Dirac δ configurations, boundary conditions, and quantum fluctuations of scalar (Higgs) fields

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Outline

- $\hbar = c = 1$

The scalar Casimir effect

One real field.

- Fluctuations of 1D scalar fields on classical backgrounds

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}U(x)\Phi^2(x,t) \ , \quad \lim_{x\pm\infty} U(x) = 0 \ , \int_{-\infty}^{\infty} dx\, U(x) < +\infty$$

$$\Phi(t,x) = \int_{\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\phi_\omega(x) \ , \quad -\phi''_\omega(x) + U(x)\phi_\omega(x) = \omega^2\phi_\omega(x)$$

$$\left(-\omega^2 - d^2/dx^2 + U(x)\right) G^{(U)}_\omega(x,x') = \delta(x-x')$$

- Fluctuation vacuum energy.

$$E_V = \sum \omega - \sum \omega_0 = \sum_{j=1}^{N} \omega_j + \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} k \left[\frac{d\delta_+}{dk} + \frac{d\delta_-}{dk}\right] \ , \quad \rho_{S_0} = \frac{L}{2\pi}$$

$$\rho_S(k) - \rho_{S_0} = \frac{1}{4\pi}\left[\frac{d\delta_+}{dk} + \frac{d\delta_-}{dk}\right] = \frac{1}{\pi}\int_{-L}^{L} dx\, \mathrm{Im}\left[G^{(U)}_\xi(x,x) - G^{(0)}_\xi(x,x)\right] , \ \xi = i\omega$$

Double-delta/delta$'$ spectral problem

- Two delta/delta$'$ potentials: mimicking two soft plates in the Casimir effect

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left(\mu_1\delta(x+a) + \lambda_1\delta'(x+a) + \mu_2\delta(x-a) + \lambda_2\delta'(x-a)\right)\Phi^2(x,t)$$

$$\left[-\frac{d^2}{dx^2} + \mu_1\delta(x+a) + \lambda_1\delta'(x+a) + \mu_2\delta(x-a) + \lambda_2\delta'(x-a)\right]\phi_\omega(x) = \omega^2\phi_\omega(x)$$

- The spectrum of $\hat{L}$: $\hat{L}\phi_\omega(x) = \omega^2\phi_\omega(x)$

$$\hat{L} = \left[-\frac{d^2}{dx^2} + \mu_1\delta(x+a) + \lambda_1\delta'(x+a) + \mu_2\delta(x-a) + \lambda_2\delta'(x-a)\right]$$

- Delta/delta$'$ matching conditions: finite step discontinuity of ψ and twisted discontinuity of ψ' at $x = \pm a$

$$\begin{pmatrix} \psi(-a_<) \\ \psi'(-a_<) \\ \psi(a_<) \\ \psi'(a_<) \end{pmatrix} = \begin{pmatrix} \frac{1+\lambda_1/2}{1-\lambda_1/2} & 0 & 0 & 0 \\ \frac{\mu_1}{1-\lambda_1^2/4} & \frac{1-\lambda_1/2}{1+\lambda_1/2} & 0 & 0 \\ 0 & 0 & \frac{1+\lambda_2/2}{1-\lambda_2/2} & 0 \\ 0 & 0 & \frac{\mu_2}{1-\lambda_2^2/4} & \frac{1-\lambda_2/2}{1+\lambda_2/2} \end{pmatrix} \begin{pmatrix} \psi(-a_>) \\ \psi'(-a_>) \\ \psi(a_>) \\ \psi'(a_>) \end{pmatrix}$$

2-δ/δ' potential: scattering I.

- The potential

$$U(x) = \mu_1\delta(x+a) + \lambda_1\delta'(x+a) + \mu_2\delta(x-a) + \lambda_2\delta'(x-a)$$

P. Kurasov, J. Math. Anal. Appl. **201** (1996) 297
M. Gadella, J. Negro, L.M. Nieto, Phys. Lett. **A373** (2009) 1310

- Scattering zones: Zone II : $x < -a$, Zone I : $-a < x < a$, Zone III : $x > a$
- Scattering waves (right-going and left-going), $\forall\, k \in \mathbb{R}^+$

$$\psi_k^r(x) = \begin{cases} e^{-ikx}\rho_r + e^{ikx} & , \quad x \in \mathrm{II} \\ A_r e^{ikx} + B_r e^{-ikx} & , \quad x \in \mathrm{I} \\ e^{ikx}\sigma_r & , \quad x \in \mathrm{III} \end{cases} \quad ; \quad \psi_k^l(x) = \begin{cases} e^{-ikx}\sigma_l & , \quad x \in \mathrm{II} \\ A_l e^{ikx} + B_l e^{-ikx} & , \quad x \in \mathrm{I} \\ e^{ikx}\rho_l + e^{-ikx} & , \quad x \in \mathrm{III} \end{cases}$$

