Exotic supersymmetries of finite-gap and reflectionless systems

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Outline

- Reminder: structure of N=2 SUSY QM
- Linear bosonized SUSY
- Nonlinear bosonized SUSY
- Hidden bosonized SUSY in local QM systems
- Extended finite-gap systems: exotic supersymmetry
- Example: electron in periodic magnetic and electric fields
- Exotic supersymmetry in the systems with first order Hamiltonian:
  - Klein effect in carbon nanostructures
  - Crystalline condensates in Gross-Neveu model
- Comments: origin of bosonized SUSY; exotic SUSY in PT-symmetric systems

Structure of Supersymmetric Quantum Mechanics

- Second order Hamiltonian operator, $H = H^\dagger$
- First order supercharges, $Q_a = Q_a^\dagger$, $a = 1, 2$
- Grading operator, $\Gamma = \sigma_3, \quad \Gamma^2 = 1$

Lie Superalgebra:

$$[H, Q_a] = 0, \quad \{Q_a, Q_b\} = 2\delta_{ab}H$$

Grading relations:

$$[\Gamma, H] = \{\Gamma, Q_a\} = 0$$

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & Q_- \\ Q_-^\dagger & 0 \end{pmatrix}, \quad Q_2 = i\sigma_3 Q_1,$$

$H_+ = Q_- Q_-^\dagger$, $\quad H_- = Q_-^\dagger Q_-$, $\quad Q_- = \frac{d}{dx} - W(x)$, $\quad H_\pm = -\frac{d^2}{dx^2} + W^2 \mp W'$,

$[H, Q_a] = 0 \Leftrightarrow Q_- H_- = H_+ Q_-,\ H_- Q_-^\dagger = Q_-^\dagger H_+,$ Darboux transformations,

- H_- and H_+ are (almost) isospectral: energy levels $E > 0$ of H are doubly degenerate (in non-periodic case; bound states); level $E = 0$, if exists, is nondegenerate.
- $\Psi_{-,E} \propto Q_-^\dagger \Psi_{+,E}$, $\qquad \Psi_{+,E} \propto Q_- \Psi_{-,E}$, $\qquad H_\pm \Psi_{\pm,E} = E\Psi_{\pm,E}$

$N=2$ supersymmetry can be realized in *non-extended* systems, bosonized SUSY QM [MP, 1996, Ann. Phys. 245, 339]:

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + W^2(x) - W'(x)R\right),$$

$$Q_1 = -i\left(\frac{d}{dx} + W(x)R\right), \qquad Q_2 = iRQ_1,$$

$W(-x) = -W(x)$ - *odd* superpotential, $\Gamma = R$, $R^2 = 1$, – reflection *(parity)* operator, $R\psi(x) = \psi(-x)$, H is a nonlocal operator,

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \quad [Q_a, H] = 0, \quad [R, H] = 0, \quad \{R, Q_a\} = 0.$$

Hidden nonlinear SUSY in harmonic parabosonic oscillator systems of the order $p = 2(k+1)$, $k = 0, 1, 2\ldots$, and in a related two-body Calogero model with exchange interaction [MP, 2000, Int. J. Mod. Phys. A 15, 3679]:

$$H = a^+a^-, \qquad Q_+ = (a^+)^{2k+1}\Pi_-, \qquad Q_- = (a^-)^{2k+1}\Pi_+,$$

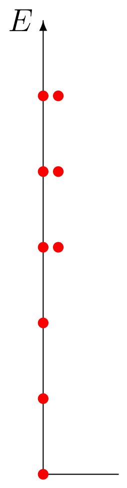
$$[Q_\pm, H] = 0, \quad Q_\pm^2 = 0, \qquad \underline{\{Q_+, Q_-\} = P_{2k+1}(H),}$$

where $\Pi_+ = \frac{1}{2}(1+R) = \cos^2 F$, $\quad \Pi_- = \frac{1}{2}(1-R) = \sin^2 F$, $\quad F = \frac{\pi}{4}\{a^+, a^-\}$,

$$[\{a^-, a^+\}, a^\pm] = \pm 2a^\pm, \quad a^-a^+|0\rangle = p|0\rangle, \quad a^-|0\rangle = 0.$$

Parabose oscillator $\rightarrow$ 2-particle Calogero model ($[a^-, a^+] = 1 + \nu R$):

$$a^\pm = \frac{1}{\sqrt{2}}(x \mp iD_\nu), \quad iD_\nu = \frac{d}{dx} - \frac{\nu}{2x}R, \quad \nu = 2k+1.$$



