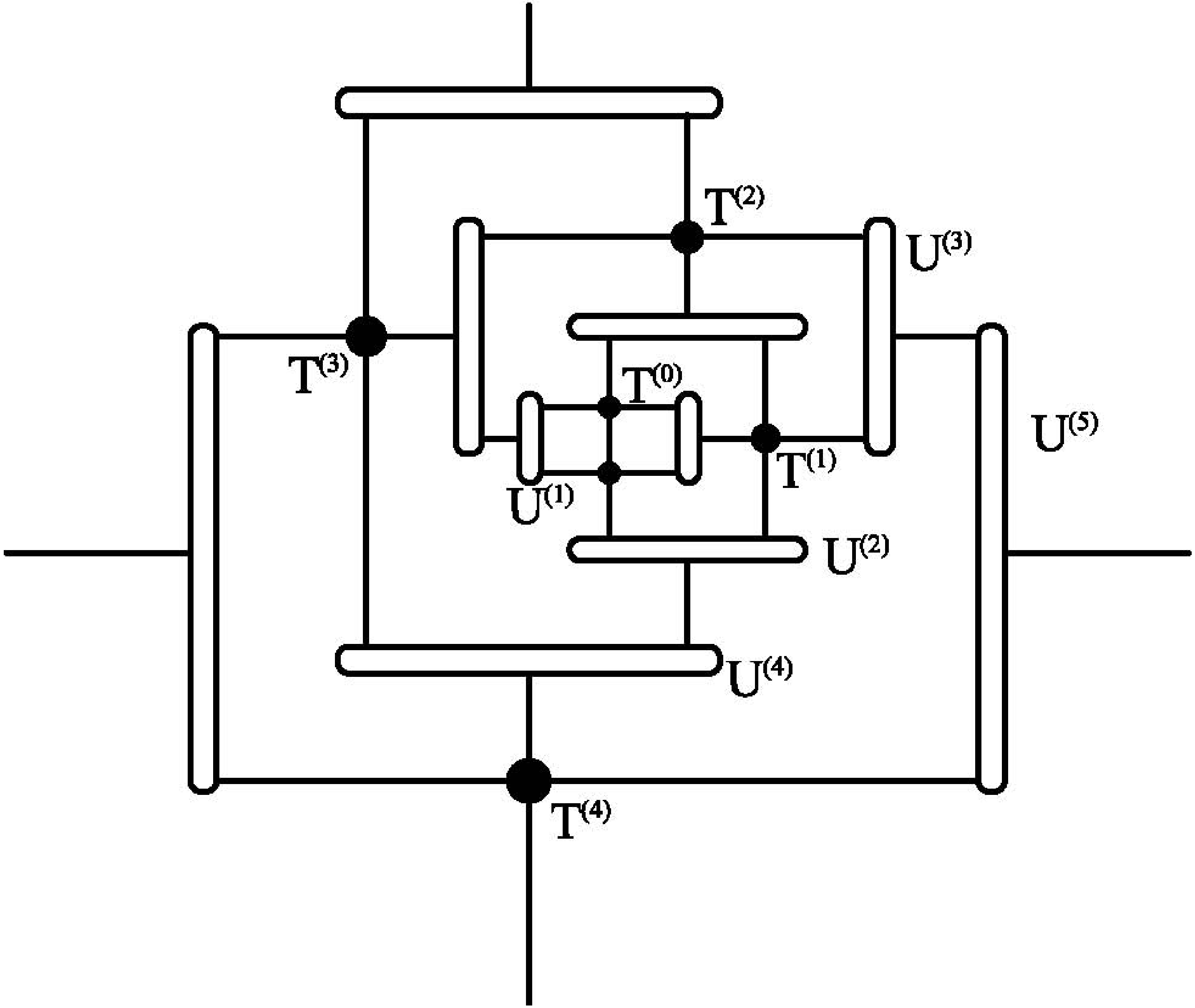
Accuracy of HOTRG

Ising model on the square lattice



D = 24

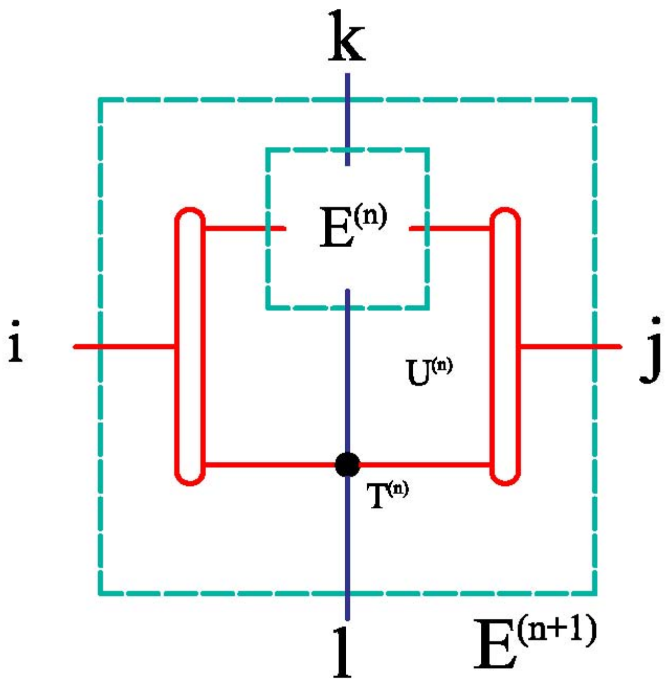
Hierarchical structure of HOTRG



Second Renormalization : HOSRG

Forward iterations: HOTRG to determine $U^{(n)}$ and $T^{(n)}$

backward iterations : evaluate the environment tensors

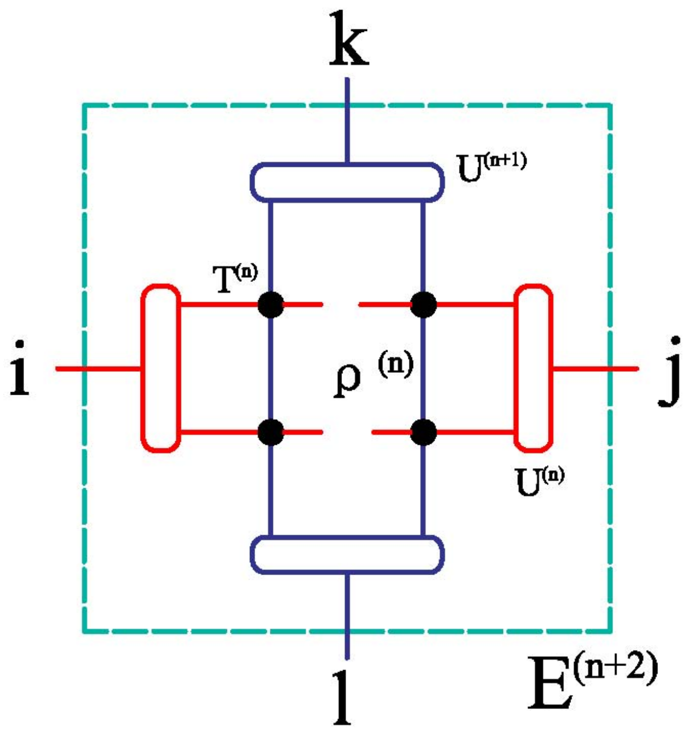


