Entropic Measures in Quantum Phase Transitions: Dicke model

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Full Dicke Model Hamiltonian (R.H. Dicke 1954)

N two-level atoms (described by spin-$\frac{1}{2}$ Pauli operators $\sigma^{(i)}_{z,+,-}$) interacting (via dipole $\vec{E}\cdot\vec{D}$) with M radiation bosonic field modes ($a_\alpha, a^\dagger_\alpha$) within an ideal cavity. Atoms are separated, so that dipole-dipole $\vec{D}\cdot\vec{D}$ interactions are negligible.

$$H = \omega_0 \sum_{i=1}^{N} \sigma_z^{(i)} + \sum_{\alpha=1}^{M} \omega_\alpha a^\dagger_\alpha a_\alpha + \sum_{i=1}^{N}\sum_{\alpha=1}^{M} \frac{\lambda_\alpha}{\sqrt{N}} (a^\dagger_\alpha + a_\alpha)(\sigma_+^{(i)} + \sigma_-^{(i)})$$

ω_0: level spliting; ω_α: photon frequencies; λ_α: coupling constants (depend on the intensity of the cavity electric field $\vec{E}$). Basic commutation relations:

$$[\sigma_z^{(i)}, \sigma_\pm^{(j)}] = \pm\delta^{ij}\sigma_\pm^{(i)}, \; [\sigma_+^{(i)}, \sigma_-^{(j)}] = 2\delta^{ij}\sigma_z^{(i)}, \; [a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}$$

Case of a single-mode radiation field $M = 1$

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(J_+ + J_-),$$

Collective atomic operators: $J_k = \sum_{i=1}^N \sigma_k^{(i)},\ k = z, +, -$

Hilbert Space: $\mathrm{Span}\{|n\rangle \otimes |j, m\rangle\}$

$n = 0, \ldots, \infty$: number of photons, $a^\dagger a|n\rangle = n|n\rangle$

$2j = N$: "number of atoms" (e^{iHt} does not mix j sectors)

$m = -j, \ldots, j$: atomic sector $\to$ $(N + 1)$-level system

$m + j$: number of excited atoms, $J_z|j, m\rangle = m|j, m\rangle$

$\hat{\mathcal{N}} = a^\dagger a + J_z + j$: total "excitation number"

Rotating Wave Approximation (valid for small λ)

"Jaynes-Cummings Hamiltonian"

$$H_{RWA} = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger J_- + a J_+),$$

neglects the "counter-rotating" terms $a^\dagger J_+$ and aJ_-. This makes H_{RWA} integrable, simplifying the analysis but removing the posibility of caos. Dicke used this model to illustrate the importance of collective effects in the atom-light interaction, leading to the concept of super-radiance: the atomic ensemble spontaneously emits above $\lambda_c^{RWA} = \sqrt{\omega\omega_0}$ with an intensity $\propto N^2$, rather than $\propto N$ (incoherent radiation).

$\hat{\mathcal{N}} = a^\dagger a + J_z + j$: is conserved in the RWA

Dicke Hamiltonian and parity

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(J_+ + J_-),$$

Hilbert Space: $\text{Span}\{|n\rangle \otimes |j,m\rangle\}$, $n =\in \mathbb{N}$, $m = -j, \ldots, j$

$$J_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle,$$

$$a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, a|n\rangle = \sqrt{n}|n-1\rangle$$

$\hat{\mathcal{N}} = a^\dagger a + J_z + j$: counts the number of excitation quanta

Parity operator $P = e^{i\pi\hat{\mathcal{N}}}$ is conserved $[P, H] = 0$

and both operators can then be jointly diagonalized.

