official title:

Rotating Casimir systems:
extracting rotation from vacuum fluctuations?

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Based on:

M.Ch., arXiv:1203.6588,
arXiv:1207.???? (to appear this month)

unofficial title:

Perpetuum mobile of the fourth kind due to zero-point fluctuations

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Based on:

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Question: **How does the Casimir energy fall?**

Answer: "… Casimir energy gravitates just as required by the equivalence principle, and therefore the inertial and gravitational masses of a system possessing Casimir energy E_C are both E_C/c^2."

S. A. Fulling, K. A. Milton, P. Parashar, A. Romeo, K. V. Shajesh, J. Wagner, Phys. Rev. D 76, 025004 (2007).

Question: **How does the Casimir energy rotate?**

Answer: It likes to rotate! ("The rotational vacuum effect")

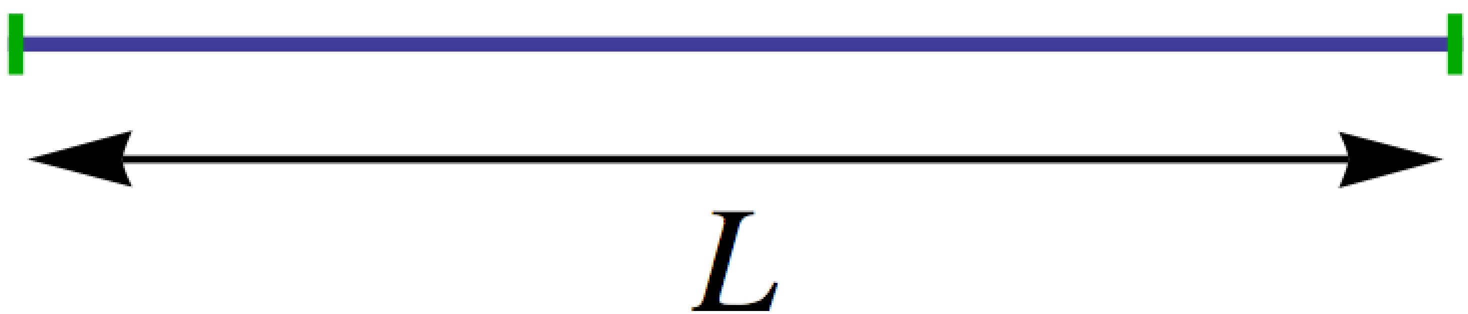
Naïve argument: Negative Casimir energy corresponds to a negative mass of zero-point fluctuations which – if they are rotating – should have a negative moment of inertia implying decrease of the zero-point energy as the angular frequency is increasing.

The Casimir effect

H. B. G. Casimir, Proc. Ron. Ned. Akad. Wetensch., **51**, 793 (1948)

Simplest version of the Casimir effect:

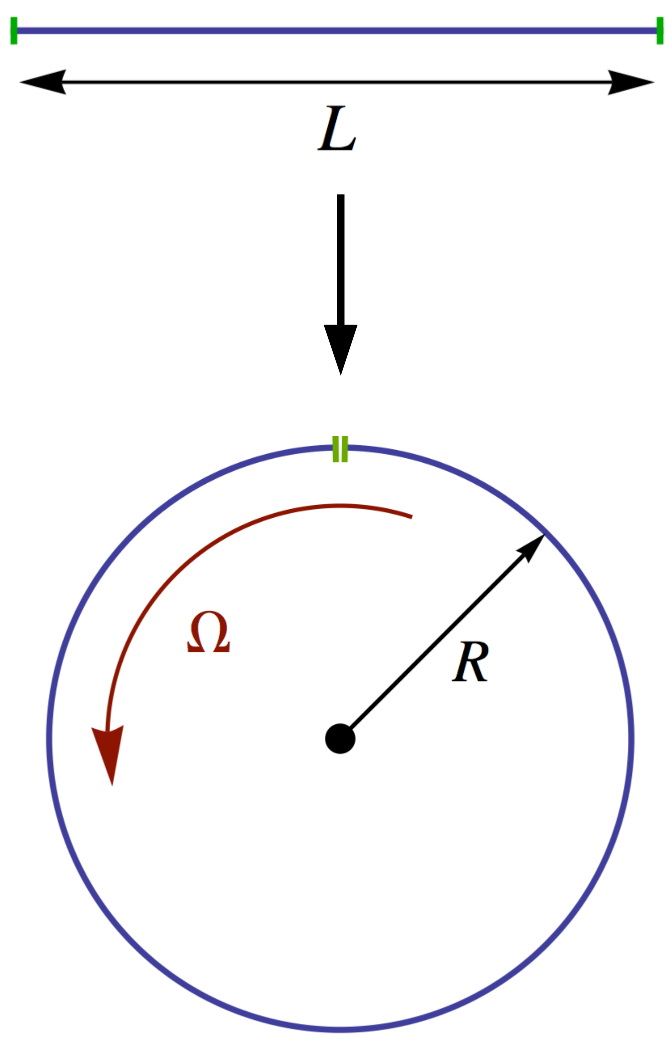
1) massless one-component scalar field
2) in one spatial dimension
3) on an interval with Dirichlet boundary conditions



The Casimir energy is negative: $E_{\text{interval}} = -\frac{\pi}{24L}\hbar c$

How to find the rotational energy of zero-point fluctuations?

Let us make a circle out of the interval:



$$E_{\text{interval}} = -\frac{\pi}{24L}\hbar c$$

Circle with the Dirichlet cut.

$$\phi(t,\varphi)\Big|_{\varphi=0} \equiv \phi(t,\varphi)\Big|_{\varphi=2\pi} = 0$$

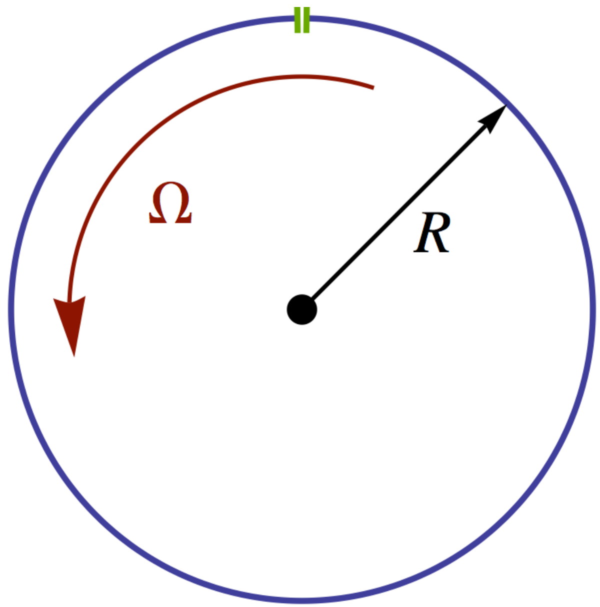
The zero-point energy in static case:

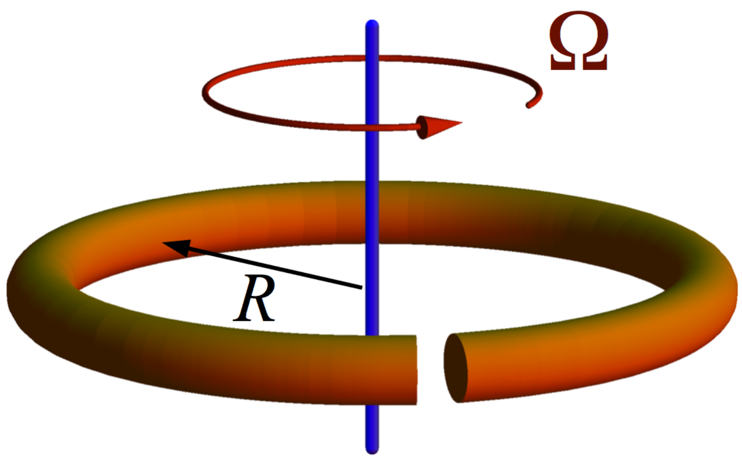
$$E_{\text{circle}} = -\frac{\pi}{24L}\hbar c \equiv -\frac{\hbar c}{48R}$$

$$L = 2\pi R$$

Rotational zero-point energy

Simplest case: a circle with the Dirichlet cut, massless scalar field





The energy density $\mathcal{E}(x) = \langle T^{00}(x) \rangle$ can be computed via the Green's function $G(x, x') = i \langle \mathrm{T}\phi(x)\phi(x') \rangle$ as follows:

$$\langle T^{\mu\nu}(x) \rangle = \left(\partial^\mu \partial'^\nu - \frac{1}{2} g^{\mu\nu} \partial^\lambda \partial'_\lambda \right) \frac{1}{i} G(x, x') \Big|_{x \to x'}$$