$$\boxed{E^{(n)}_{kaj_1i_1} = \sum_{ijli_2j_2a} E^{(n+1)}_{ijkl} T^{(n)}_{i_2j_2al} U^{(n)}_{i_1i_2,i} U^{(n)}_{j_1j_2,j}}$$

HOSRG: sweeping

Forward iterations:

- Evaluate the bond density matrix
- Find new $U^{(n)}$ and $T^{(n)}$

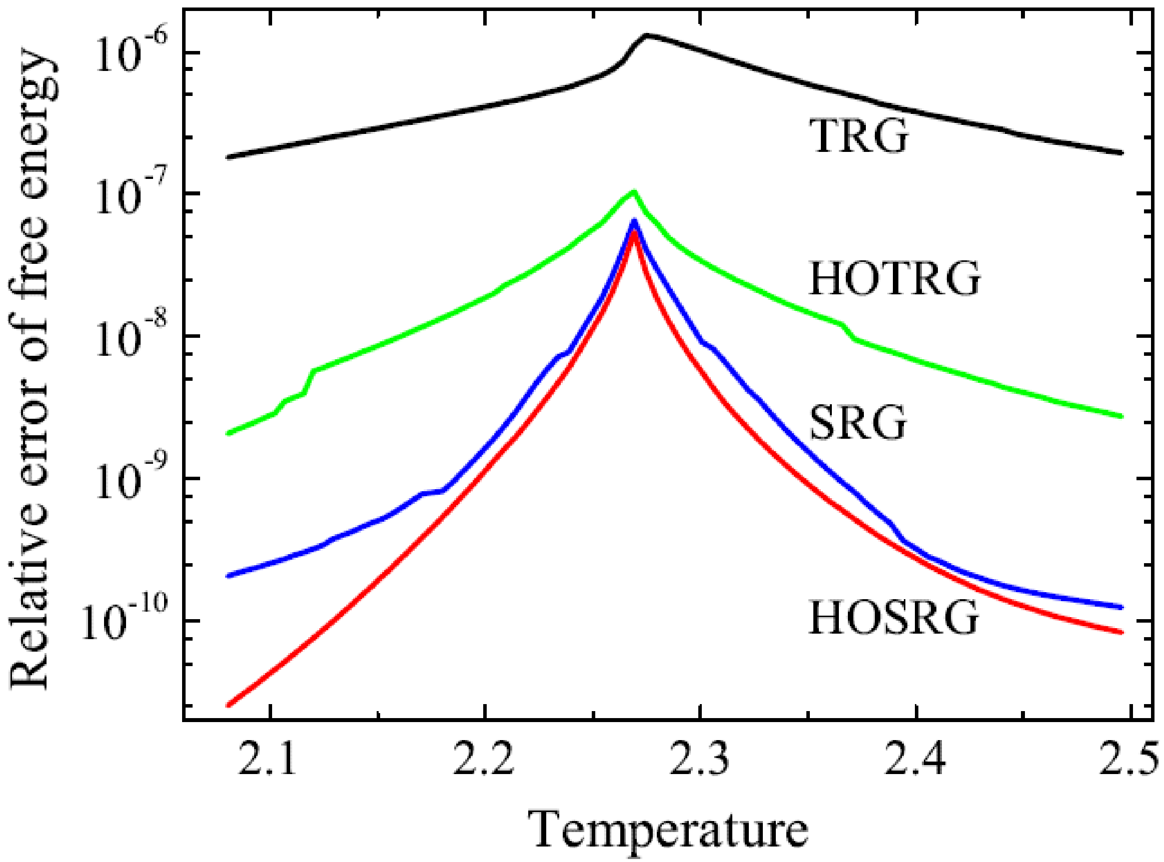


$$\boxed{\rho^{(n)} = \sum E^{(n+2)} U^{(n)} U^{(n)} U^{(n+1)} U^{(n+1)} T^{(n)} T^{(n)} T^{(n)} T^{(n)}}$$