The ground state is even

The matrix elements of the Hamiltonian in this basis are:

$$\begin{aligned}\langle n'; j, m'|H|n; j, m\rangle &= (n\omega + m\omega_0)\delta_{n',n}\delta_{m',m} \\ &+ \frac{\lambda}{\sqrt{2j}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}) \\ &\times(\sqrt{j(j+1) - m(m+1)}\delta_{m',m+1} \\ &+ \sqrt{j(j+1) - m(m-1)}\delta_{m',m-1}).\end{aligned}$$

$|n\rangle \otimes |j, m\rangle$ and $|n'\rangle \otimes |j, m'\rangle$ are connected by temporal evolution iff $n + m + j$ and $n' + m' + j$ have the same parity (even or odd)

$e^{i\pi(n+m+j)}$: good quantum number. The ground state is even

Variational approximation: coherent states

The following variational approximation to the ground state is usually considered:

$$|\alpha, z\rangle = |\alpha\rangle \otimes |z\rangle$$

where

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$
$$|z\rangle = (1+|z|^2)^{-j} e^{zJ_+}|j,-j\rangle = (1+|z|^2)^{-j} \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} z^{j+m}|j,m\rangle,$$

are the ordinary (canonical) and spin-j coherent states for the photon and the atom sectors, respectively.

Ground-State Energy Surface (for $N \to \infty$), Equilibrium Points and critical atom-field coupling strength: $\lambda_c = \frac{1}{2}\sqrt{\omega\omega_0}$

The equilibrium points

$$\alpha = \alpha_0 = \begin{cases} 0, & \text{if } \lambda < \lambda_c \\ -\sqrt{N}\sqrt{\frac{\omega_0}{\omega}}\frac{\lambda}{\lambda_c}\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^{-4}}, & \text{if } \lambda \geq \lambda_c \end{cases}$$

$$z = z_0 = \begin{cases} 0, & \text{if } \lambda < \lambda_c \text{ (normal phase)} \\ \sqrt{\frac{\frac{\lambda}{\lambda_c}-\left(\frac{\lambda}{\lambda_c}\right)^{-1}}{\frac{\lambda}{\lambda_c}+\left(\frac{\lambda}{\lambda_c}\right)^{-1}}} & \text{if } \lambda \geq \lambda_c \text{ (superradiant phase)} \end{cases}$$

minimize the energy surface ($z \equiv e^{-i\phi}\tan(\theta/2)$):

$$\begin{aligned} E(\alpha, z) &= \langle \alpha, z|H|\alpha, z\rangle \\ &= \omega|\alpha|^2 - \frac{1}{2}N\omega_0\cos(\theta) + \sqrt{2N}\lambda\frac{\alpha+\alpha^*}{\sqrt{2}}\sin(\theta)\cos(\phi) \end{aligned}$$

Mean number of photons in the ground state:

$$\langle \alpha_0, z_0 | a^\dagger a | \alpha_0, z_0 \rangle = \begin{cases} 0, & \text{if } \lambda < \lambda_c \text{ (normal phase)} \\ |\alpha_0|^2, & \text{if } \lambda \geq \lambda_c \text{ (superradiant phase)} \end{cases}$$

Mean number of excited atoms in the ground state:

$$\langle \alpha_0, z_0 | J_z + j | \alpha_0, z_0 \rangle = \begin{cases} 0, & \text{if } \lambda < \lambda_c \text{ (normal phase)} \\ N \frac{|z_0|^2}{1+|z_0|^2}, & \text{if } \lambda \geq \lambda_c \text{ (superradiant phase)} \end{cases}$$

Holstein-Primakoff representation

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(J_+ + J_-),$$

Angular momentum operators J in terms of the bosonic operators, $[b, b^\dagger] = 1$ ($N = 2j$):

$$\begin{aligned} J_+ &= b^\dagger \sqrt{N - b^\dagger b} = \sqrt{N}\, b^\dagger + O(1/N), \\ J_- &= \sqrt{N - b^\dagger b}\, b = \sqrt{N}\, b + O(1/N), \\ J_z/N &= (b^\dagger b - N/2)/N = -1/2 + O(1/N). \end{aligned}$$

