Multipartite entanglement: An algebraic criterion

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Introduction

- **In spite of the strong interest and deep studies, detection and classification of entanglement in multipartite systems are unsolved problems up to date;**

- **An important quantity that has been used to reveal entanglement is the purity of the reduced density matrix the more two systems are entangled, the less pure is the reduced state describing each one of the two systems.**

Bipartite Systems

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\{|k\rangle_A \otimes |j\rangle_B\}$$

Superposition principle

$$|\psi\rangle = \sum_{k,j} c_{kj} |k\rangle_A \otimes |j\rangle_B = \sum_{k,j} c_{kj} |kj\rangle$$

Separable state

$$|\psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B$$

Otherwise $\longrightarrow$ **entangled**

Examples

Entagled states

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$\rho_A = Tr_B(|\psi^+\rangle\langle\psi^+|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Separable states

$$|\chi^+\rangle = \frac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes (|0\rangle_B + |1\rangle_B)$$

$$\rho_A = Tr_B(|\chi^+\rangle\langle\chi^+|) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Multipartite systems

$$\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$$

- **Completely separable**
  - **Separable**
    - **Completely entangled**

Matrix of amplitude probabilities

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$|\phi\rangle = \sum_{i,j} a_{ij} |ij\rangle$$

Matrix of amplitude probabilities

$$A = (a_{ij})$$

Bipartite case

Theorem:

A pure state $|\phi\rangle$ of a bipartite $m \times n$ system is separable iff the rank of the corresponding matrix of the amplitudes probabilities $A = (a_{ij})$ is one.

Multipartite???

Multipartite case

Tripartite systems!

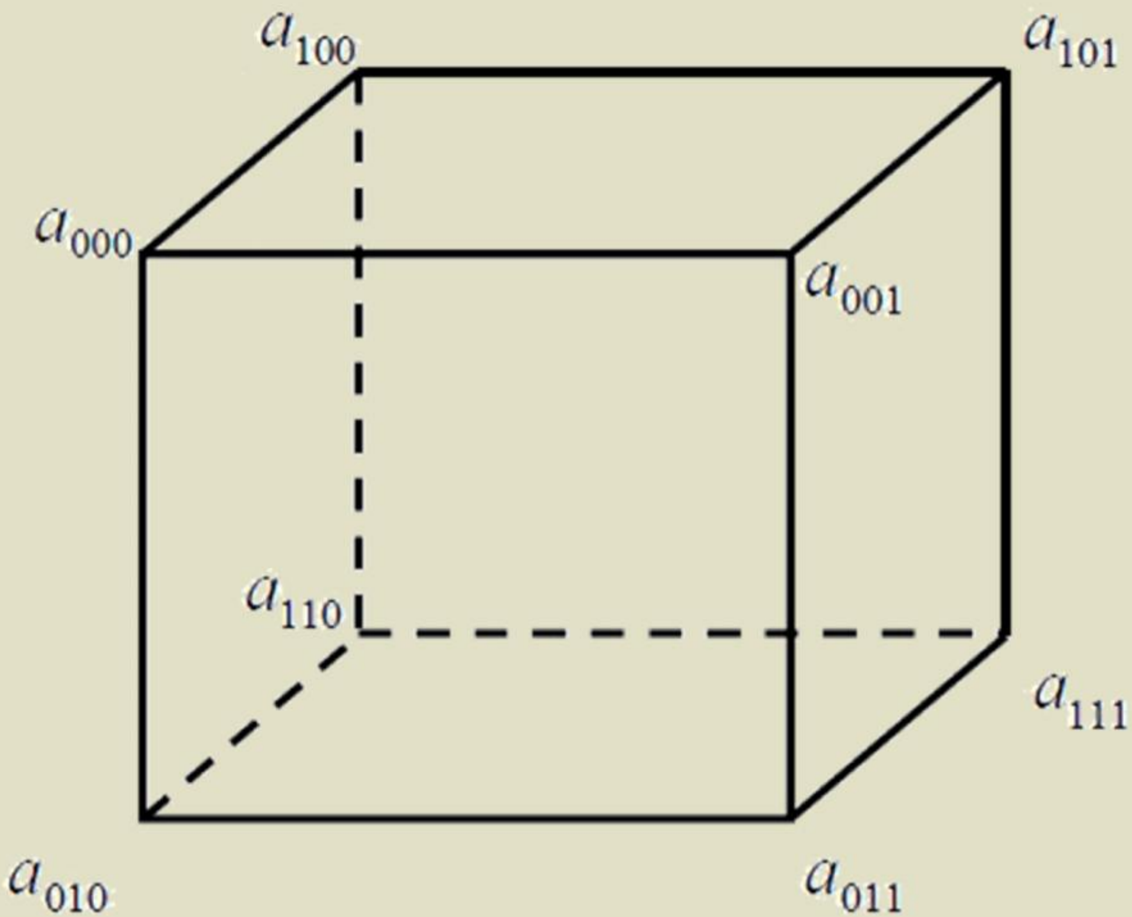
Theorem: A pure state $|\phi\rangle$ in a $n \times m \times p$ Hilbert space is separable iff at least one of the following is true:

$$M_1 := \sum |a_{ijk}a_{i'j'k'} - a_{ij'k'}a_{i'jk}|^2 = 0;$$

$$M_2 := \sum |a_{ijk}a_{i'j'k'} - a_{i'jk'}a_{ij'k}|^2 = 0;$$

$$M_3 := \sum |a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk'}|^2 = 0.$$