$$\boxed{\rho^{(n)} = U^{(n)} \lambda^{(n)} U^{\dagger(n)}}$$

Accuracy of HOSRG

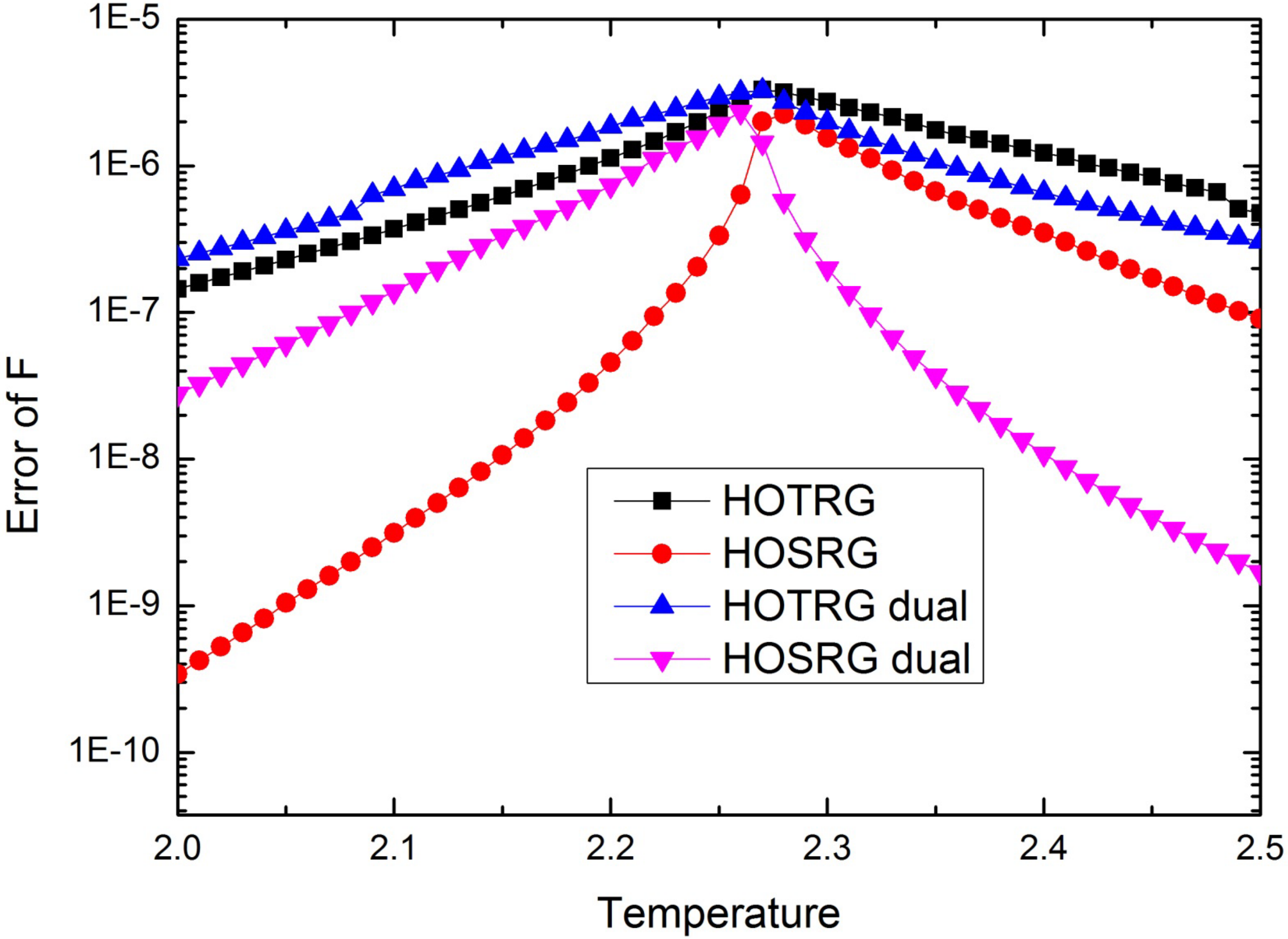
Ising model on the square lattice



D = 24

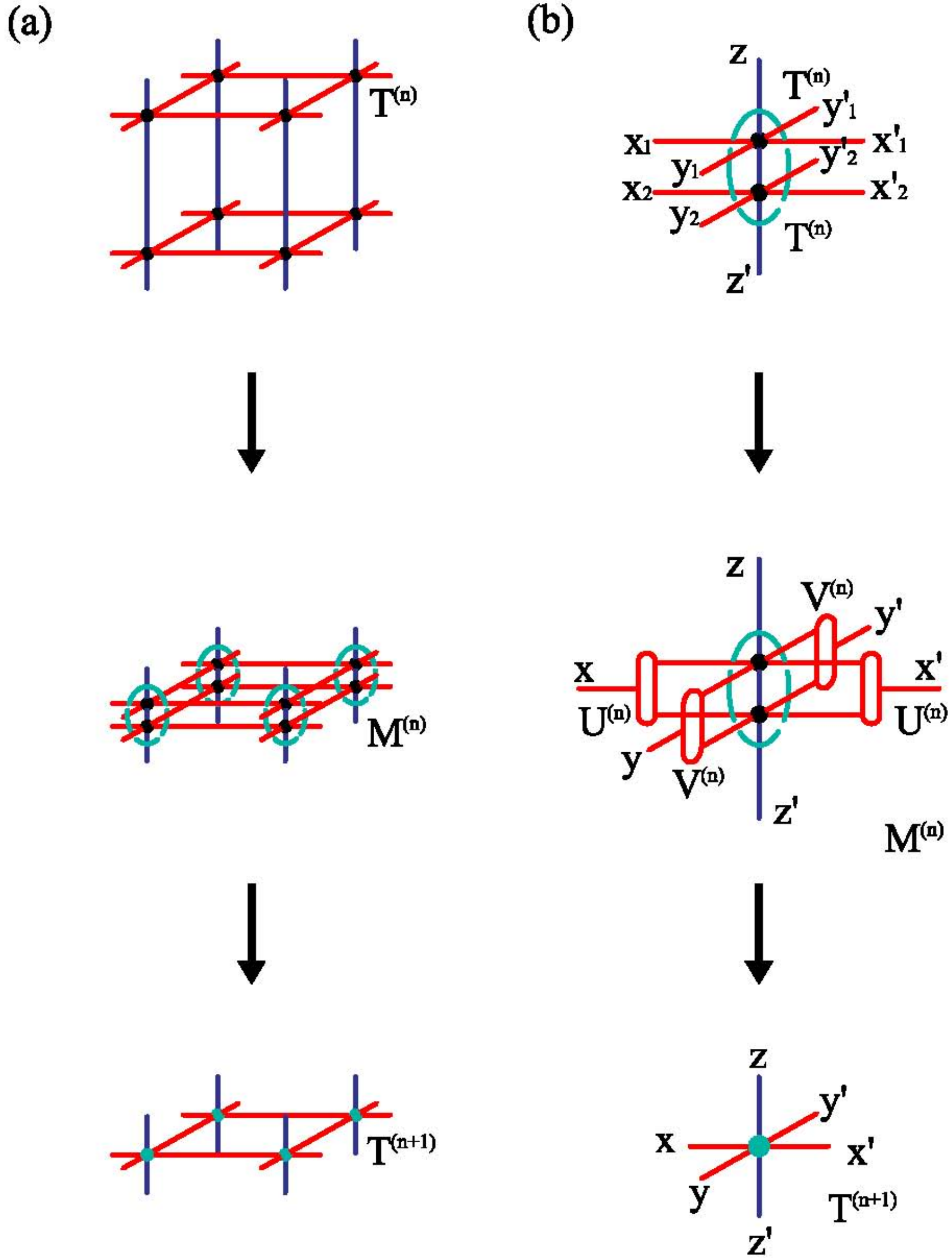
HOSRG with forward-backward sweeping

Ising model on the square lattice