Energy density of zero-point fluctuations

Energy density:

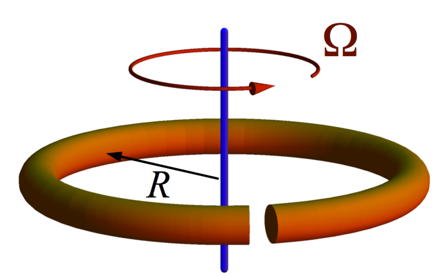
$$\langle T^{00}(t,\varphi)\rangle = \left(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + \frac{1}{R^2}\frac{\partial}{\partial \varphi}\frac{\partial}{\partial \varphi'}\right)\frac{1}{2i}G(t,t';\varphi,\varphi')\Big|_{\substack{t'\to t\\ \varphi'\to\varphi}}$$

The Green's function satisfies the following equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2}\right)G(t,\varphi;t',\varphi') = \frac{1}{R}\delta(\varphi-\varphi')\delta(t-t')$$

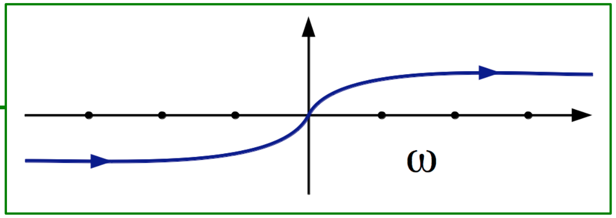
with the boundary condition corresponding to rotation:

$$\phi(t,\varphi)\Big|_{\varphi=[\Omega t]_{2\pi}} = 0$$



The Green's function

$$G(t,t';\varphi,\varphi') = \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \frac{\phi_{\omega,m}(t,\varphi)\phi^{\dagger}_{\omega,m}(t',\varphi')}{\lambda_{\omega,m} - i\epsilon}$$



can be expressed via eigenfunctions

$$\phi_{m,\omega}(t,\varphi) = \sqrt{\frac{1}{\pi R}} \sin\left[\frac{m}{2}[\varphi - t\Omega]_{2\pi}\right] \exp\left\{-i\omega\left(t - \frac{\Omega R^2\,[\varphi - t\Omega]_{2\pi}}{1-\Omega^2 R^2}\right)\right\}$$

(orthonormal and complete system of eigenfunctions)

and eigenvalues

$$\lambda_{\omega,m} = \frac{1-\Omega^2 R^2}{4R^2} m^2 - \frac{\omega^2}{1-\Omega^2 R^2}$$

of the corresponding equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2}\right)\phi_{\omega,m}(t,\varphi) = \lambda_{\omega,m}\,\phi_{\omega,m}(t,\varphi)$$

Green's function, explicitly

$$G_\Omega(t,t';\varphi,\varphi') = \frac{i}{\pi}\mathcal{G}\left(\frac{[\varphi-\Omega t]_{2\pi}}{2}, \frac{[\varphi'-\Omega t']_{2\pi}}{2}, \frac{\left|(1-\Omega^2R^2)(t-t') - \Omega R^2\left([\varphi-\Omega t]_{2\pi} - [\varphi'-\Omega t']_{2\pi}\right)\right|}{2R}\right)$$

where

$$\mathcal{G}(x,y,z) = \sum_{m=1}^{\infty} \frac{\sin(mx)\sin(my)}{m} e^{-imz} = \frac{1}{4}\ln\frac{\left[1-e^{i(x+y-z)}\right]\left[1-e^{-i(x+y+z)}\right]}{\left[1-e^{i(x-y-z)}\right]\left[1-e^{i(-x+y-z)}\right]}$$

$$\equiv \frac{1}{4}\ln\left|\frac{\cos(x+y)-\cos z}{\cos(x-y)-\cos z}\right| - \frac{i}{8}\left([z-x-y]_{2\pi} + [z+x+y]_{2\pi} - [z-x+y]_{2\pi} - [z+x-y]_{2\pi}\right)$$

since

$$-\sum_{m=1}^{\infty}\frac{e^{imx}}{m} = \ln\left(1-e^{ix}\right) \equiv \ln\left(2\left|\sin\frac{x}{2}\right|\right) + \frac{i}{2}\left([x]_{2\pi}-\pi\right)$$

Time-splitting regularization:

Energy density:

$$\langle T^{00}(t,\varphi)\rangle = \frac{1}{2i}\lim_{t'\to t}\lim_{\varphi'\to\varphi}\left(\partial_t\partial'_t + \frac{1}{R^2}\partial_\varphi\partial'_\varphi\right)G(t,t';\varphi,\varphi')$$

$$= -\frac{1}{2\pi}\lim_{t'\to t}\underbrace{\frac{1}{(t'-t)^2}}_{\text{divergent}} - \underbrace{\frac{1+\Omega^2R^2}{96\pi R^2}}_{\text{finite}}$$

Physical energy density:

$$\mathcal{E}^{\mathrm{ZP}}_\Omega(t,\varphi) \equiv \langle T^{00}(t,\varphi)\rangle^{\mathrm{phys}} = -\frac{1+\Omega^2R^2}{96\pi R^2}$$

Physical energy:

$$E^{\mathrm{ZP}}_\Omega \equiv R\int_0^{2\pi} d\varphi\, \mathcal{E}^{\mathrm{ZP}}_\Omega(t,\varphi) = -\frac{1+R^2\Omega^2}{48R}$$

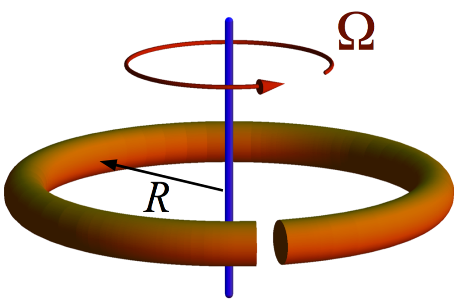
No rotation, Ω = 0

$$E_{\mathrm{circle}} = -\frac{\pi}{24L}\hbar c \equiv -\frac{\hbar c}{48R}$$

General features of rotational energy (I)

The energy gets smaller with increase of angular frequency!

$$E_{\Omega}^{\mathrm{ZP}} = -\frac{1 + R^2\Omega^2}{48R}$$



Note: this is a relativistic expression.