- Right-going waves (*diestro*) scattering amplitudes

$$\rho_r = \frac{-\frac{2e^{2iak}\left(k\left(\lambda_1^2+4\right)-2i\mu_1\right)(2k\lambda_2+i\mu_2)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)} - }{-\frac{2e^{-2iak}(2k\lambda_1+i\mu_1)\left(k\left(\lambda_2^2+4\right)+2i\mu_2\right)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)}} \quad , \quad \sigma_r = \frac{(\lambda_1^2-4)(\lambda_2^2-4)k^2}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)}$$

$$A_r = \frac{k(\lambda_1^2-4)(k(\lambda_2^2+4)+2i\mu_2)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)} \quad , \quad B_r = -\frac{2e^{2iak}k(-4+\lambda_1^2)(2k\lambda_2+i\mu_2)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)}$$

2-δ/δ' potential: scattering II.

- Left-going waves (*zurdo*) scattering amplitudes

$$\rho_l \quad = \quad \frac{2e^{-2iak}\left(k\left(\lambda_1^2+4\right)+2i\mu_1\right)(2k\lambda_2-i\mu_2)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)} + \atop + \frac{2e^{2iak}(2k\lambda_1-i\mu_1)\left(k\left(\lambda_2^2+4\right)-2i\mu_2\right)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)} \quad , \quad \sigma_l = \frac{(\lambda_1^2-4)(\lambda_2^2-4)k^2}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)}$$

$$A_l = -\frac{2e^{2iak}k(\lambda_2^2-4)(2k\lambda_1-i\mu_1)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)} \quad , \quad B_l = -\frac{k(\lambda_2^2-4)(k(\lambda_1^2+4)+2i\mu_1)}{\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)}$$

- Denominator of scattering amplitudes:

$$\begin{aligned}\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a) &= 4e^{4iak}(2k\lambda_1-i\mu_1)(2k\lambda_2+i\mu_2)+\\ &+ \left(k\left(\lambda_1^2+4\right)+2i\mu_1\right)\left(k\left(\lambda_2^2+4\right)+2i\mu_2\right)\end{aligned} \tag{1}$$

- Phase shifts and spectral density

$$e^{2i\delta_\pm} = \sigma \pm \sqrt{\rho_l\rho_r} \qquad , \qquad \rho_S(k) = \frac{1}{2\pi}\frac{d(\delta_+ + \delta_-)}{dk} + \rho_{S_0}$$

2-δ/δ' potential: Bound states.

Study the roots of Δ in the positive imaginary axis of the k-complex plane.

$$\Delta(i\kappa;\mu_1,\lambda_1,\mu_2,\lambda_2,a)=0 \quad , \quad \kappa\in\mathbb{R}^+$$

$$4e^{-4a\kappa}(2\kappa\lambda_1-\mu_1)(2\kappa\lambda_2+\mu_2)+\left(\kappa\left(\lambda_1^2+4\right)+2\mu_1\right)\left(\kappa\left(\lambda_2^2+4\right)+2\mu_2\right)=0$$

- Switch-off the δ''s: $\lambda_1=\lambda_2=0$:

$$e^{-4a\kappa}=\frac{4\kappa^2}{\mu_1\mu_2}+2\frac{\mu_1+\mu_2}{\mu_1\mu_2}\kappa+1$$

3 possibilities of intersections on the positive real κ-axis of the two families of curves: 0, no bound states, 1, one bound state or 2, two bound states.

•NO BOUND STATES :

- $\mu_1,\mu_2>0$
- $\mu_1\cdot\mu_2<0$ and $-2a<\frac{\mu_1+\mu_2}{\mu_1\mu_2}<0$

•ONE BOUND STATE:

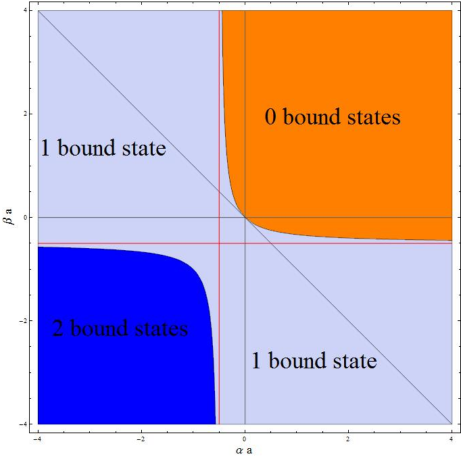
- $\mu_1,\mu_2<0$ and $\frac{\mu_1+\mu_2}{\mu_1\mu_2}<-2a$
- $\mu_1+\mu_2<0$ and $\mu_1\cdot\mu_2<0$
- $\mu_1+\mu_2>0$ and $-2a<\frac{\mu_1+\mu_2}{\mu_1\mu_2}<0$