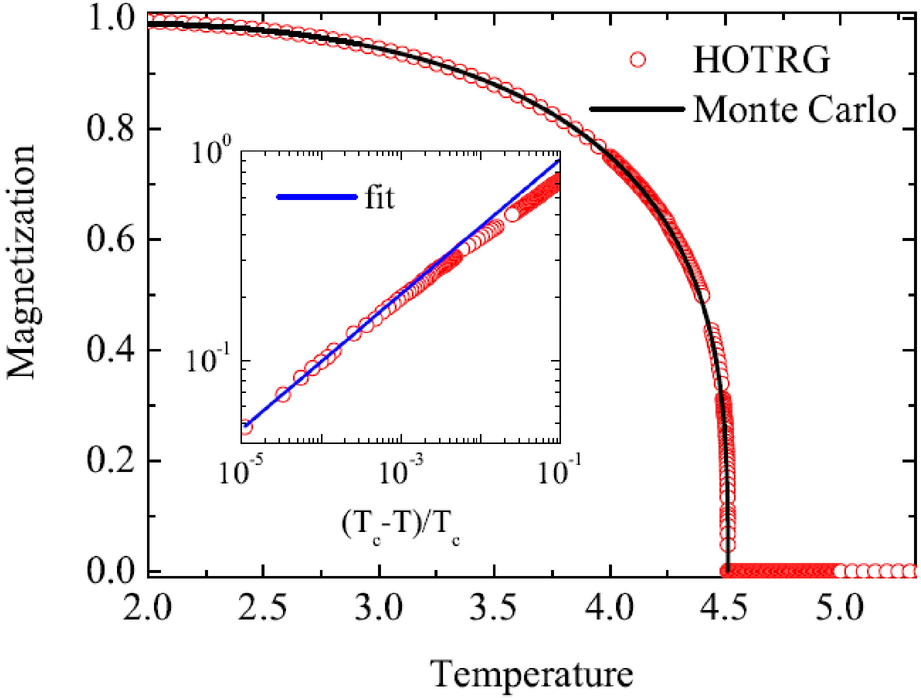


D = 10

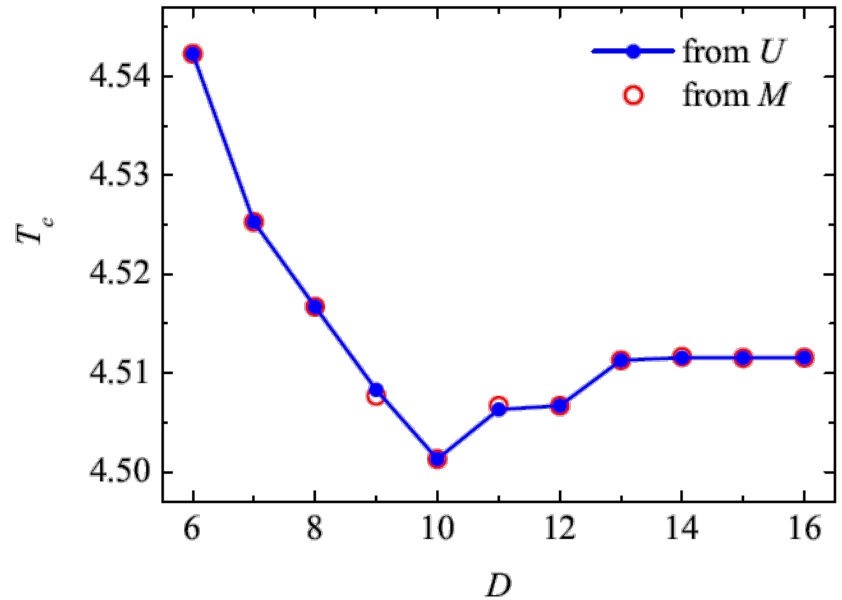
Three dimensions



Benchmark result for the 3D Ising model



3D Ising model



Critical exponent

TRG:	**0.3237**
MC:	**0.3262**
HTSE:	**0.3265**

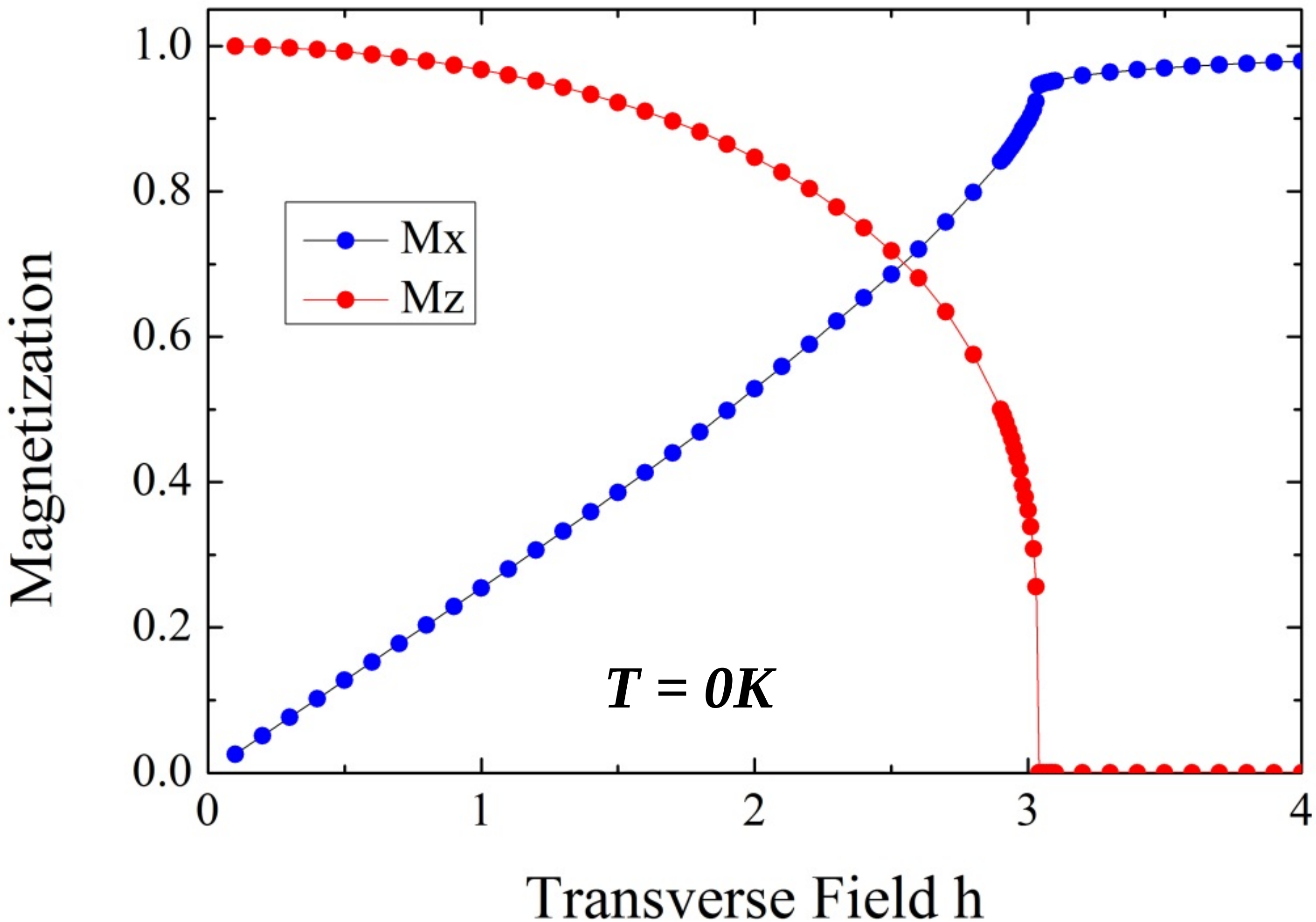
T_c = 4.511546

Relative Error