Linear aproximation (valid in the thermodynamic limit $N \to \infty$)

$$H_\infty = \omega_0 b^\dagger b + \omega a^\dagger a + \lambda(a^\dagger + a)(b^\dagger + b),$$

two coupled oscillators (exactly integrable)

Ground state wave function in position and momentum

Position and momentum operators for the two bosonic modes:

$$X = \frac{1}{\sqrt{2\omega}}(a^\dagger + a), \quad P_X = i\sqrt{\frac{\omega}{2}}(a^\dagger - a),$$
$$Y = \frac{1}{\sqrt{2\omega_0}}(b^\dagger + b), \quad P_Y = i\sqrt{\frac{\omega_0}{2}}(b^\dagger + b),$$

$$|\psi\rangle = \sum_{n=0}^{n_c}\sum_{m=-j}^{j} c_{nm}^{(j)}|n;j,m\rangle, \quad n_c: \text{ cutoff}$$

$$\psi(x,y) = \sqrt{\omega\omega_0}e^{-\frac{1}{2}(\omega x^2+\omega_0 y^2)}\sum_{n=0}^{n_c}\sum_{m=-j}^{j} c_{nm}^{(j)}\frac{H_n(\sqrt{\omega}x)H_{j+m}(\sqrt{\omega_0}y)}{2^{(n+m+j)/2}\sqrt{n!(j+m)!}}$$

$$\tilde{\psi}(p_x,p_y) = \frac{e^{-\frac{1}{2}(\frac{p_x^2}{\omega}+\frac{p_y^2}{\omega_0})}}{\sqrt{\omega\omega_0}}\sum_{n=0}^{n_c}\sum_{m=-j}^{j}\frac{c_{nm}^{(j)}}{(-i)^{-n-m-j}}\frac{H_n(\frac{p_x}{\sqrt{\omega}})H_{j+m}(\frac{p_y}{\sqrt{\omega_0}})}{2^{(n+m+j)/2}\sqrt{n!(j+m)!}}.$$

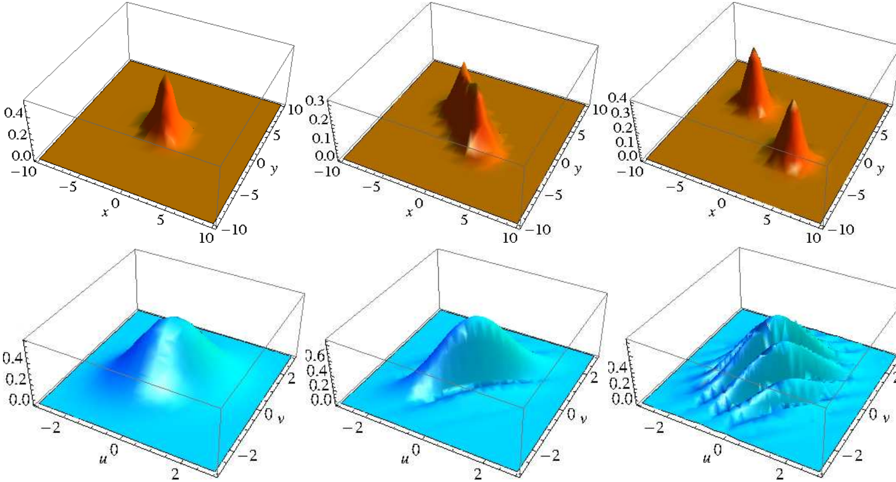
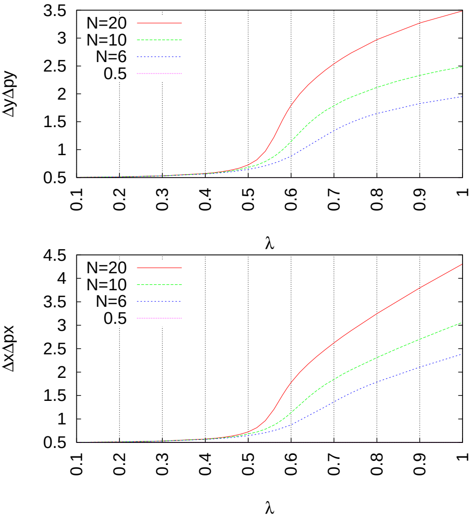