•TWO BOUND STATES:

- $\mu_1,\mu_2<0$ and $-2a<\frac{\mu_1+\mu_2}{\mu_1\mu_2}<0$

2-δ potential: Bound states II.

- The plane of parameters $(\mu_1 \cdot a, \mu_2 \cdot a)$ is divided into three zones by the hyperbola $\frac{\mu_1+\mu_2}{\mu_1\mu_2} = -2a$:



Probability flux across the δ/δ' plates

- The wave function between plates

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad , \quad \psi'(x) = ik\left(Ae^{ikx} - Be^{-ikx}\right)$$

- Probability flux through plates

$$\psi^*(x)\psi'(x) = ik\left(|A|^2 - |B|^2 + AB^*e^{2ikx} - A^*Be^{-2ikx}\right)$$

$$\Phi_\psi = \psi^*(-a)\psi'(-a) + \psi^*(a)\psi'(a) = 2ik\left[|A|^2 - |B|^2 + (AB^* - A^*B)\left(e^{-2ika} + e^{2ika}\right)\right]$$

- Fluxes of the waves incoming from either the right or the left

$$\Phi^r_\psi[k;\mu_1,\lambda_1,\mu_2,\lambda_2,a] = \frac{2ik}{|\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2 +$$
$$+\,4ik^2(\lambda_1^2-4)\cos(2ka)\left\{k(\lambda_2-2)^2\mu_2\cos(2ak) + 2(k^2\lambda_2(\lambda_2^2+4)+\mu_2^2)\sin(2ak)\right\}\Big]$$

$$\Phi^l_\psi[k;\mu_1,\lambda_1,\mu_2,\lambda_2,a] = \frac{-2ik}{|\Delta(k;\mu_1,\lambda_1,\mu_2,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2 +$$
$$+\,4ik^2(\lambda_2^2-4)\cos(2ka)\left\{-k(\lambda_1-2)^2\mu_1\cos(2ak) + 2(k^2\lambda_2(\lambda_1^2+4)-\mu_1^2)\sin(2ak)\right\}\Big]$$

The ultra-strong limit I:

Impenetrable walls: $\mu_1 = \mu_2 = \mu \to \infty$

●For $\mu_1 = \mu_2 = \mu$:

$$\Delta(k;\lambda_1,\mu,\lambda_2,\mu,a) = 4e^{4ika}\,(2k\lambda_2 - i\mu)\,(2k\lambda_2 - i\mu)\cdot(2k\lambda_1 + i\mu) +$$

$$+ \quad \left(k(\lambda_1^2+4)+2i\mu\right)\cdot\left(k(\lambda_2^2+4)+2i\mu\right) \quad , \quad \Delta_2(k;a) = 4\left(e^{4iak}-1\right)$$

$$\Delta(k;\lambda_1,\mu,\lambda_2,\mu,a) = \Delta_2(k;a)\mu^2 + \Delta_1(k;\lambda_1,\lambda_2,a)\mu + \Delta_0(k;\lambda_1,\lambda_2,a)$$

$$\rho_r = -2\frac{e^{2iak}[k(\lambda_1^2+4)-2i\mu][2k\lambda_2+i\mu] + e^{-2ika}[k(\lambda_2^2+4)+2i\mu][2k\lambda_1+i\mu]}{\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)},$$

$$\sigma_r = \frac{(\lambda_1^2-4)(\lambda_2^2-4)k^2}{\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)} \quad , \quad A_r = \frac{k(\lambda_1^2-4)(k(\lambda_2^2+4)+2i\mu)}{\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)}$$

$$B_r = -2\frac{ke^{2iak}(\lambda_1^2-4)(2k\lambda_2+i\mu)}{\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)}$$

● Opaque limit for arbitrary $k>0$

$$\lim_{\mu\to\infty}\rho_r = -e^{-2iak}; \quad \lim_{\mu\to\infty}\sigma_r = \lim_{\mu\to\infty}A_r = \lim_{\mu\to\infty}B_r = 0$$

●There are no quantum fluctuations between plates in the ultra-strong limit for arbitrary $k>0$. There is total reflection.