TRG < 10^{-6}

other RG methods ~ 10^{-2}

Thermodynamics of 2D QuantumTransverse Ising Model



Phase Transition with Partial Symmetry Breaking

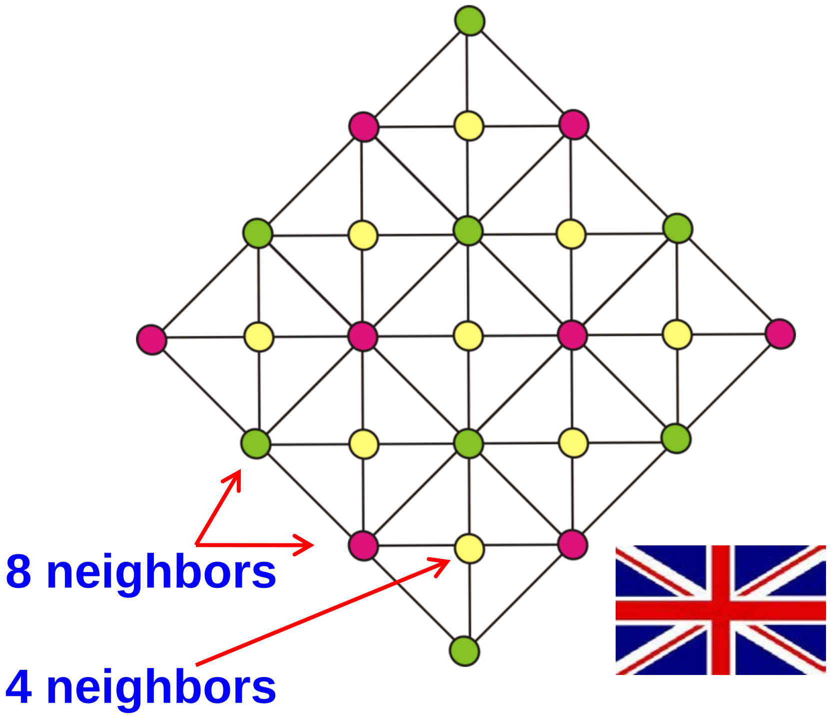
QN Chen et al, PRL 107, 165701 (2011)

q=4 Potts Model on the UnionJack Lattice

Is there any phase transition?