Energy of a classical rotating body: $E_{\mathrm{cl}}(\Omega) = \dfrac{I_{\mathrm{cl}}\Omega^2}{2}$ positive

Classical moment of inertia: $I_{\mathrm{cl}} \equiv \dfrac{\partial^2 E_{\mathrm{cl}}}{\partial \Omega^2} = mR^2$

Moment of inertia of zero-poinf fluctuations

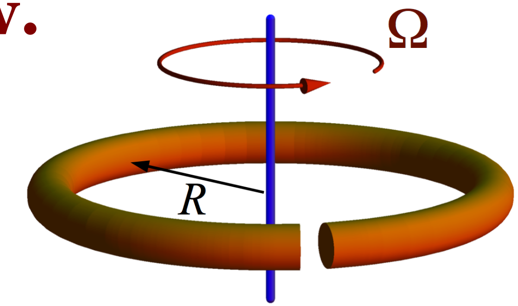
$$I^{\mathrm{ZP}} \equiv \frac{\partial^2}{\partial \Omega^2} E_{\Omega}^{\mathrm{ZP}} = -\frac{\hbar R}{24c}$$

negative

General features of rotational energy (II)

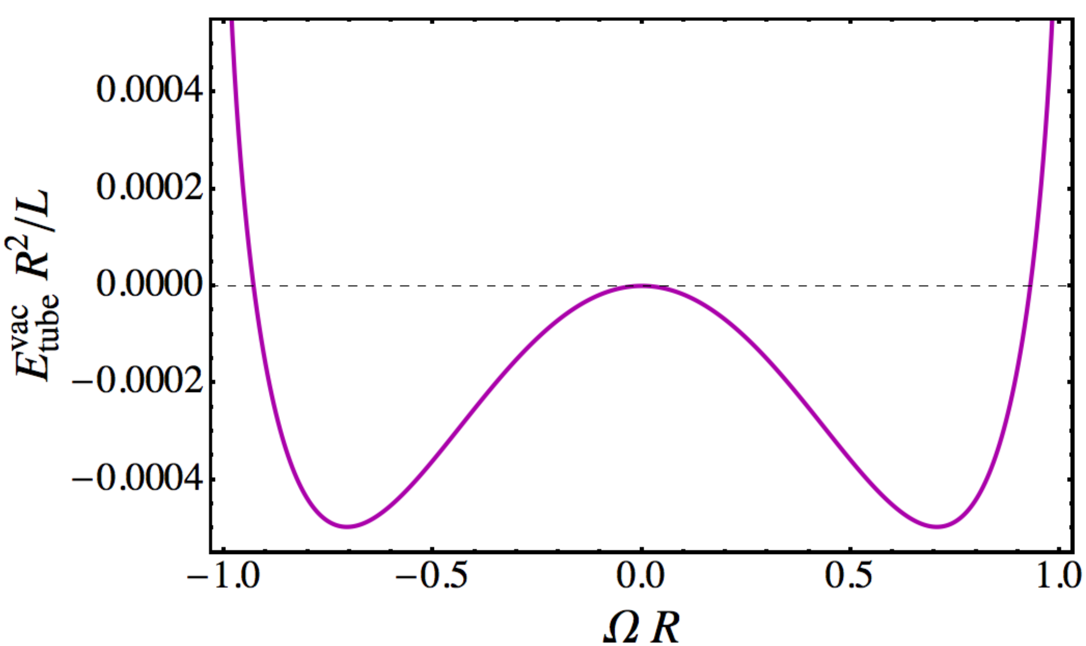
The energy is unbounded from below.

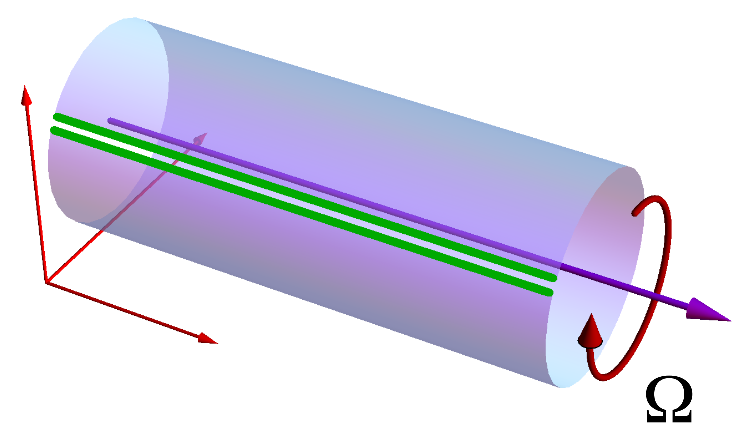
$$E_\Omega^{\mathrm{ZP}} = -\frac{1+R^2\Omega^2}{48R}$$



Reason: mathematically (infinitely) thin circle.
In a real, spatially extended device the energy is bounded.

Example: rotating long cylinder of radius *R*





General features of rotational energy (III)

The rotation of an isolated device is not self-accelerating!

$$E_\Omega^{\mathrm{ZP}} = -\frac{1 + R^2\Omega^2}{48R}$$

Physical reason: conservation of the angular momentum.

Check [following K. A. Milton, J. Phys. A **37**, R209 (2004)]:

The force acting on the Dirichlet cut:

$$\mathcal{F} = \frac{1}{R^2}\left(\langle T^{\varphi\varphi}(t,\varphi)\rangle \Big|_{\varphi=[\Omega t - 0]_{2\pi}} - \langle T^{\varphi\varphi}(t,\varphi)\rangle \Big|_{\varphi=[\Omega t + 0]_{2\pi}} \right)$$

does not depend on φ

$$\mathcal{F} = 0$$

No force!

General features of rotational energy (IV)

It is very small!

An analog of the moment of inertia for zero-point fluctuations:

$$I^{\mathrm{ZP}} \equiv \frac{\partial^2}{\partial \Omega^2} E_{\Omega}^{\mathrm{ZP}} = -\frac{\hbar R}{24 c}$$

In physical units:

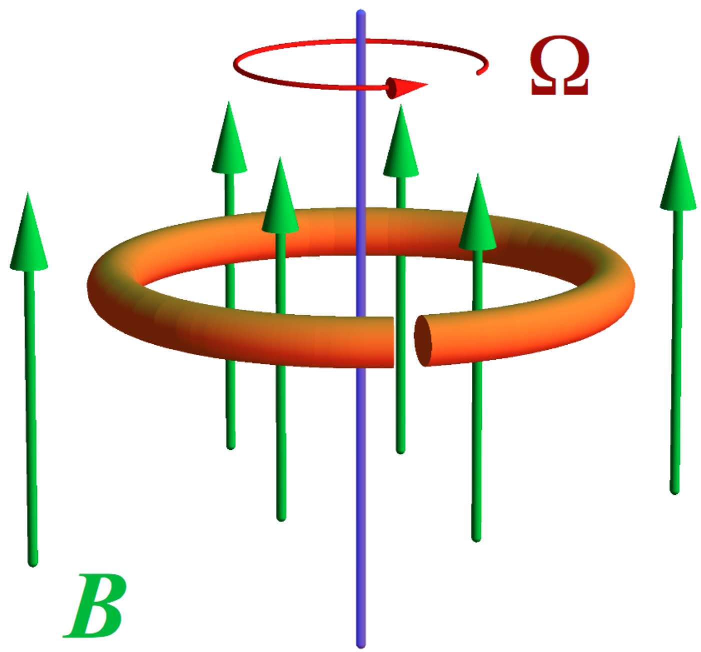
$$I^{\mathrm{ZP}} = -2.2 \times 10^{-42} \cdot \left(\frac{R}{\mathrm{m}}\right) \cdot \mathrm{kg}\,\mathrm{m}^2$$

... very-very small number ... can we make it larger?

Yes, we can!