The ultra-strong limit II:

Unitary QFT between plates

- Surviving fluctuations between plates in the ultra-strong limit:

$$\Delta_2(k;a) = (e^{2ika}-1)(e^{2ika}+1) = 0 \Rightarrow k_n = \frac{\pi}{2a}n \quad , \quad n \in \mathbb{Z}^+$$

$\Delta_2(k;a)$ is the Dirichlet spectral function obtained by Asorey and Muñoz- Castañeda (JMMC Ph. D. Thesis)

- For $k_n \in \ker(\Delta_2)$ and $\mu = \mu = \infty$:

$$\sigma_r(k_n, \mu=\infty) = 0 \quad , \quad \rho_r(k_n, \mu=\infty) = -(-1)^n$$

$$-(-1)^n A_r(k_n, \mu=\infty) = -(-1)^n \frac{4-\lambda_1^2}{8+4(\lambda_1-\lambda_2)+\lambda_1^2+\lambda_2^2} = B_r(k_n, \mu=\infty)$$

$$\Rightarrow \psi(x; k_n, \mu=\infty) = \frac{4-\lambda_1^2}{8+4(\lambda_1-\lambda_2)+\lambda_1^2+\lambda_2^2}\left(e^{ikx} - (-1)^n e^{-ikx}\right)$$

Dirichlet boundary conditions at both plates are satisfied by $\psi(x,k_n)\mid_{\mu=\infty}$ for all $n \in \mathbb{Z}^+$ in the ultra-strong limit. $\psi_{2p+1}\mid_{\mu=\infty} \sim \cos\left(k_{2p+1}x\right)$, $n = 2p+1$ and $\psi_{2p}\mid_{\mu=\infty} \sim \sin\left(k_{2p}x\right)$, $n = 2p$, $p \in \mathbb{Z}^+$

$$\psi_{2p+1}(-a)\mid_{\mu=\infty} = 0 = \psi_{2p+1}(a)\mid_{\mu=\infty} \quad , \quad \psi_{2p}(-a)\mid_{\mu=\infty} = 0 = \psi_{2p}(a)\mid_{\mu=\infty}$$

The ultra-strong limit III:

Casimir energy

- No probability flux across plates

$$\Phi^r_\psi[k;\mu,\lambda_1,\mu,\lambda_2,a] = \frac{2ik}{|\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2+$$

$$+\,4ik^2(\lambda_1^2-4)\cos(2ka)\left\{k(\lambda_2-2)^2\mu\cos(2ak)+2(k^2\lambda_2(\lambda_2^2+4)+\mu^2)\sin(2ak)\right\}\Big]$$

$$\Phi^l_\psi[k;\mu,\lambda_1,\mu,\lambda_2,a] = \frac{-2ik}{|\Delta(k;\mu,\lambda_1,\mu,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2+$$

$$+\,4ik^2(\lambda_2^2-4)\cos(2ka)\left\{-k(\lambda_1-2)^2\mu\cos(2ak)+2(k^2\lambda_2(\lambda_1^2+4)-\mu^2)\sin(2ak)\right\}\Big]$$

$$\lim_{\mu\to+\infty}\Phi^r_\psi = \lim_{\mu\to+\infty}\Phi^l_\psi = 0$$

- Zeta function prescription for regularized vacuum energy:

$$E_d(s) = \frac{1}{2}\sum_{n=1}^{\infty}\left(n^2\pi^2/(2a)^2\right)^{-s} = \frac{1}{2}\left(\frac{\pi}{2a}\right)^{-2s}\zeta(2s)\,,\quad s\in\mathbb{C}.$$

Physical limit $s=-\frac{1}{2}$: $E_d=\frac{\pi}{4a}\zeta(-1)=-\pi/(48a)$.

Quasi-conformal limit $\mu_1 = \mu_2 = 0$

- Scattering amplitudes

$$\sigma_r = \frac{(-4+\lambda_1^2)(-4+\lambda_2^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)} \quad , \quad \rho_r = -4\frac{e^{4iak}(4+\lambda_1^2)\lambda_2+\lambda_1(4+\lambda_2^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)}$$

$$A_r = -\frac{(-4+\lambda_1^2)(4+\lambda_2^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)} \quad , \quad B_r = \frac{4e^{2iak}(-4+\lambda_1^2)\lambda_2}{\Delta(k;0,\lambda_1,0,\lambda_2,a)}$$