$$H = J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}$$

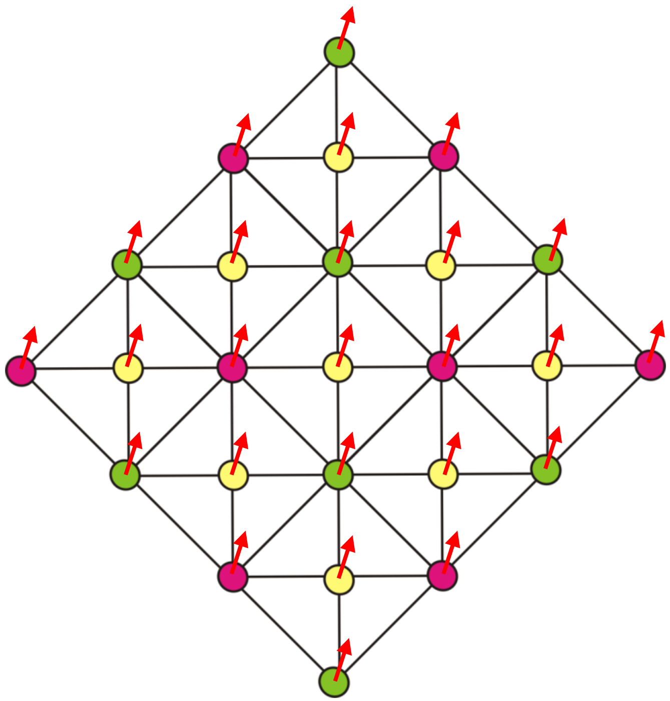
$$\sigma_i = 1,\ldots,4$$

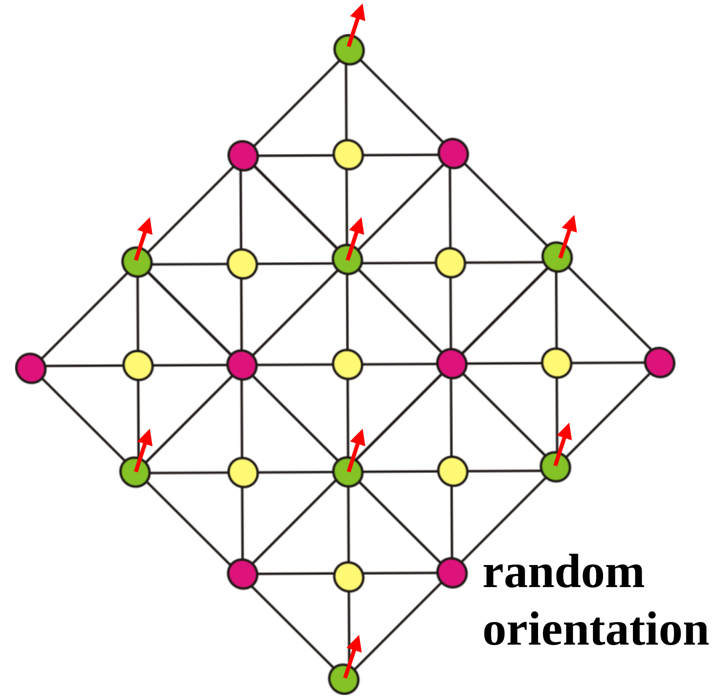


The Potts model is a basic model of statistical physics

It has been intensively studied for more than 70 years

Full versus Partial Symmetry Breaking





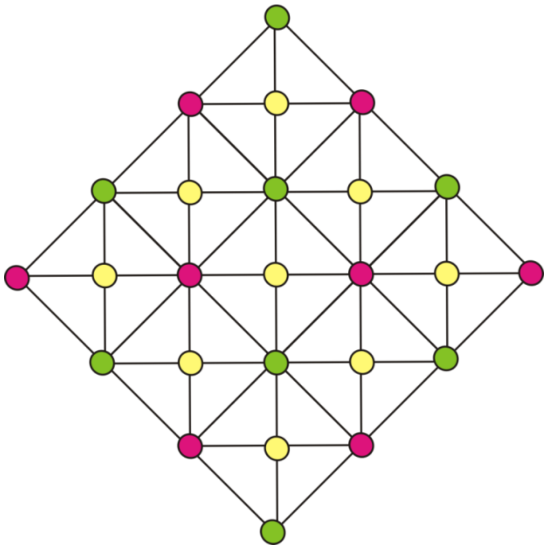
full symmetry breaking

Entropy = 0

partial symmetry breaking

Entropy is finite

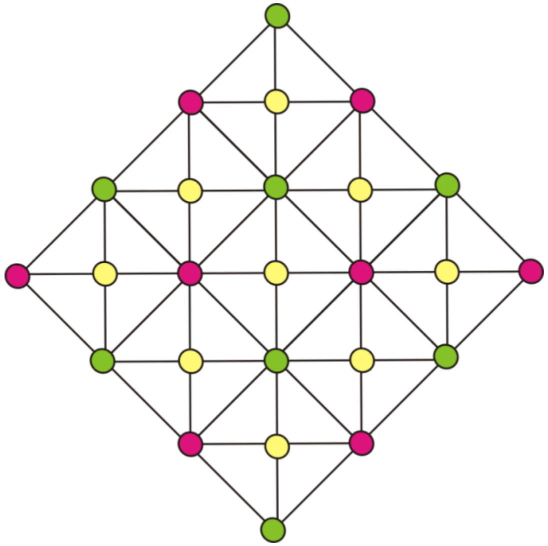
Ground States and Their Entropies

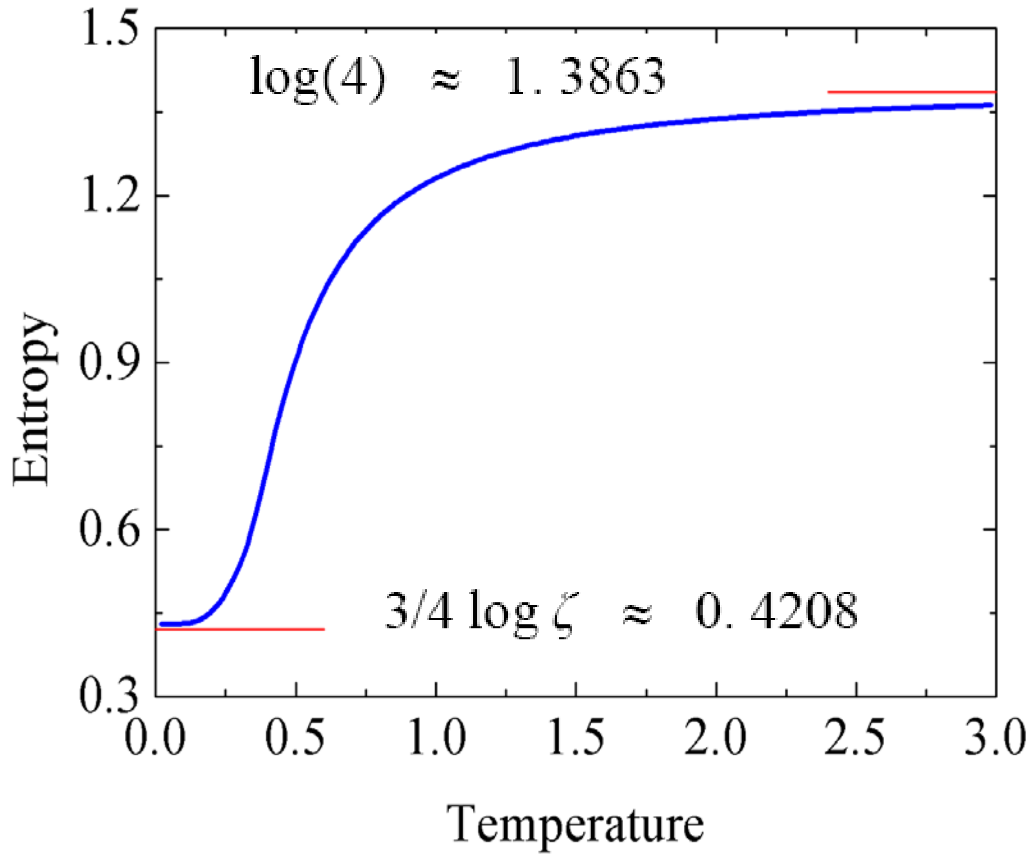


$$\boxed{S = (N/2) \ln 2 + 2 * (3N/4) \ln \xi}$$

- **If red or green sublattice is ordered, the ground states are $\xi^{3N/4}$-fold degenerate** $S = (3N/4) \ln \xi$
- **both red and green sublattices are ordered, the ground states are $2^{N/2}$-fold degenerate:** $S = (N/2) \ln 2$