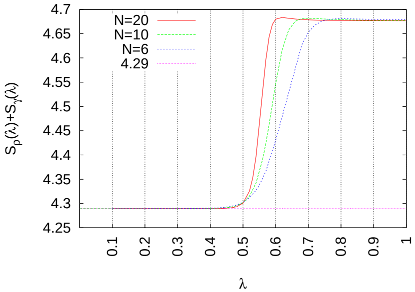
Figure: Ground state density function in position space (top) and momentum space (bottom) for different values of λ (from left to right $\lambda = 0.3$, $\lambda = 0.55$ and $\lambda = 0.7$) for $\omega_0 = \omega = 1$ and $N = 20$.

Uncertainty principle

Variances diverge in the superradiant phase for $j \to \infty$



$$S_f \equiv -\int f(x,y)\ln f(x,y)dxdy, \quad \rho = |\psi|^2, \; \gamma = |\tilde{\psi}|^2$$



Not compatible with $|\alpha_0, z_0\rangle$, which gives constant entropy

$$4.29 \simeq 2\ln(e\pi) \leq S_\rho + S_\gamma,$$

Ground (+) and first-excited (–) states (even and odd parity):

$$|\alpha, z, \pm\rangle = \frac{|\alpha\rangle \otimes |z\rangle \pm |-\alpha\rangle \otimes |-z\rangle}{\mathcal{N}_\pm(\alpha, z)},$$

where $\mathcal{N}_\pm$ is a normalization factor

$$\mathcal{N}_\pm(\alpha, z) = \sqrt{2}\left(1 \pm e^{-2|\alpha|^2}\left(\frac{1-|z|^2}{1+|z|^2}\right)^{2j}\right)^{1/2}$$

We shall use the fact that

$$\mathrm{Min}(\langle \alpha, z, \pm|H|\alpha, z, \pm\rangle) \simeq \langle \alpha_0, z_0, \pm|H|\alpha_0, z_0, \pm\rangle$$

far from λ_c, and the approximation $|z\rangle \simeq |\beta\rangle = e^{-|\beta|^2/2} e^{zb^\dagger}|0\rangle$ with $\beta = \sqrt{2j}\, z$ for $j \gg 1$

Even-Ground State Wave Function in Position and Momentum Representation

Taking into account the position and momentum representation of an ordinary (canonical) CS:

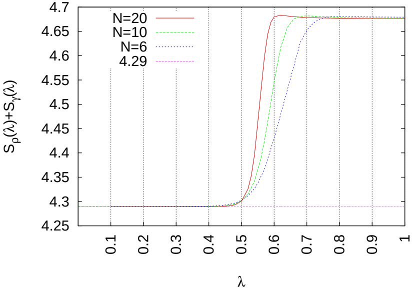
$$\begin{aligned}\langle x|\alpha\rangle &= \left(\frac{\omega^2}{\pi}\right)^{1/4} e^{i\sqrt{2\omega}\,\alpha_2 x} e^{-(\sqrt{\omega}x-\sqrt{2}\,\alpha_1)^2/2},\\ \langle p|\alpha\rangle &= \left(\frac{1}{\pi\omega^2}\right)^{1/4} e^{i\sqrt{\frac{2}{\omega}}\,\alpha_1 p} e^{-(\frac{p}{\sqrt{\omega}}-\sqrt{2}\,\alpha_2)^2/2},\end{aligned}$$