Charged massless particles in magnetic field

Circle with the Dirichlet cut is pierced by a magnetic flux F_B



Lagrangian:

$$\mathcal{L} = [D_\mu \Phi]^* \, D^\mu \Phi$$
$$\equiv [D_t \Phi]^* \, D_t \Phi - \frac{1}{R^2} [D_\varphi \Phi]^* \, D_\varphi \Phi$$

with the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$

Boundary condition at the cut:

$$\Phi(t,\varphi)\Big|_{\varphi = [\Omega t]_{2\pi}} = 0$$

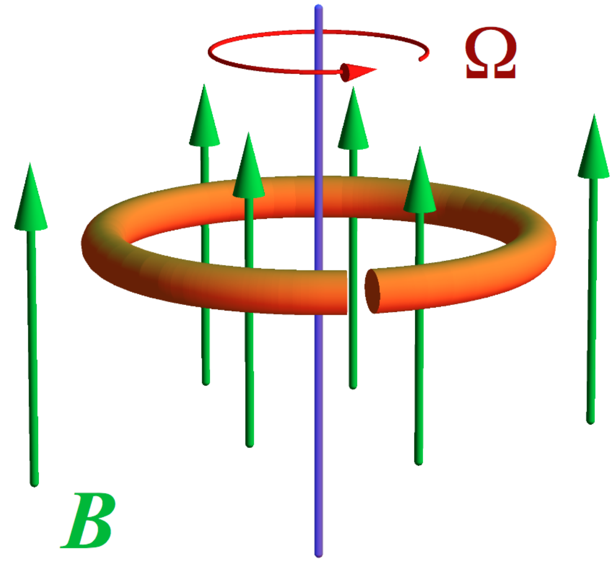
Symmetry: $U(1): \quad \Phi \to e^{ie\omega} \Phi \,, \quad A_\mu \to A_\mu + \partial_\mu \omega$

Using the gauge symmetry we choose the gauge potential in the form:

$$A_\varphi = \frac{\gamma_B}{e}\,, \qquad A_t = 0 \qquad \text{with} \qquad \gamma_B = \frac{eF_B}{2\pi}$$

where F_B is the magnetic flux going through the circle:

$$F_B = \iint_S d^2\boldsymbol{s}\cdot\boldsymbol{B} \equiv \oint_C d\boldsymbol{x}\cdot\boldsymbol{A} = R\int_0^{2\pi} d\varphi\, A_\varphi$$



Eigensystem

Equation:

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\left(\frac{\partial}{\partial \varphi} - i\gamma_B\right)^2\right]\Phi_{\omega,m}(t,\varphi) = \Lambda_{\omega,m}\,\Phi_{\omega,m}(t,\varphi)$$

Eigenvalues:

$$\Lambda_{\omega,m} = \frac{1-\Omega^2 R^2}{4R^2}m^2 - \frac{(\omega+\gamma_B\Omega)^2}{1-\Omega^2R^2}$$

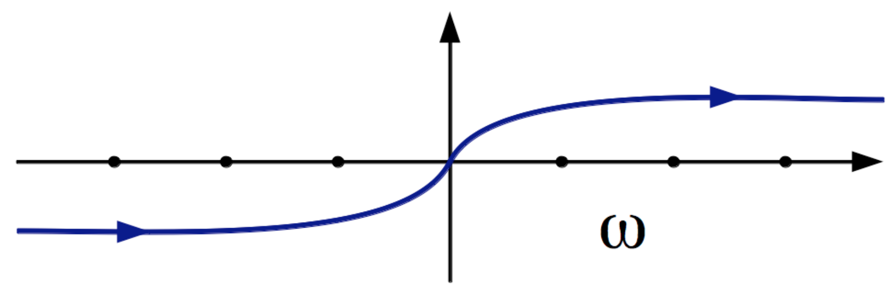
Eigenfunctions:

$$\Phi_{\omega,m}(t,\varphi) = \sqrt{\frac{1}{\pi R}}\sin\left(\frac{m}{2}[\varphi - t\Omega]_{2\pi}\right)\exp\left\{-i\omega t + i\frac{\gamma_B + \omega\,\Omega R^2}{1-\Omega^2R^2}[\varphi - t\Omega]_{2\pi}\right\}$$

where $\gamma_B = \dfrac{eF_B}{2\pi}$

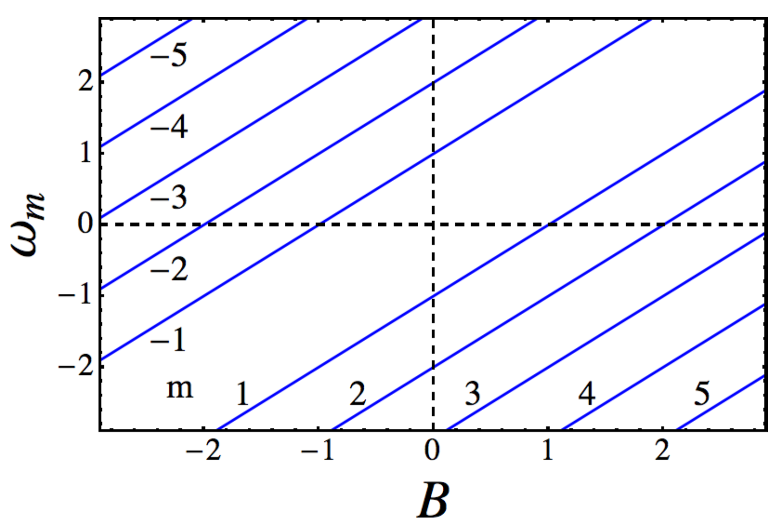
Green's function:

$$G_{\Omega,B}(t,t';\varphi,\varphi') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \frac{\Phi_{\omega,m}(t,\varphi)\Phi^*_{\omega,m}(t',\varphi')}{\Lambda_{\omega,m} - i\epsilon}$$



The magnetic flux shifts the poles:

$$\omega_m = \frac{1-\Omega^2 R^2}{2R} m - \gamma_B \Omega \qquad \text{where} \qquad \gamma_B = \frac{eF_B}{2\pi}$$



The shift the poles depends both on the magnetic flux F_B and the angular frequency of rotation Ω

Emerging discontinuities

Due to the dependence of the position of the poles on the angular frequency Ω and on the magnetic flux F_B, the Green's function experiences discontinuities, which depend on the integer number:

$$M_{\Omega,B} = \left\lfloor \frac{2\gamma_B \Omega R}{1-\Omega^2 R^2} \right\rfloor = \left\lfloor \frac{\Omega R}{1-\Omega^2 R^2} \frac{e F_B}{\pi} \right\rfloor$$

$$\equiv \left\lfloor \frac{eB\Omega R^2}{c^2-\Omega^2 R^3} \frac{c}{\hbar} \right\rfloor = \left\lfloor \frac{\Omega}{\Omega_{\mathrm{ch}}(B)} \frac{1}{1-\Omega^2 R^2} \right\rfloor$$

where the characteristic frequency is:

$$\Omega_{\mathrm{ch}}(B) = \frac{\pi}{e F_B R} \equiv \frac{\hbar c}{e B R^3}$$

Magnetic flux

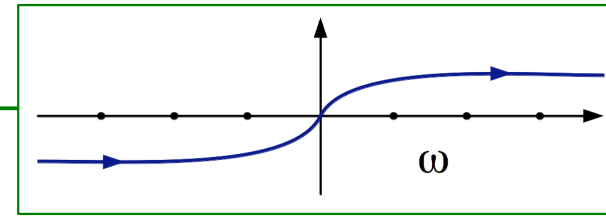
$$F_B = \pi R^2 B$$

Floor operation: $\lfloor 0.1 \rfloor = 0, \ \lfloor 1.9 \rfloor = 1$

Technicalities, calculation of the Green's function

The integral involved in calculation of the Green's function,

$$G_{\Omega,B}(t,t';\varphi,\varphi') = \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \frac{\Phi_{\omega,m}(t,\varphi)\Phi^*_{\omega,m}(t',\varphi')}{\Lambda_{\omega,m} - i\epsilon}$$



has poles

$$\omega_m = \frac{1-\Omega^2 R^2}{2R} m - \gamma_B \Omega \equiv \mu_m - \gamma_B \Omega$$

where the eigenvalues $\Lambda_{\omega,m} = \frac{1-\Omega^2R^2}{4R^2} m^2 - \frac{(\omega+\gamma_B\Omega)^2}{1-\Omega^2R^2}$ vanish.