Entropy and Partial Order



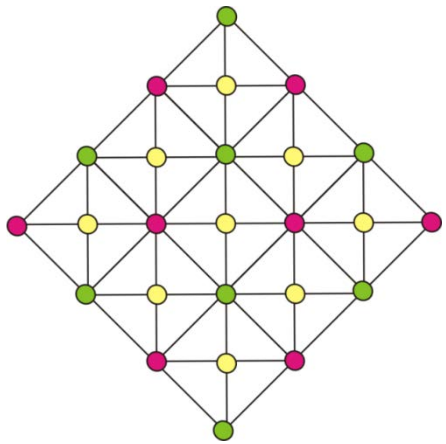


The red or green sublattice is ordered

$$\zeta = 1.7525 > 2^{2/3} \approx 1.5874$$

Conjecture: There is a Finite T Phase Transition

There is a partial symmetry breaking at 0K

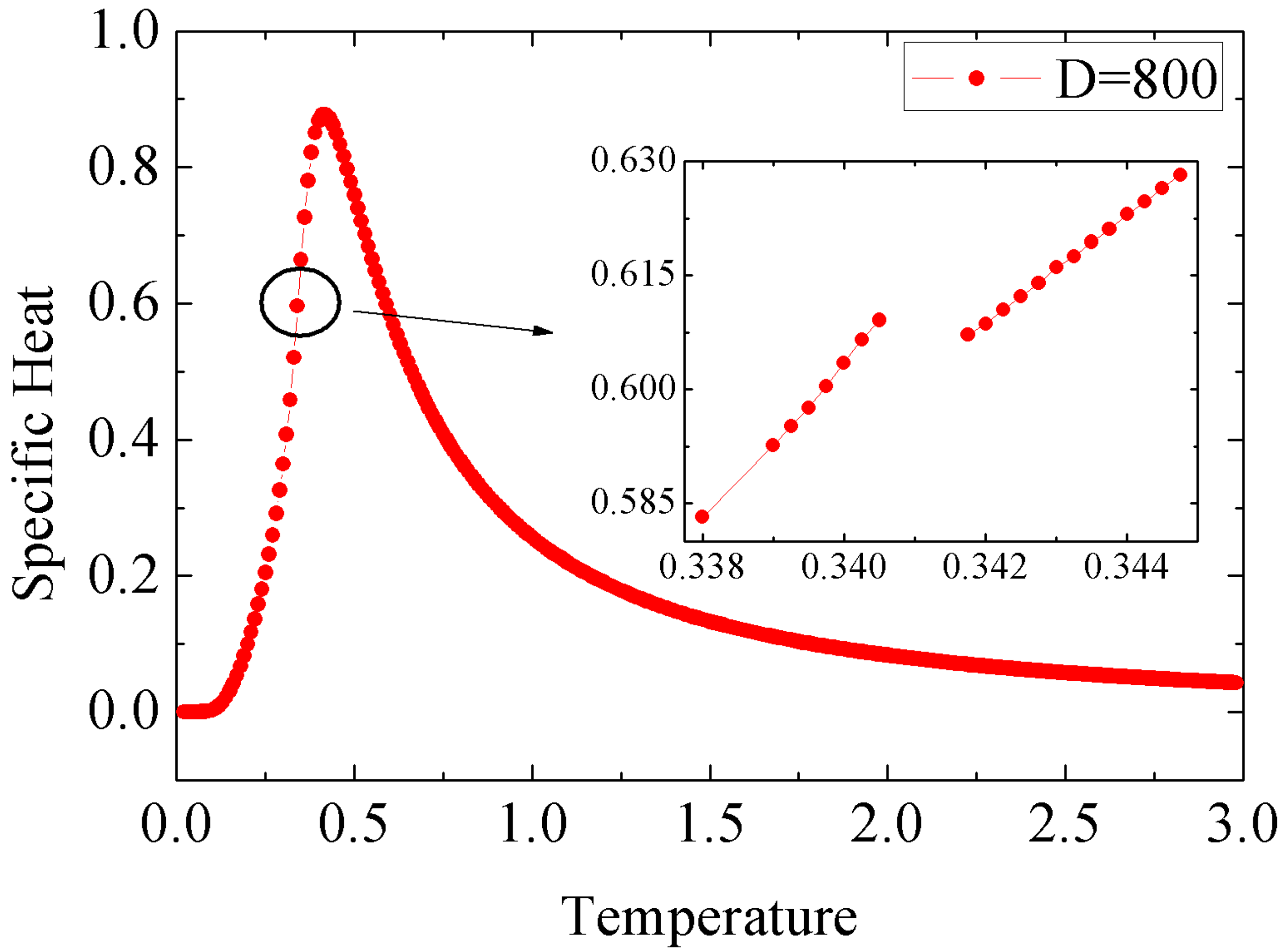


q = 4 Potts model

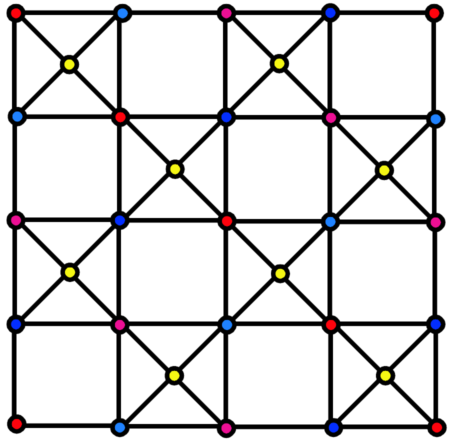
There is a finite T phase transition with two singularities:

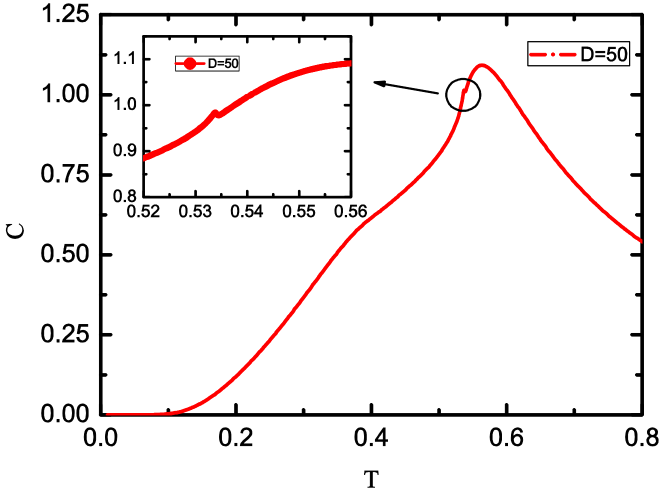
1. **ordered and disordered states**
2. **Z_2 between green and red**

Phase Transition: Specific Heat Jump