Spectrum of the parabosonic oscillator system at $\nu = 5$ $(k = 2, \quad p = 6)$

⇒ Some non-extended, purely bosonic quantum mechanical systems with a *local* Hamiltonian may have a hidden, bosonized supersymmetry [F. Correa, MP, Ann. Phys. 2007, 322, 2493]:

• *Examples of the systems with a hidden, bosonized supersymmetry*: free particle, Dirac delta potential problem, bound state Aharonov-Bohm effect, planar Aharonov-Bohm effect, finite-gap reflectionless Pöschl-Teller system, finite-gap periodic Lamé and associated Lamé systems

• non-periodic Pöschl-Teller and periodic Lamé systems are particular examples of finite-gap systems related to the KdV equation; corresponding *higher order Lax operators* play a role of one of the *supercharges of the bosonized supersymmetry*

Extended finite-gap systems: exotic supersymmetry

- Second order extended (matrix) Hamiltonian, H
- Two Lax operators = integrals of order $2n+1$
- Two supercharges of order $2k$ and two supercharges of order $2(n-k)+1$
- Alternatives for Γ
- $\Rightarrow$ Exotic *nonlinear* supersymmetry reflects peculiarities of the spectrum:
- 2-fold degeneration of each of $(2n+1)$ *edge* band energies
- 4-fold degeneration of energy levels inside the allowed bands

F. Correa, V. Jakubský, L.M. Nieto, M.P., 2008, PRL, **101**, 030403 :

Exotic supersymmetry can be realized by a non-relativistic electron in periodic magnetic and electric fields of a special form

$$H_e = (p_x + A_x)^2 + (p_y + A_y)^2 + \sigma_3 B_z - \phi,$$

$A_x = 0, A_y = w(x), B_z = \frac{dw}{dx}; w(x) = \alpha \frac{d}{dx} \ln(\mathrm{dn} x), \phi(x) = \beta w^2(x) + \gamma w(x) + \delta$
$\Rightarrow$

$$H^{\pm}_{m,l} = -\frac{d^2}{dx^2} + V^{\pm}_{m,l}(x), \qquad V^{+}_{m,l}(x) = V^{-}_{m,l}(x + L),$$

$$V^{-}_{m,l}(x) = -C_m \mathrm{dn}^2 x - C_l \frac{k'^2}{\mathrm{dn}^2 x} + c, \qquad C_l = l(l+1)$$

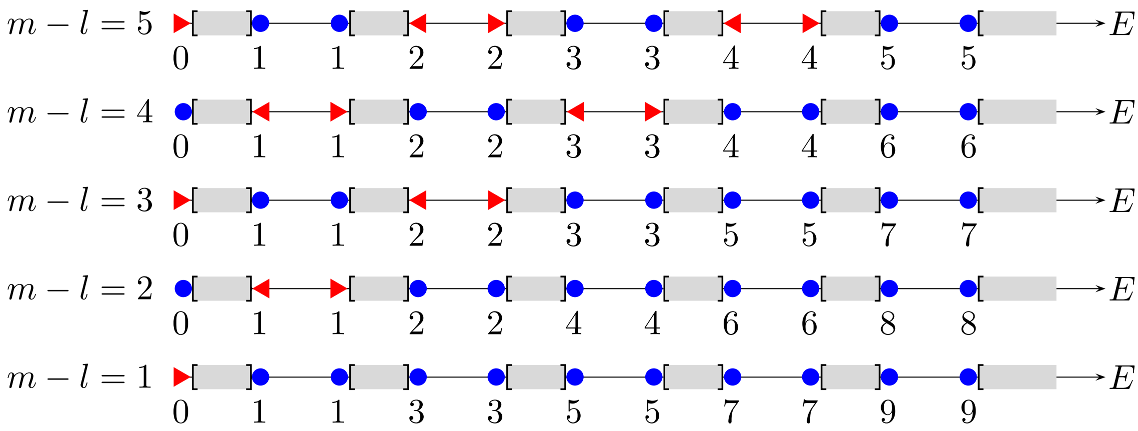


Figure 1: The blue dots and red triangles represent the band-edge states of $H_{m,l}$. The number of nodes of each band-edge state is indicated. For each setting, the ground state is periodic. The lower order operator, Q_+ or Q_-, annihilates the band-edge states represented by triangles while the singlet states indicated by dots are annihilated by the higher order operator.