We use the following property (valid for even vector f_m with $f_m = f_{-m}$ and $f_0 = 0$):

$$\int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{m=1}^{\infty} \frac{e^{-i\alpha\omega} f_m}{\Lambda_{\omega,m} - i\epsilon} = \int\limits_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sum_{m\in\mathbb{Z}}{}' \frac{e^{-i\alpha\omega} f_m}{\Lambda_{\omega,m} - i\epsilon}$$

$$= \frac{i e^{i\alpha\Omega\gamma_B}}{2\Omega_0} \sum_{m\in\mathbb{Z}}{}' f_m \frac{e^{-i\alpha\mu_m}}{m} \mathrm{sign}\,(\alpha)\,\Theta(\alpha\omega_m)$$

$$= \frac{i e^{i\alpha\Omega\gamma_B}}{2\Omega_0} \sum_{m=N_{\Omega,B}(\alpha)}^{\infty}{}' f_m \frac{e^{-i|\alpha|\mu_m}}{m}$$

The m = 0 term is excluded

$$N_{\Omega,B}(\alpha) = \frac{1}{2} + \left[M_{\Omega,B} + \frac{1}{2}\right] \mathrm{sign}\,(\alpha)$$

$$\equiv \begin{cases} M_{\Omega,B} + 1\,, & \alpha > 0 \\ -M_{\Omega,B}\,, & \alpha < 0 \end{cases}$$

Green's function, explicitly

$$G_{\Omega,B}(t,t';\varphi,\varphi') = \frac{i}{\pi} e^{i\left(\Omega(t-t')+[\varphi-t\Omega]_{2\pi}-[\varphi'-t'\Omega]_{2\pi}\right)\gamma_B} \sum_{m=N_{\Omega,B}(\alpha)}^{\infty}{}' \; h_m(t,t';\varphi,\varphi')$$

Magnetic flux enters here

$$= e^{i\left(\Omega(t-t')+[\varphi-t\Omega]_{2\pi}-[\varphi'-t'\Omega]_{2\pi}\right)\gamma_B} \left\{ G_\Omega(t,t';\varphi,\varphi') + \frac{i}{\pi} S\left[h_m(t,t';\varphi,\varphi'), N_{\Omega,B}\big(\alpha(t,t';\varphi,\varphi')\big)\right] \right\}$$

Magnetic flux enters here $\gamma_B = \dfrac{eF_B}{2\pi}$

The Green's function for a neutral scalar particle (= in absence of the magnetic field)

$$G_\Omega(t,t';\varphi,\varphi') = \frac{i}{\pi} \mathcal{G}\left(\frac{[\varphi-\Omega t]_{2\pi}}{2}, \frac{[\varphi'-\Omega t']_{2\pi}}{2}, \frac{\left|(1-\Omega^2R^2)(t-t') - \Omega R^2\left([\varphi-\Omega t]_{2\pi} - [\varphi'-\Omega t']_{2\pi}\right)\right|}{2R} \right)$$

with $\displaystyle \mathcal{G}(x,y,z) = \sum_{m=1}^{\infty} \frac{\sin(mx)\sin(my)}{m} e^{-imz} = \frac{1}{4} \ln \frac{\left[1-e^{i(x+y-z)}\right]\left[1-e^{-i(x+y+z)}\right]}{\left[1-e^{i(x-y-z)}\right]\left[1-e^{i(-x+y-z)}\right]}$

$$S[K_m, N] = \begin{cases} -\sum\limits_{m=1}^{N-1} K_m, & N > 1 \\ 0, & N = 0,1 \\ \sum\limits_{m=N}^{-1} K_m, & N < 0 \end{cases}$$

$$h_m(t,t';\varphi,\varphi') = e^{-\frac{i(1-\Omega^2R^2)}{2R}\left|\alpha(t,t';\varphi,\varphi')\right| m} \cdot \frac{1}{m} \sin\left(\frac{m}{2}[\varphi - t\Omega]_{2\pi}\right) \sin\left(\frac{m}{2}[\varphi'-t'\Omega]_{2\pi}\right)$$

$$\alpha(t,t';\varphi,\varphi') = t - t' - \frac{\Omega R^2\left([\varphi-t\Omega]_{2\pi} - [\varphi'-t'\Omega]_{2\pi}\right)}{1-\Omega^2R^2}$$

Energy density of zero-point fluctuations (uniform rotation in magnetic field)

The stress-energy tensor:

$$\langle T^{\mu\nu}(x)\rangle = \left(D^\mu D'^{\nu *} + D^\nu D'^{\mu *} - g^{\mu\nu} D^\lambda D'^{*}_\lambda\right) \frac{1}{i} G(x,x')\Big|_{x\to x'}$$

The energy density:

$$\langle T^{00}(t,\varphi)\rangle = \left[\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + \frac{1}{R^2}\left(\frac{\partial}{\partial\varphi} - i\gamma_B\right)\left(\frac{\partial}{\partial\varphi'} + i\gamma_B\right)\right]\frac{1}{i}G(t,t';\varphi,\varphi')\Big|_{\substack{t\to t'\\ \varphi\to\varphi'}}$$

Explicit calculation gives us:

$$\mathcal{E}^{\mathrm{ZP}}_{\Omega,B} \equiv \langle T^{00}\rangle^{\mathrm{phys}}_{\Omega,B} = -\Big[1 + 6M_{\Omega,B}(M_{\Omega,B}+1)\Big]\frac{1+\Omega^2R^2}{48\pi R^2}$$

with

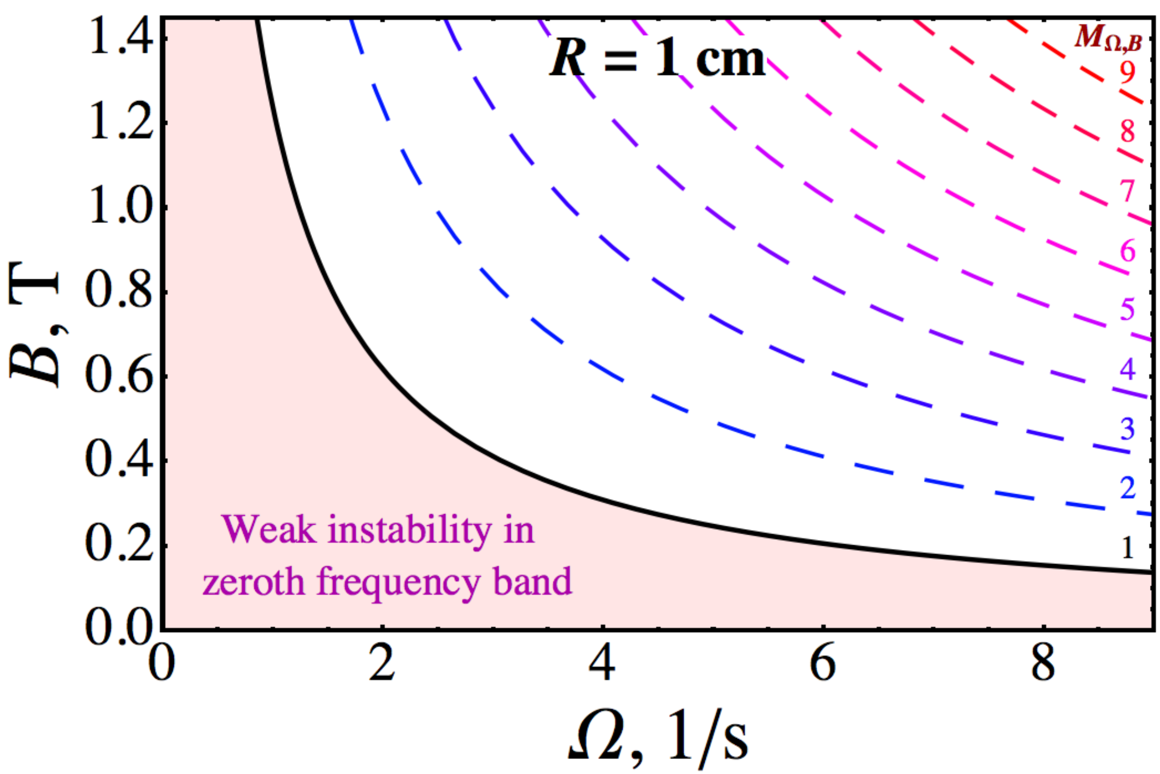
$$M_{\Omega,B} \equiv \left\lfloor \frac{eB\Omega R^2}{c^2 - \Omega^2 R^3}\,\frac{c}{\hbar}\right\rfloor = \left\lfloor \frac{\Omega}{\Omega_{\mathrm{ch}}(B)}\,\frac{1}{1-\Omega^2R^2}\right\rfloor \quad \text{and} \quad \Omega_{\mathrm{ch}}(B) = \frac{\pi}{eF_BR} \equiv \frac{\hbar c}{eBR^3}$$