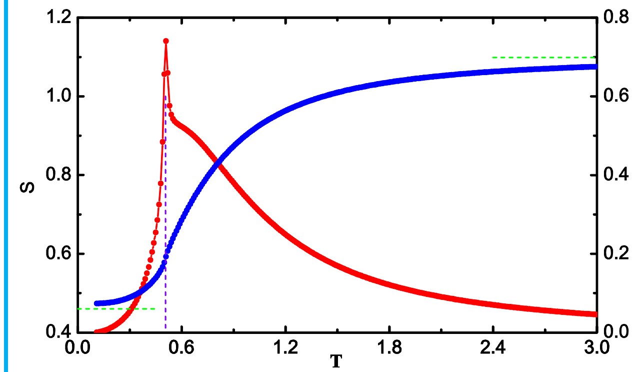


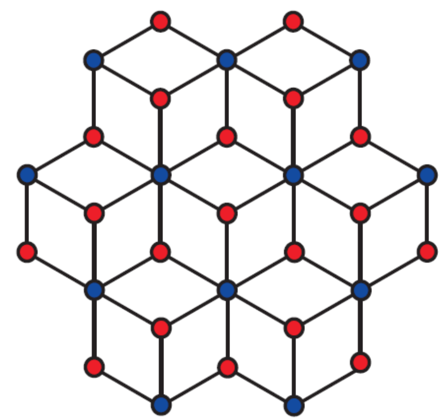
Partial Order Phase Transition in Other Irregular Lattices



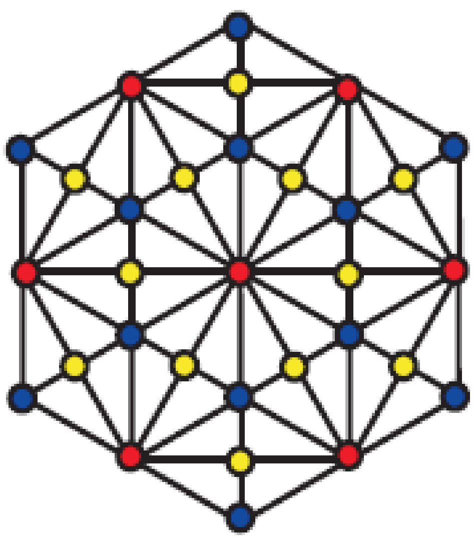


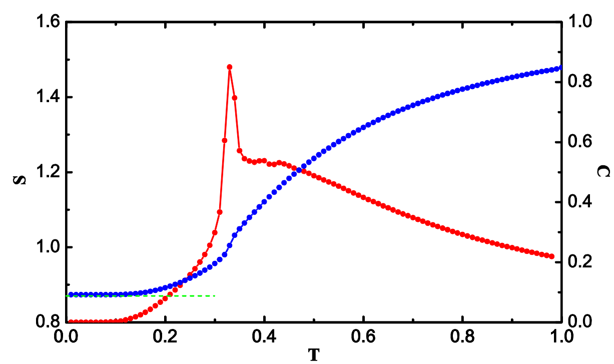
Checkerboard Lattice





Diced Lattice





Centered Diced Lattice

Summary

1. **HOSRG provides an accurate and efficient numerical method for studying 2D or 3D classical/quanutum lattice models**

2. **AF Potts Model has an entropy driven partial ordering and finite T phase transition on irregular lattices**