$$\Delta(k;0,\lambda_1,0,\lambda_2,a) = 16e^{4iak}\lambda_1\lambda_2 + (4+\lambda_1^2)(4+\lambda_2^2)$$

$$\sigma_l = \frac{(-4+\lambda_1^2)(-4+\lambda_2^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)} \quad , \quad \rho_l = 4\frac{e^{4iak}(4+\lambda_2^2)\lambda_1+\lambda_2(4+\lambda_1^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)}$$

$$A_l = -\frac{4e^{2iak}(-4+\lambda_2^2)\lambda_1}{\Delta(k;0,\lambda_1,0,\lambda_2,a)} \quad , \quad B_l = -\frac{(-4+\lambda_2^2)(4+\lambda_1^2)}{\Delta(k;0,\lambda_1,0,\lambda_2,a)}$$

- Fluxes of the waves incoming from either the right or the left

$$\Phi^r_\psi[k;0,\lambda_1,0,\lambda_2,a] = \frac{2ik}{|\Delta(k;0,\lambda_1,0,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2+$$
$$+\,4ik^2(\lambda_1^2-4)\cos(2ka)\left\{2k^2\lambda_2(\lambda_2^2+4)\sin(2ak)\right\}\Big]$$

$$\Phi^l_\psi[k;0,\lambda_1,0,\lambda_2,a] = \frac{-2ik}{|\Delta(k;0,\lambda_1,0,\lambda_2,a)|^2}\cdot\Big[4k^2(\lambda_1^2-4)^2(\lambda_2-4)^2+$$
$$+\,4ik^2(\lambda_2^2-4)\cos(2ka)\left\{2k^2\lambda_2(\lambda_1^2+4)\sin(2ak)\right\}\Big]$$

Magic points: $\lambda_1 = \pm 2, \lambda_2 = \pm 2$

$\mu_1 = \mu_2 = 0$

- More opaque couplings

$$\lambda_1 = \pm 2, \lambda_2 \neq 2 : \ \Phi^r_\psi = \sigma_r = A_r = B_r = \sigma_l = 0 \ , \ x = -a \text{ plate} , \quad \text{opaque}$$

$$\lambda_1 \neq 2, \lambda_2 = \pm 2 : \ \Phi^l_\psi = \sigma_l = A_l = B_l = \sigma_r = 0 \ , \ x = a \text{ plate} , \quad \text{opaque}$$

- The wave function between plates

$$\psi^r_k(x; 0, \lambda_1, 0, 2, a) = \frac{\lambda_1^2 - 4}{4 + 4e^{4ika}\lambda_1 + \lambda_1^2}\left(e^{-ik(x-2a)} - e^{ikx}\right)$$

$$(\psi')^r_k(x; 0, \lambda_1, 0, 2, a) = \frac{-ik(\lambda_1^2 - 4)}{4 + 4e^{4ika}\lambda_1 + \lambda_1^2}\left(e^{-ik(x-2a)} + e^{ikx}\right)$$

- Dirichlet ($e^{4ika} = 1$) or mixed Dirichlet-Neumann ($e^{4ika} = -1$) boundary conditions

$$\psi^r_k(a; 0, \lambda_1, 0, 2, a) = 0 \quad , \quad (\psi')^r_k(a; 0, \lambda_1, 0, 2, a) = -2ik\frac{e^{ika}(\lambda_1^2 - 4)}{4 + 4e^{4ika}\lambda_1 + \lambda_1^2}$$

$$\psi^r_k(-a; 0, \lambda_1, 0, 2, a) = \frac{(\lambda_1^2 - 4)e^{-ika}}{4 + 4e^{4ika}\lambda_1 + \lambda_1^2}\left(e^{4ika} - 1\right)$$

$$(\psi')^r_k(-a; 0, \lambda_1, 0, 2, a) = -ik\frac{e^{-ika}(\lambda_1^2 - 4)}{4 + 4e^{4ika}\lambda_1 + \lambda_1^2}\left(e^{4ika} + 1\right)$$

New spectral function: $h^{(2a)}_{dn}(k) \propto 2\cos(2ak) \equiv k_n = \frac{\pi}{4a}n, n \in \mathbb{Z}^+ : \qquad E_{dn} = -\frac{\pi}{96a}.$