Energy of zero-point fluctuations

$$E^{\mathrm{ZP}}_{\Omega,B} = -\Big[1 + 6 M_{\Omega,B}(M_{\Omega,B}+1)\Big]\frac{1+\Omega^2 R^2}{24R}$$

Strong enhancement of rotational energy

Enhancement bands, illustration:



$$M_{\Omega,B} = \left\lfloor \frac{\Omega}{\Omega_{\mathrm{ch}}(B)} \frac{1}{1-\Omega^2 R^2} \right\rfloor$$

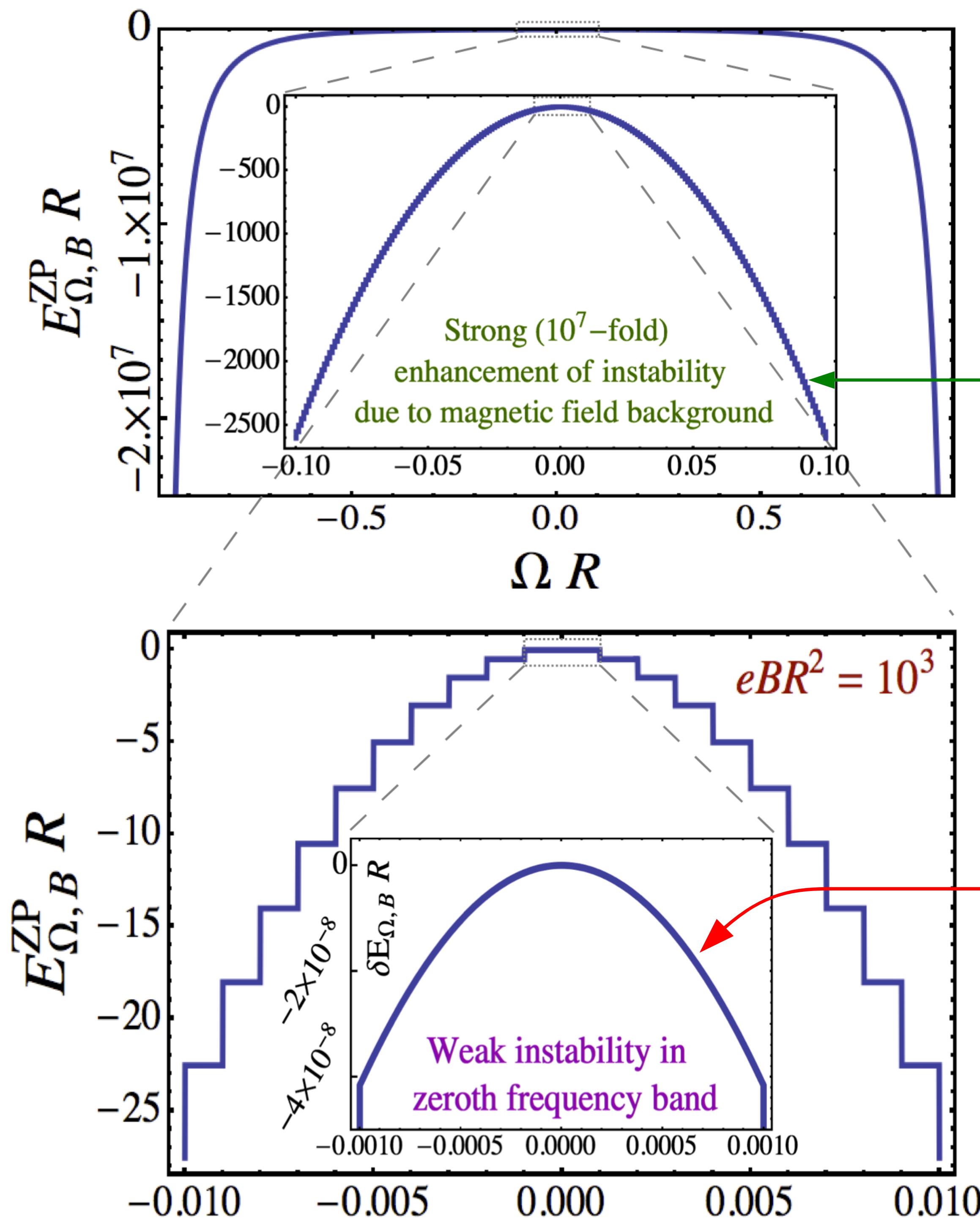
$$\equiv \left\lfloor \frac{eB\Omega R^2}{c^2 - \Omega^2 R^3} \frac{c}{\hbar} \right\rfloor$$

In nonrelativistic rotation, the enhancement depends on the product $B\,\Omega$ only.