$$\psi(x,y) = \frac{\sqrt{\omega_0\omega/\pi}}{\mathcal{N}_+(\alpha_0,\beta_0)}\left(e^{-\frac{1}{2}(\sqrt{\omega}\,x-\sqrt{2}\,\alpha_0)^2-\frac{1}{2}(\sqrt{\omega_0}\,y-\sqrt{2}\,\beta_0)^2} + e^{-\frac{1}{2}(\sqrt{\omega}\,x+\sqrt{2}\,\alpha_0)^2-\frac{1}{2}(\sqrt{\omega_0}\,y+\sqrt{2}\,\beta_0)^2}\right),$$

$$\tilde{\psi}(p_x,p_y) = \frac{2/\sqrt{\pi\omega_0\omega}}{\mathcal{N}_+(\alpha_0,\beta_0)} e^{-\frac{p_x^2}{2\omega}-\frac{p_y^2}{2\omega_0}} \cos\left(\sqrt{2}\left(\frac{p_x}{\sqrt{\omega}}\alpha_0 + \frac{p_y}{\sqrt{\omega_0}}\beta_0\right)\right),$$

where now $\mathcal{N}_+(\alpha_0,\beta_0) = \left(2(1+e^{-2\alpha_0^2-2\beta_0^2})\right)^{1/2}$.

Step Behavior of Entropy in the Thermodynamic Limit



$$S_\rho + S_\gamma = \begin{cases} S^{\text{normal}} = \ln (e\pi)^2 \simeq 4.29, & \text{if } \lambda < \lambda_c \\ S^{\text{super}} = \ln((2\pi)^2 e) \simeq 4.68, & \text{if } \lambda \geq \lambda_c \end{cases}$$

Rényi entropy

$$R_f^\mu \equiv \frac{1}{1-\mu} \ln \int f^\mu(\mathbf{r}) d\mathbf{r}, \quad \text{for} \quad 0 < \mu < \infty, \quad \mu \neq 1,$$

Hausdorff-Young inequality:

$$R_\rho^\mu + R_\gamma^\nu \geq g(\mu,\nu), \quad \frac{1}{\mu} + \frac{1}{\nu} = 2$$

$$g(\mu,\nu) = \frac{D}{2}\left[\frac{1}{\mu-1}\ln\left(\frac{\mu}{\pi}\right) + \frac{1}{\nu-1}\ln\left(\frac{\nu}{\pi}\right)\right],$$

Rényi entropy in position and momentum representation

$$R_\rho^\mu \simeq \begin{cases} \ln\left(\mu^{\frac{1}{\mu-1}}\pi\right), & \text{if } \lambda < \lambda_c \\ \ln\left(2\mu^{\frac{1}{\mu-1}}\pi\right), & \text{if } \lambda \geq \lambda_c \end{cases}$$

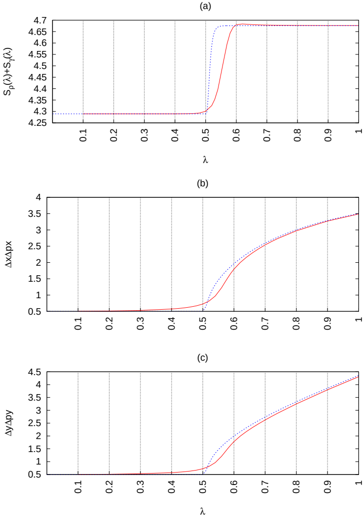
$$R_\gamma^\nu \simeq \begin{cases} \ln\left(\nu^{\frac{1}{\nu-1}}\pi\right), & \text{if } \lambda < \lambda_c \\ \ln\left(\left(\frac{\Gamma(2\nu+1)}{\Gamma^2(\nu+1)2^\nu\nu}\right)^{\frac{1}{1-\nu}}\pi\right), & \text{if } \lambda \geq \lambda_c \end{cases}$$