A visualization



Proof

$$(\Rightarrow)$$

$$a_{ijk} = \alpha_i \beta_{jk}$$

$$M_1 = \sum_{\substack{i,j,k \\ i',j',k'}} |a_{ijk}a_{i'j'k'} - a_{ij'k'}a_{i'jk}|^2 =$$

$$= \sum_{\substack{i,j,k \\ i',j',k'}} |\alpha_i \beta_{jk} \alpha_{i'} \beta_{j'k'} - \alpha_i \beta_{j'k'} \alpha_{i'} \beta_{jk}|^2 = 0$$

$$(\Leftarrow)$$

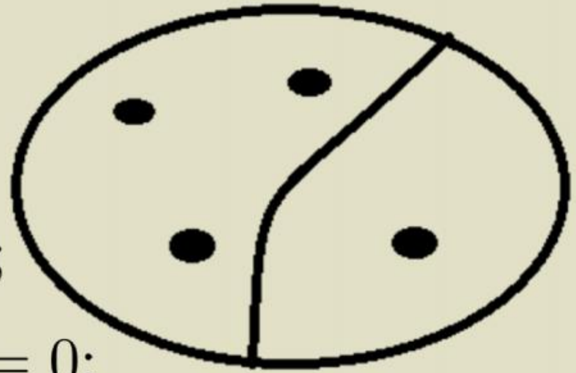
$$|a_{ijk}a_{i'j'k'} - a_{ij'k'}a_{i'jk}|^2 = 0, \quad \forall i, j, k, i', j', k'$$

$$a_{i'j'k'} = \frac{a_{i'jk}}{a_{ijk}} a_{ij'k'} = \alpha_{i'} \beta_{j'k'}$$

$\blacksquare$

Quadripartite systems

$$M_1 := \sum |a_{ijkl}a_{i'j'k'l'} - a_{ij'k'l'}a_{i'jkl}|^2 = 0;$$



$$M_2 := \sum |a_{ijkl}a_{i'j'k'l'} - a_{i'jk'l'}a_{ij'kl}|^2 = 0;$$

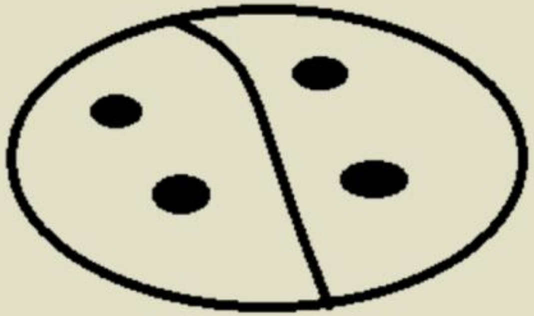
$$M_3 := \sum |a_{ijkl}a_{i'j'k'l'} - a_{i'j'kl'}a_{ijk'l}|^2 = 0;$$

$$M_4 := \sum |a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l}a_{ijkl'}|^2 = 0;$$

$$M_{12} := \sum |a_{ijkl}a_{i'j'k'l'} - a_{ijk'l'}a_{i'j'kl}|^2 = 0;$$

$$M_{13} := \sum |a_{ijkl}a_{i'j'k'l'} - a_{ij'kl'}a_{i'jk'l}|^2 = 0;$$

$$M_{23} := \sum |a_{ijkl}a_{i'j'k'l'} - a_{i'jkl'}a_{ij'k'l}|^2 = 0.$$



Examples

GHZ-state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$M_1 = M_2 = M_3 = 1.$$

Werner-state

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle),$$

$$M_1 = M_2 = M_3 = \frac{8}{9}.$$

Some properties

- Invariant under local unitary transformations;
- Bounded as:

$$0 \leq M_k \leq \frac{2(D-1)}{D}$$

(D dimension of the smaller subsystem)

Maximization

Maximization is reached when the whole-system state can be written as

$$|\psi\rangle = D^{-1/2} \sum_{j=1}^{D} e^{i\chi_j} |j\rangle_k |\Phi_j\rangle_{\bar{k}}$$

where

$$\langle j|\, l\rangle = \langle \Phi_j|\, \Phi_l\rangle = \delta_{jl}$$

Maximization

All pure states of three qubits for which $M_k = 1 \, \forall k$ are equivalent to the GHZ-state, i.e. there exists a local and unitary transformation that maps this state into GHZ-state.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle_k|\phi_0\rangle_{\bar{k}} + \frac{1}{\sqrt{2}}e^{i\chi}|1\rangle_k|\phi_1\rangle_{\bar{k}}$$

Conclusion

- Multipartite entanglement;
  - Multipartite entanglement via simultaneous maximization of the relevant functionals;
    - Similar proof of the GHZ-state for generic multipartite maximally entangled state

Thank you for your attention!