$$\Omega_{\mathrm{ch}}(B) = \frac{\pi}{eF_B R} \equiv \frac{\hbar c}{eBR^3}$$

An example of the enhancement

500 elementary fluxes pass through the circle



The (modest) enhancement due to magnetic flux

The “usual”, non-enhanced rotational energy

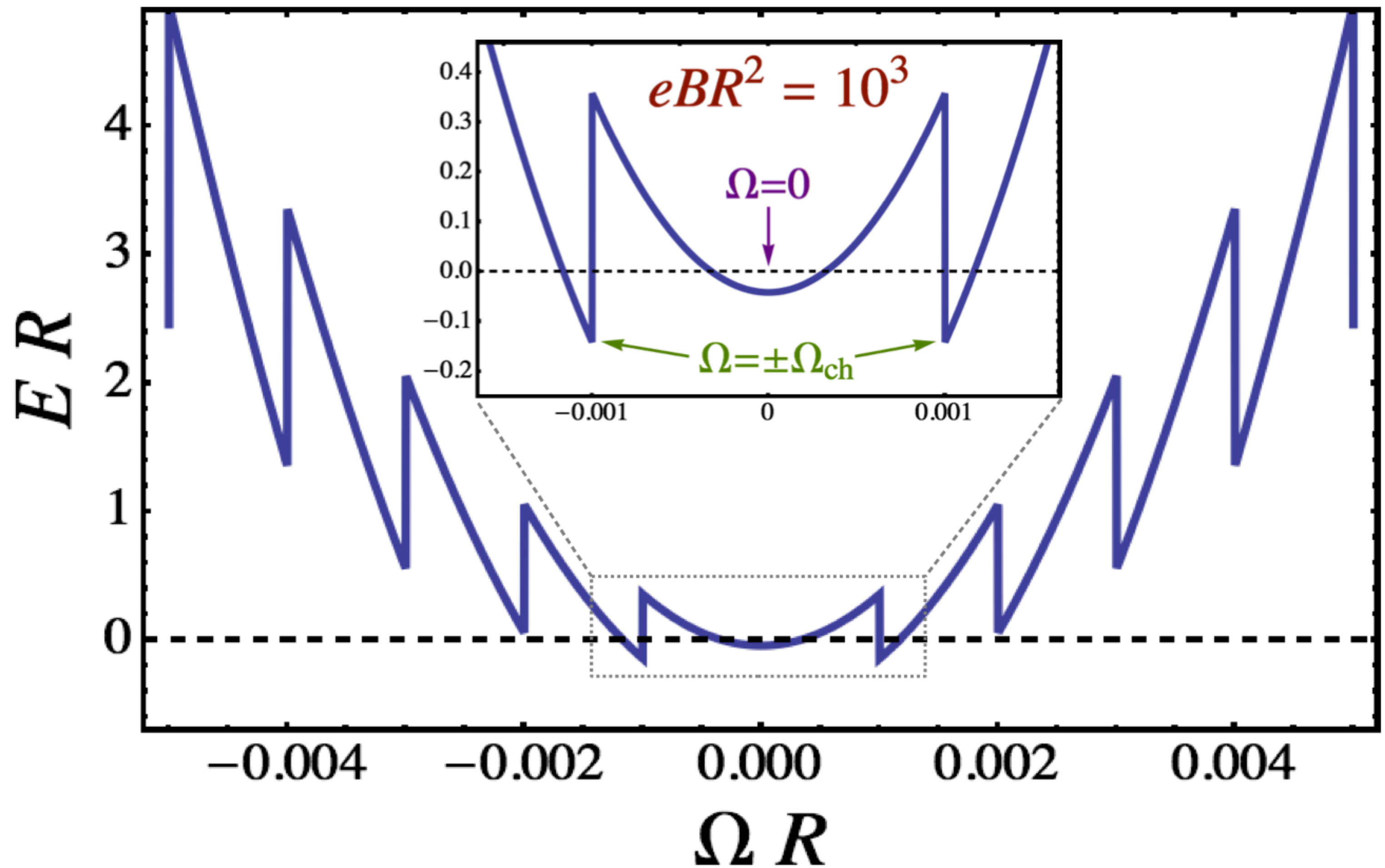
Scales of enhancement

B, T	R	$\Omega_{\rm ch}$, s^{-1}	$\tau_{\rm ch}$, s	$\Omega_{\rm ch}R/c$	f	note
100	1 cm	2×10^{-3}	3200 (53 min)	7×10^{-14}	4×10^{27}	max B in a laboratory
1	1 cm	0.2	32	7×10^{-12}	4×10^{23}	generic-scale B and R
1	1 mm	200	0.03	7×10^{-10}	4×10^{19}	
1	0.1 mm	2×10^{6}	3×10^{-5}	7×10^{-8}	4×10^{15}	width of a human hair

The enhancement factor:

$$f(B) \equiv \left.\frac{E^{\rm ZP}_{\Omega,B}}{E^{\rm ZP}_{\Omega,B=0}}\right|_{\Omega\gg\Omega_{\rm ch}(B)} = \frac{6}{(\Omega_{\rm ch}R)^2} \equiv 6\,e^2B^2R^4$$

Total energy, an example:



Total energy: $$E(\Omega) = E^{\mathrm{ZP}}_{\Omega,B} + E_{\mathrm{cl}}(\Omega)$$

The classical part of total energy: $$E_{\mathrm{cl}}(\Omega) = \frac{I_{\mathrm{cl}}\Omega^2}{2}$$

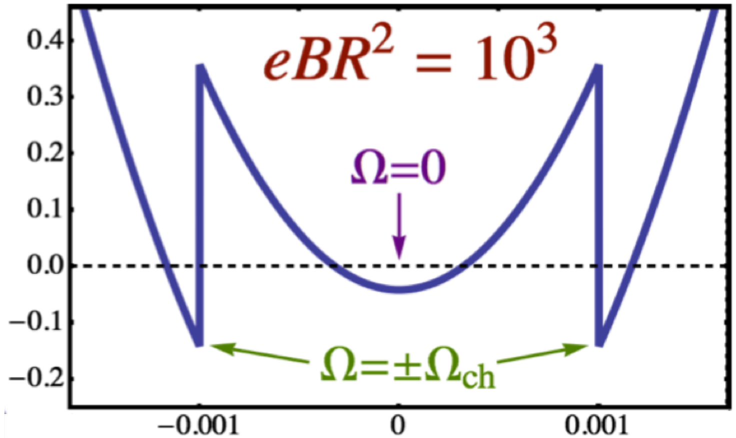
Perpetuum mobile of the fourth kind

If for some angular frequency Ω

$$E(\Omega \neq 0) < E(\Omega = 0)$$

then the ground state corresponds to a perpetual rotation.

Even in the presence of environment (e.g., of a thermal bath) the device will always tend to be in the rotating state, since it corresponds to the minimum of energy.

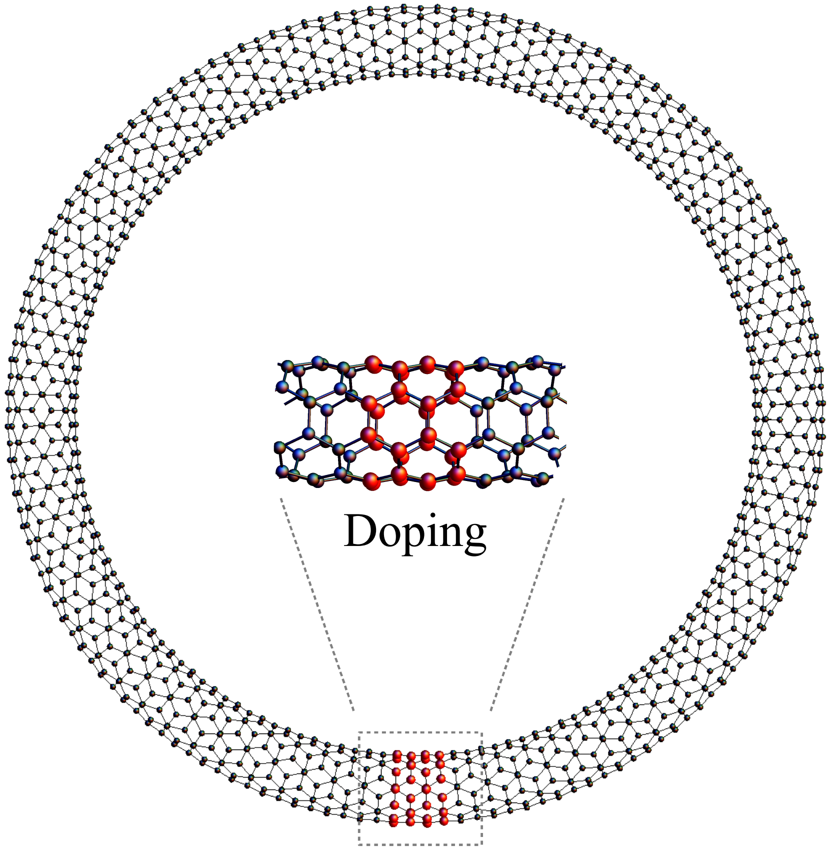


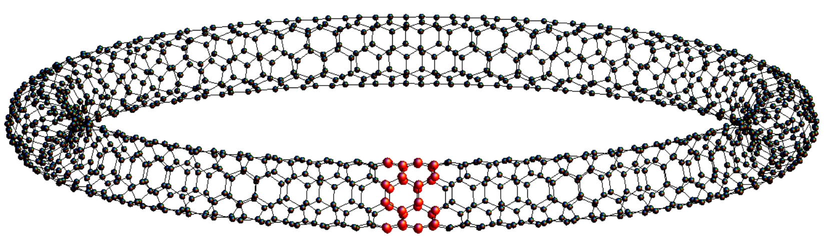
Our example corresponds to the most modest (10-fold) enhancement at the frequency

$$\Omega = \Omega_{\mathrm{ch}}$$

Device made of doped carbon nanotubes?