Cat variances

$$\Delta x = \sqrt{\frac{\frac{1}{2} + \frac{4\pi\alpha_0^2}{\mathcal{N}_+(\alpha_0,\beta_0)^2}}{\omega}}, \quad \frac{\Delta p_x}{\sqrt{\omega}} = \sqrt{2\alpha_0^2\left(\frac{2\pi}{\mathcal{N}_+(\alpha_0,\beta_0)^2} - 1\right) + \frac{1}{2}},$$

$$\Delta y = \sqrt{\frac{\frac{1}{2} + \frac{4\pi\beta_0^2}{\mathcal{N}_+(\alpha_0,\beta_0)^2}}{\omega_0}}, \quad \frac{\Delta p_y}{\sqrt{\omega_0}} = \sqrt{2\beta_0^2\left(\frac{2\pi}{\mathcal{N}_+(\alpha_0,\beta_0)^2} - 1\right) + \frac{1}{2}}.$$

Numerical versus cat approximation, $N = 20$



Entanglement

Ground state wave function (real)

$$\psi(x,y)=\sum_{n=0}^{n_c}\sum_{m=-j}^{j} c_{nm}^{(j)}\phi_{n,m}(x,y), \quad \langle\phi_{n,m}|\phi_{n',m'}\rangle=\delta_{n,n'}\delta_{m,m'}$$

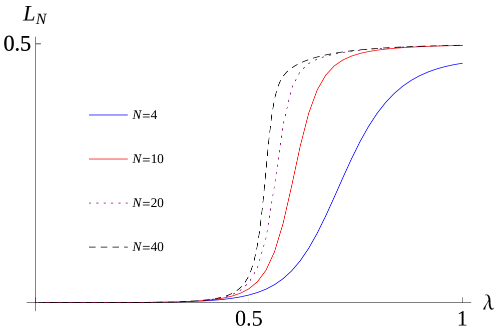
Reduced density matrix:

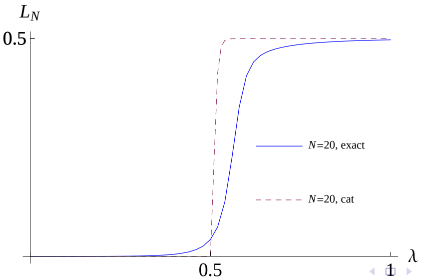
$$\rho_1(x,x')=\int dy\,\psi(x,y)\psi(x',y)$$

Purity:

$$\mathrm{Tr}(\rho_1^2)=\int\int dx dx' \rho_1(x,x')\rho_1(x',x)=\sum_{n,n'=0}^{n_c}\left(\sum_{m=-j}^{j} c_{nm}^{(j)}c_{n'm}^{(j)}\right)^2$$

Entanglement Linear Entropy $L_N(\lambda) = 1 - \mathrm{Tr}(\rho_1^2)$





Husimi distribution

Husimi distribution of the basis (photon times Dicke) states:

$$\varphi^{(j)}_{n,m}(\alpha, z) = \langle n|\alpha\rangle\langle j, m|z\rangle = \frac{e^{-|\alpha|^2/2}\alpha^n}{\sqrt{n!}} \frac{\sqrt{\binom{2j}{j+m}}z^{j+m}}{(1+|z|^2)^j}.$$

Husimi distribution of the ground state

$$\begin{aligned}\Psi(\alpha, z) &= |\langle\alpha, z|\psi\rangle|^2 \\ &= \sum_{n,n'=0}^{n_c} \sum_{m,m'=-j}^{j} c^{(j)}_{nm}\bar{c}^{(j)}_{n'm'}\varphi^{(j)}_{n,m}(\alpha, z)\varphi^{(j)}_{n,m}(\bar{\alpha}, \bar{z})\end{aligned}$$

and normalized according to:

$$\frac{2j+1}{\pi^2}\int_{\mathbb{R}^4} \Psi(\alpha, z)d^2\alpha \frac{d^2z}{(1+|z|^2)^2} = 1.$$