Robin boundary conditions I

The spectral function

- More magics at the $\lambda_2 = -2$ opaque point. The amplitudes between plates are:

$$A_r(k;\mu_1,\lambda_1,\mu_2,-2,a) = \frac{2ik(\lambda_1^2-4)\mu_2}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)}$$

$$B_r(k;\mu_1,\lambda_1,\mu_2,-2,a) = -\frac{2ike^{2ika}(\lambda_1^2-4)(i\mu_2-4k)}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)}$$

$$\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a) = 4e^{4ika}(2k\lambda_1-i\mu_1)(i\mu_2-4k)+\left(k(\lambda_1^2+4)+2i\mu_1\right)\cdot(8k+2i\mu_2)$$

- No probability flux across the plates. Robin BC spectral function

$$\Phi_\psi^r(k;\mu_1,\lambda_1,\mu_2,-2,a) = 64ik^2(\lambda_1^2-4)\cos(2ka)\left\{k\mu_2\cos(2ka)-2(k^2-\frac{\mu_2^2}{16})\sin(2ka)\right\}$$

$$k_n\mu_2\cos(2k_na)-2(k_n^2-\frac{\mu_2^2}{16})\sin(2k_na)=0 \;\Rightarrow\; \tan\alpha=\frac{\mu_2}{4}$$

- The AIM unitary matrix and the AMC spectral condition

$$\mathbb{U}(\alpha)=e^{2i\alpha}\mathbb{I}\;,\quad h_R(k;\alpha)\propto\left\{\begin{array}{c}-2k\sin(2\alpha)\cos(2ka)+\\+\left(k^2(1+\cos(2\alpha))-(1-\cos(2\alpha))\right)\sin(2ka)\end{array}\right.$$

Robin boundary conditions II

The wave functions

- The right movers wave function between plates and its derivative

$$\psi_k(x) = \frac{-2k(\lambda_1^2 - 4\left[e^{-ik(x-2a)}(i\mu_2 - 4k) - e^{ikx}(i\mu_2 + 4k)\right]}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)}$$

$$\psi_k'(x) = \frac{2ik^2(\lambda_1^2 - 4\left[e^{-ik(x-2a)}(i\mu_2 - 4k) + e^{ikx}(i\mu_2 + 4k)\right]}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)}$$

- Values at the boundary points

$$\psi_k(a) = 2\frac{k^2 e^{ika}(\lambda_1^2 - 4)}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)} \ , \ \psi_k'(a) = -4\frac{k^2 e^{ika}(\lambda_1^2 - 4)\mu_2}{\Delta(k;\mu_1,\lambda_1,\mu_2,-2,a)}$$

$$\psi_k(-a) = \frac{4k\cos(2ak) + \mu_2\sin(2ak)}{4k}\psi_k(a) \ , \ \psi_k'(-a) = \frac{\mu_2\cos(2ak) - 4k\sin(2ak)}{\mu_2}\psi_k'(a)$$

- Robin boundary conditions

$$\mathrm{R1}: \psi_k(a) + \frac{4}{\mu_2}\psi_k'(a) = 0 \quad , \quad \frac{4k_n\cos(2k_n a) + \mu_2\sin(2k_n a)}{k_n(\mu_2\cos(2k_n a) - 4k_n\sin(2k_n a))} = -\frac{4}{\mu_2}$$

$$\mathrm{R2}: \psi_k(-a) + \frac{4k\cos(2ka) + \mu_2\sin(2ka)}{k(\mu_2\cos(2ka) - 4k\sin(2ka))}\psi_k'(-a) = 0$$

TGTG formula for vacuum energy

- *T*-matrix remainder

$$\begin{aligned} T &= U(1-G_o U)^{-1} = (1-G_o U)^{-1} U = U + UG_0U + UG_0UG_0U + \cdots \\ T(x,y) &= U(x)\delta(x-y) + U(x)G_0(x-y)U(y) + \\ &+ \int dz\, U(x)G_0(x-z)U(z)G_0(z-y)U(y) + \cdots \\ G_\omega^{(U)}(x,y) &= G_\omega^{(0)}(x,y) - \int dz_1 dz_2 G_\omega^{(0)}(x,z_1)T_\omega^{(U)}(z_1,z_2)G_\omega^{(0)}(z_2,y) \end{aligned}$$

- Compact objects in one dimension.

$$U(x) = U_1(x) + U_2(x)$$

$U_i(x)$ smooth functions with disjoint compact supports on the real line.