Super-extended system:

$$\mathcal{H}=\begin{pmatrix}\tilde{H} & 0\\ 0 & H\end{pmatrix},\quad \mathcal{Q}_{\pm}=\begin{pmatrix}0 & Q_{\pm}\\ Q_{\pm}^{\dagger} & 0\end{pmatrix},\quad \mathcal{Z}=\begin{pmatrix}\tilde{Z} & 0\\ 0 & Z\end{pmatrix},$$

$$[\mathcal{H},\mathcal{Z}]=[\mathcal{H},\mathcal{Q}_{\pm}]=[\mathcal{Q}_{+},\mathcal{Q}_{-}]=[\mathcal{Z},\mathcal{Q}_{\pm}]=0,$$

$$\mathcal{Z}=\mathcal{Q}_{-}\mathcal{Q}_{+}=\mathcal{Q}_{+}\mathcal{Q}_{-},$$

$$\mathcal{Z}^2=P_{\mathcal{Z}}(\mathcal{H}),\quad \mathcal{Q}_{+}^2=P_{+}(\mathcal{H}),\quad \mathcal{Q}_{-}^2=P_{-}(\mathcal{H}),$$

$$P_{\mathcal{Z}}(\mathcal{H})=\prod_{j=1}^{2n+1}(\mathcal{H}-E_j),\quad P_{+}(\mathcal{H})=\prod_{j=1}^{2r}(\mathcal{H}-E_j^{+}),\quad P_{-}(\mathcal{H})=\prod_{j=1}^{2(n-r)+1}(\mathcal{H}-E_j^{-}),$$

$$P_{\mathcal{Z}}=P_{+}(\mathcal{H})P_{-}(\mathcal{H})\quad = \quad \text{spectral polynomial}$$

$\underline{\Gamma = \sigma_3}$: a SUSY subalgebra generated by local integrals $\mathcal{H}$, $\mathcal{Z}$ and

$$\mathcal{Q}_\pm^{(1)} = \mathcal{Q}_\pm, \quad \mathcal{Q}_\pm^{(2)} = i\sigma_3\mathcal{Q}_\pm,$$

is identified as a centrally extended nonlinear $N=4$ supersymmetry,

$$\{\mathcal{Q}_+^{(a)}, \mathcal{Q}_+^{(b)}\} = 2\delta^{ab}P_+(\mathcal{H}), \quad \{\mathcal{Q}_-^{(a)}, \mathcal{Q}_-^{(b)}\} = 2\delta^{ab}P_-(\mathcal{H}),$$
$$\{\mathcal{Q}_+^{(a)}, \mathcal{Q}_-^{(b)}\} = 2\delta^{ab}\mathcal{Z},$$
$$[\mathcal{H}, \mathcal{Q}_\pm^{(a)}] = [\mathcal{H}, \mathcal{Z}] = [\mathcal{Z}, \mathcal{Q}_\pm^{(a)}] = 0,$$

in which $\mathcal{Z}$ plays a role of the bosonic central charge.

The supercharges $\mathcal{Q}_+^{(a)}$ annihilate a part of band-edge states organized in supersymmetric doublets, another part of supersymmetric doublets is annihilated by $\mathcal{Q}_-^{(a)}$. The band-edge states which do not belong to the kernel of the supercharges $\mathcal{Q}_+^{(a)}$ (or $\mathcal{Q}_-^{(a)}$) are transformed (rotated) by these supercharges within the corresponding supersymmetric doublet. The bosonic central charge $\mathcal{Z}$ annihilates all the band-edge states.

$\Rightarrow$ Spontaneously partially broken centrally extended nonlinear $N=4$ supersymmetry, cf. partial supersymmetry breaking in supersymmetric field theories with BPS-monopoles.

• In some extended systems, the odd order supercharge is a matrix differential operator of the first order that may be identified as a Dirac type Hamiltonian.

• A corresponding system may be characterized then by the supersymmetric structure with the first order matrix Hamiltonian.

$\Rightarrow$ Such an exotic supersymmetry may be used to explain:

• Klein tunneling in carbon nanostructures: V. Jakubsky, L.-M- Nieto, MP, 2011, PRD, 83, 047702

• some peculiarities of the kink-antikink and kink crystalline condensates in the Gross-Neveu model (they appear, particularly, in the physics of conducting polymers): MP, A. Arancibia, L.-M. Nieto, 2011, PRD, 83, 065025

G. Semenoff (1984): tight binding description of *graphene* is reduced to the $2D$ *massless Dirac equation* in the low-energy approximation (the role of c plays $v_F \sim c/300$)

$+$ boundary conditions $\Rightarrow$ single wall nanotubes: $\Psi(\mathbf{x} + \mathbf{C_h}) = \Psi(\mathbf{x})$, where $\mathbf{C_h} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$ is a circumference (chiral) vector, $\mathbf{a_1}$ and $\mathbf{a_2}$ are the primitive translation vectors of the Bravais lattice, $n_{1,2} \in \mathbb{Z}$.