(naïve estimation)





Classical part of the total energy:

$$E_{\rm cl}(\Omega) = \frac{I_{\rm cl}\Omega^2}{2} \equiv \pi\mu R^3\Omega^2$$

Moment of inertia of the device:

$$I_{\rm cl} \equiv \frac{\partial^2 E_{\rm cl}}{\partial\Omega^2} = mR^2\,, \qquad m = 2\pi\mu R$$

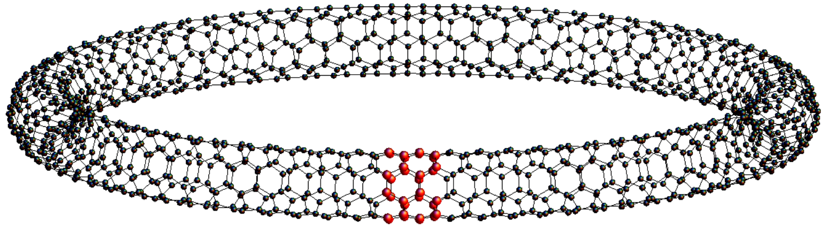
Mass density per unit length

$$\mu = 3.24 \times 10^{-15}\,{\rm kg/m}$$

Speed of light → Fermi velocity

$$c \to v_F \approx 8.1 \times 10^5\,{\rm m/s} \approx \frac{c}{300}$$

Nanotube torus



Zero-point energy:

$$E^{\mathrm{ZP}}_{\Omega,B} = -\Big[1 + 6M_{\Omega,B}(M_{\Omega,B}+1)\Big]\frac{\Omega_0^2 v_F^2 + \Omega^2}{12\Omega_0 v_F}\hbar, \quad M_{\Omega,B} = \left\lfloor \frac{eB\Omega R^2}{v_F^2 - \Omega^2 R^2}\frac{v_F}{\hbar} \right\rfloor$$

Characteristic angular frequency: $\Omega_{\mathrm{ch}} = \dfrac{\hbar v_F}{eBR^3}$

Physical units:

$$\Omega_{\mathrm{ch}} \simeq 3.35 \times 10^{-9} \left(\frac{B}{\mathrm{T}}\right)^{-1} \left(\frac{R}{\mathrm{m}}\right)^{-3} \mathrm{s}^{-1}$$

Minimal radius needed for perpetual motion: $\left(\dfrac{R_{\min}}{\mathrm{m}}\right) \simeq 0.015 \left(\dfrac{B}{\mathrm{T}}\right)^{-1}$

One needs: $R > R_{\min}$

$$R_{\min}(B = 100\mathrm{T}) \simeq 1.5 \times 10^{-4}\,\mathrm{m} \equiv 0.15\,\mathrm{mm}$$
$$R_{\min}(B = 1\mathrm{T}) \simeq 1.5 \times 10^{-2}\,\mathrm{m} \equiv 1.5\,\mathrm{cm}\,.$$

Rotational time periods:

$$\tau_{\min}(B = 100\mathrm{T}) \simeq 0.63\,\mathrm{s}\,,$$
$$\tau_{\min}(B = 1\mathrm{T}) \simeq 6330\,\mathrm{s} \approx 1.75\,\mathrm{hrs}$$

Laws of thermodynamics: no contradiction

First law ("work from nothing is forbidden"): Our device produces no work, therefore its existence is consistent with the First Law.

Second law ("entropy of any isolated system not in thermal equilibrium increases"). For rotating systems this law gives the following condition for the thermal equilibrium:

$$\Omega = \left(\frac{\partial E}{\partial L}\right)_S$$

"L" is the angular momentum

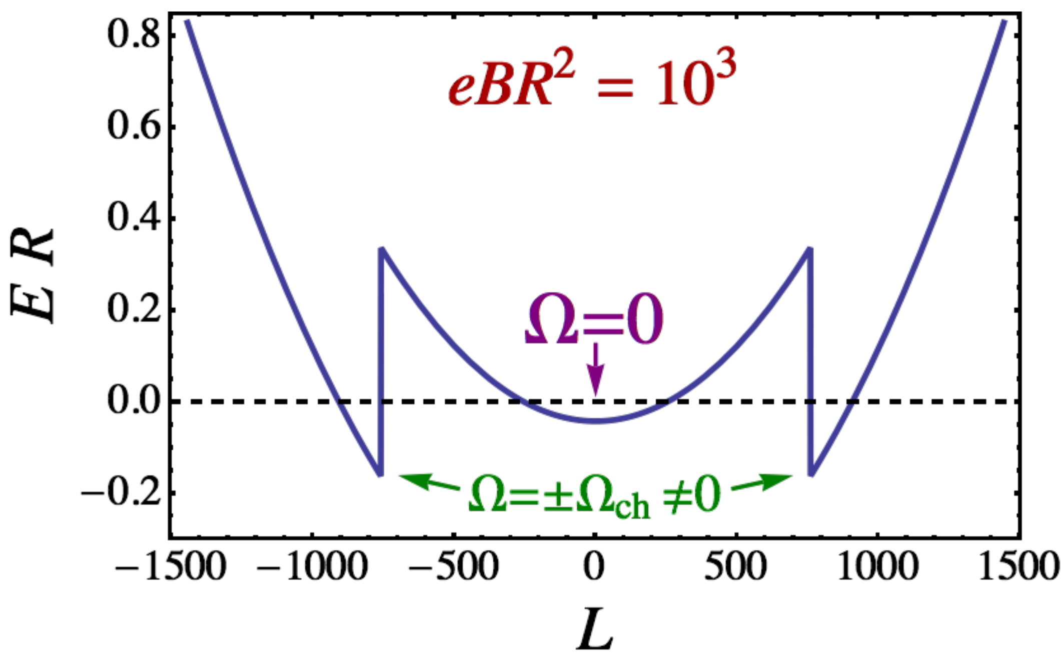
"S" is the entropy (constant at T=0)

For any classical system $E(L)$ is a convex function → Ω=0

Simplest example: the classical energy, $E_{\rm cl} = \dfrac{L^2}{2I_{\rm cl}}$, takes its minimum in the ground state, $\dfrac{\partial E_{\rm cl}}{\partial L} = 0$, thus $\Omega = 0$.

Why the rotation of the device is possible in a thermal equilibrium?

Because the dependence of the zero-point energy on its angular momentum is a discontinuous function!
(a situation, which is impossible in standard thermodynamics)



At the minimum, $L=L_{min}$, the first derivative is nonzero:

$$\Omega = \lim_{L \to L_{\min} \pm 0} \left(\frac{\partial E}{\partial L} \right)_S$$

and the angular frequency is nonzero as well.

Consistent with the Second Law.

Other works on perpetual motion:

1) Classical time crystals

Alfred Shapere and Frank Wilczek, [arXiv:1202.2537].

A proposal to make a permanently rotating state in classical and semiclassical systems, including traveling density waves.

2) Quantum time crystals, Frank Wilczek, [arXiv:1202.2539].

A proposal to make a permanently rotating system in quantum mechanics and in imaginary time.

3) Space-time crystals of trapped ions

Tongcang Li, Zhe-Xuan Gong, Zhang-Qi Yin, H. T. Quan, Xiaobo Yin, Peng Zhang, L.-M. Duan, and Xiang Zhang [arXiv:1206.4772]

A proposal to make a permanently rotating state in a cold ion system on a ring in magnetic field.

Conclusions:

1. We have calculated for a first time the rotational energy of the zero-point fluctuations.

2. We have demonstrated that this rotational energy may take its minimum at nonzero values of the angular frequency (“the rotational vacuum effect”).

3. We have shown that the magnetic field should drastically (astronomically) enhance the energy of zero--point fluctuations at certain conditions.

4. We have proposed a general design of a device, for which a ground state is a rotating state. The device has no internally moving parts.

5. This device prefers to rotate forever even in the presence of an external environment (“perpetuum mobile of the fourth type”).