- Vacuum interaction energy *TGTG* formula

$$\begin{aligned} E_0^{\text{int}} &= -\frac{i}{2}\int \frac{d\omega}{2\pi} \mathrm{Tr}_{L^2} \ln(\mathbf{1} - M_\omega) \\ M_\omega &= G_\omega^{(0)} T_\omega^{(1)} G_\omega^{(0)} T_\omega^{(2)} \\ M_\omega(x,y) &= \int dz_1 dz_2 dz_3 G_\omega^{(0)}(x,z_1)T_\omega^{(1)}(z_1,z_2)G_\omega^{(0)}(z_2,z_3)T_\omega^{(2)}(z_3,y) \end{aligned}$$

Kenneth and Klich, Phys. Rev. B 78, 014103 (2008). Bordag *et al*, "Advances in the Casimir effect", Oxford, UK: Oxford Univ. Pr. (2009).

Single δ/δ' pair T matrix I

Scattering data

- Scattering waves and matching conditions

$$\begin{aligned} \psi_k^r(x) &= \left(e^{ikx} + \rho_r e^{-ikx}\right) H(-x) + \sigma_r e^{ikx} H(x) \\ \psi_k^l(x) &= \sigma_l e^{-ikx} H(-x) + \left(e^{ikx} + \rho_l e^{ikx}\right) H(x) \\ \psi_k(0_>) &= \frac{1+\lambda/2}{1-\lambda/2}\psi_k(0_<)\ ,\ \psi_k'(0_>) = \frac{1-\lambda/2}{1+\lambda/2}\psi_k'(0_<) + \frac{\mu}{1-\lambda^2/4}\psi_k(0_<) \end{aligned}$$

- Scattering amplitudes the S-matrix

$$S = \begin{pmatrix} \sigma_r & \rho_r \\ \rho_l & \sigma_l \end{pmatrix} = \frac{1}{2ik(1+\frac{\lambda^2}{4}) - \mu} \begin{pmatrix} 2ik(1-\frac{\lambda^2}{4}) & -2ik\lambda + \mu \\ 2ik\lambda + \mu & 2ik(1-\frac{\lambda^2}{4}) \end{pmatrix}$$

- The bound/anti-bound state: $\kappa_b = -\frac{\mu}{2(1+\lambda^2/4)}$, $E_b = -\frac{\mu^2}{4(1+\lambda^2/4)^2}$

$$\psi_b(x) = \sqrt{-\frac{\mu}{2}}\frac{1}{1+\lambda^2/4}\left[(1+\lambda/2)e^{-\frac{\mu}{2(1+\lambda^2/4)}x} H(-x) + (1-\lambda/2)e^{\frac{\mu}{2(1+\lambda^2/4)}x} H(x)\right]$$

-

$$G^{(\mu,\lambda)}(x,y) = \frac{1}{W[\psi_k^r,\psi_k^l]}\left(\psi_k^r(x)\psi_k^l(y)H(x-y) + \psi_k^r(y)\psi_k^l(x)H(y-x)\right)$$

Single δ/δ' pair T matrix II.

- The δ/δ' Green's function :

$$G_\omega^{(\mu,\lambda)}(x,y) = G_\omega^{(0,0)}(x-y) - \begin{cases} \begin{cases} \frac{\rho_r}{2ik} e^{-ik(x+y)} \ , \ x,y<0 \\ \frac{\rho_l}{2ik} e^{ik(x+y)} \ , \ x,y>0 \end{cases} \\ \frac{\sigma-1}{2ik} e^{ik|x-y|} \ , \ \mathrm{sgn}(x) \neq \mathrm{sgn}(y) \end{cases}$$

$$G_\omega^{(0,0)}(x-y) = -\frac{1}{2ik} e^{ik|x-y|} \ , \ \omega=+k$$

- The δ/δ' T-matrix:

$$T_\omega^{(\mu,\lambda)}(x,y) = -\left(-\frac{d^2}{dx^2}-\omega^2\right)\left(-\frac{d^2}{dy^2}-\omega^2\right)\left[G_\omega^{(\mu,\lambda)}(x,y) - G_\omega^{(0,0)}(x-y)\right]$$
$$= 2ik(\sigma-1)\delta(x)\delta(y) = 2ik\frac{\mu-ik\lambda^2}{2ik(1+\lambda^2/4)-\mu}\delta(x)\delta(y)$$

- Euclidean rotation: $k=i\kappa, \omega=i\xi$

$$G^{(0,0)}(x,y) = \frac{1}{2\kappa}e^{-\kappa|x-y|} \ , \ T_\xi^{(\mu,\lambda)}(x,y) = 2\kappa\frac{\mu+\kappa\lambda^2}{2\kappa(1+\lambda^2/4)+\mu}\delta(x)\delta(y)$$