Moments and Rényi-Wehrl entropies

ν-th moments of the Husimi distribution ($|z\rangle \simeq |\beta\rangle, \quad \beta = \sqrt{2j}\, z$):

$$M_{j,\nu}(\lambda) = \int_{\mathbb{R}^4} \frac{d^2\alpha d^2\beta}{\pi^2} (\Phi(\alpha,\beta))^\nu \stackrel{j\to\infty}{\longrightarrow} \begin{cases} \nu^{-2}, & \text{if } \lambda < \lambda_c \\ 2^{1-\nu}\nu^{-2}, & \text{if } \lambda \geq \lambda_c. \end{cases} .$$

Rényi-Wehrl entropy:

$$W_{j,\nu}(\lambda) = \frac{1}{1-\nu} \ln(M_{j,\nu}(\lambda)),$$

Wehrl entropy ($\nu \to 1$)

$$W_j(\lambda) = -\int_{\mathbb{R}^4} \frac{d^2\alpha d^2\beta}{\pi^2}\, \Phi(\alpha,\beta) \ln \Phi(\alpha,\beta) \stackrel{j\to\infty}{\longrightarrow} \begin{cases} 2, & \text{if } \lambda < \lambda_c \\ 2 + \ln(2), & \text{if } \lambda \geq \lambda_c. \end{cases}$$

Growth of zeros of the Husimi distribution and QPT

Husimi density is determined by its zeros through the Weierstrass-Hadamard factorization. The distribution of zeros differs for classically regular or chaotic systems and can be considered as a quantum indicator of classical chaos.

$$\Psi(\alpha, z) = 0 \Rightarrow 2\bar{\alpha}\alpha_e + 2j \ln \frac{1+\bar{z}z_e}{1-\bar{z}z_e} = i\pi(2l+1),\ l \in \mathbb{Z}.$$

Using the approximation $|z\rangle \simeq |\beta\rangle, \quad \beta = \sqrt{2j}\,z$:

$$\Phi(\alpha, \beta) = 0 \Rightarrow 2\bar{\alpha}\alpha_e + 2\bar{\beta}\beta_e = i\pi(2l+1),\ l \in \mathbb{Z},$$

$$\begin{aligned} \alpha_1 &= -\frac{\beta_e}{\alpha_e}\beta_1, \\ \alpha_2 &= -\frac{\beta_e}{\alpha_e}\beta_2 - \frac{\pi}{2\alpha_e}(2l+1). \end{aligned}$$

Growth of zeros of the Husimi distribution and QPT

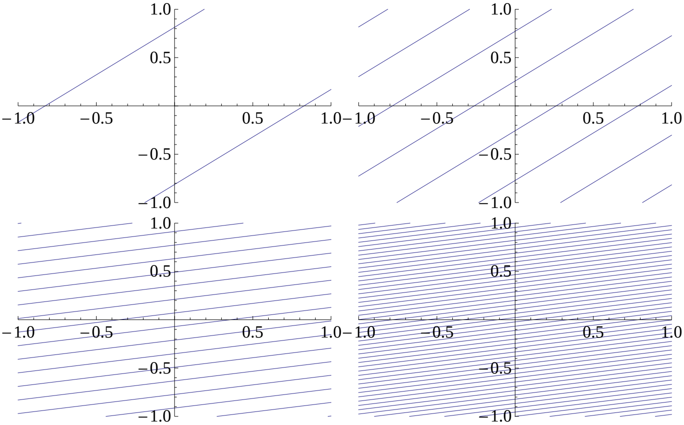


Figure: Zeros of the Husimi distribution $\Phi(\alpha, \beta)$ in the cell $\alpha_2, \beta_2 \in [-1, 1]$ of the momentum plane for $\lambda = 0.6, j = 10$ (top-left), $\lambda = 0.6, j = 100$ (top-right), $\lambda = 10, j = 10$ (bottom-left) and $\lambda = 10, j = 100$ (bottom-right) for $\lambda_c = 0.5$.

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