The *TGTG* Casimir energy I

T-matrices for two pairs of δ/δ' configurations

$$
\begin{aligned}
T_\xi^{(\mu_1,\lambda_1)}(x,y) &= \delta(x+a)\delta(y+a)\cdot\frac{2\kappa(\mu_1+\kappa\lambda_1^2)}{2\kappa(1+\lambda_1^2/4)+\mu_1}\\
T_\xi^{(\mu_2,\lambda_2)}(x,y) &= \delta(x-a)\delta(y-a)\cdot\frac{2\kappa(\mu_2+\kappa\lambda_2^2)}{2\kappa(1+\lambda_2^2/4)+\mu_2}
\end{aligned}
$$

The M-matrix

$$
\begin{aligned}
&M_\xi^{(\mu_1,\lambda_1,\mu_2,\lambda_2)}(x,y)=\\
&=\int dz_1\int dz_2\int dz_3\, G_\xi^{(0,0)}(x-z_1)T_\xi^{(\mu_1,\lambda_1)}(z_1,z_2)G_\xi^{(0,0)}(z_2-z_3)T_\xi^{(\mu_1,\lambda_1)}(z_3,y)\\
&=\frac{(\mu_1+\kappa\lambda_1^2)(\mu_2+\kappa\lambda_2^2)}{[2\kappa(1+\lambda_1^2/4)+\mu_1][2\kappa(1+\lambda_2^2/4)+\mu_2]}\cdot e^{-2\kappa|x+a|}\cdot\delta(y-a)
\end{aligned}
$$

- The trace of the M-matrix

$$
\begin{aligned}
&\mathrm{Tr}_{L^2}\, M_\xi^{(\mu_1,\lambda_1,\mu_2,\lambda_2)}(x,y)=\int_{-\infty}^{\infty}dx\, M_\xi^{(\mu_1,\lambda_1,\mu_2,\lambda_2)}(x,x)=\\
&=\frac{(\mu_1+\kappa\lambda_1^2)(\mu_1+\kappa\lambda_1^2)e^{-4\kappa a}}{[2\kappa(1+\lambda_1^2/4)+\mu_1][2\kappa(1+\lambda_2^2/4)+\mu_2]}
\end{aligned}
$$

The *TGTG* Casimir energy II.

- The *TGTG* integrand

$$\mathrm{Tr}_{L^2}\ln\left(1-M_\xi^{(\mu_1,\lambda_1,\mu_2,\lambda_2)}(x,y)\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\mathrm{Tr}_{L^2}M_\xi^{(\mu_1,\lambda_1,\mu_2,\lambda_2)}(x,y)\right]^n$$

$$=\ln\left(1-\frac{(\mu_1+\kappa\lambda_1^2)(\mu_2+\kappa\lambda_2^2)e^{-4\kappa a}}{[2\kappa(1+\lambda_1^2/4)+\mu_1][2\kappa(1+\lambda_2^2/4)+\mu_2]}\right)$$

- The Casimir energy and force

$$E_{\mathrm{int}}(\mu_1,\lambda_1,\mu_2,\lambda_2;a)=$$

$$=\frac{1}{2\pi}\int_0^\infty d\kappa\ln\left(1-\frac{(\mu_1+\kappa\lambda_1^2)(\mu_2+\kappa\lambda_2^2)e^{-4\kappa a}}{[2\kappa(1+\lambda_1^2/4)+\mu_1][2\kappa(1+\lambda_2^2/4)+\mu_2]}\right)$$

$$F_{\mathrm{int}}(\mu_1,\lambda_1,\mu_2,\lambda_2;a)=-\frac{1}{2}\frac{dE_{\mathrm{int}}(a)}{da}=$$

$$=-\int_0^\infty\frac{d\kappa}{4\pi}\frac{4\kappa(\mu_1+\kappa\lambda_1^2)(\mu_2+\kappa\lambda_2^2)}{e^{4a\kappa}[2\kappa(1+\lambda_1^2/4)+\mu_1][2\kappa(1+\lambda_2^2/4)+\mu_2]-(\mu_1+\kappa\lambda_1^2)(\mu_2+\kappa\lambda_2^2)}$$

- Ultra-strong limit $\mu_1=\mu_2=\infty$:

$$E_{\mathrm{int}}^{\infty}(a)=\frac{1}{2\pi}\int_0^\infty d\kappa\ln\left(1-e^{-4\kappa a}\right)=-\frac{\pi}{24}\frac{1}{2a}$$

$$F_{\mathrm{int}}^{\infty}(a)=-\frac{1}{4\pi}\int_0^\infty d\kappa\frac{4\kappa}{e^{4a\kappa}-1}=\frac{\pi}{24}\frac{1}{4a^2}$$