Let $\mathbf{C_h}$ be parallel to $y \Rightarrow$ low energy behavior of charge carriers is approximated by

$$H_\epsilon \psi = v_F \left(-i\sigma_1 \partial_x + \epsilon \sigma_2\right) \psi = E\psi \,,$$

where $\epsilon = 0$ for metallic nanotubes and $\epsilon = \pm\frac{2\pi}{3|\mathbf{C_h}|}$ for semiconducting nanotubes.

For metallic nanotubes in the presence of impurities, and for the particles in graphene normally incident on potential barrier $V(x)$ (see Fig. 1):

$$H_V\psi = (-iv_F\sigma_1\partial_x + V(x))\psi = E\psi.$$

Define

$$\mathcal{H} = \begin{pmatrix} H_V & 0 \\ 0 & H_0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Besides Γ, the Hamiltonian $\mathcal{H}$ has two other symmetries,

$$\mathcal{U}_1 = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix}, \quad \mathcal{U}_2 = i\,\Gamma\,\mathcal{U}_1,$$

where $U = U(x)$ is a *unitary* operator of a *local* chiral rotation,
$U = e^{i\alpha\sigma_1} = \cos\alpha\,\mathbf{1} + i\sin\alpha\,\sigma_1, \qquad \alpha(x) = \frac{1}{v_F}\int^x V(\tau)d\tau.$

They satisfy relations

$$[\mathcal{H}, \mathcal{U}_a] = 0, \quad \{\mathcal{U}_a, \mathcal{U}_b\} = 2\delta_{ab}\mathbb{1}, \quad [\Gamma, \mathcal{H}] = \{\Gamma, \mathcal{U}_a\} = 0.$$

• Extended *first order* system composed from H_V and H_0 possesses the $N = 2$ *zero-order* supersymmetry extended by the central charge $\mathbb{1}$ and graded by Γ.

• Like in the non-relativistic case of a reflectionless system with the n-gap, second order Hamiltonian, this structure underlies the absence of the backward scattering in the 0-gap system given by the first order Hamiltonian H_V.

$\Rightarrow$ The relation $UH_V = H_0U$, implied by $[\mathcal{H}, \mathcal{U}_a] = 0$ and the unitarity of U, reveal the unitary equivalence of H_V with the free massless Dirac Hamiltonian H_0.

$\Rightarrow$ the setting given by H_V is unitary equivalent to the free massless particle system H_0 and, hence, it shares its trivial scattering properties.

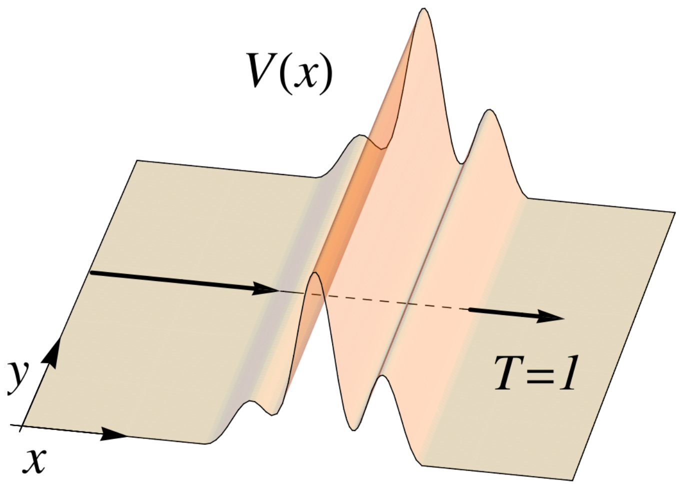


Figure 2: For $\delta k_y = 0$, the transmission coefficient T of the particle bouncing on the y-independent barrier is equal to one. Thick black arrows illustrate the particle approaching and penetrating the barrier.

• In massive case (semiconducting nanotubes: $m = \epsilon = \pm 2\pi/|3\mathbf{C_h}|$; other than normal incidence in graphene: $m = \delta k_y \neq 0$), unitary transformation of H_V yields

$$UH_V = H_m U - 2v_F m \sin\alpha\, \sigma_3,$$

where $H_V = H_m + V$, $H_m = H_0 + v_F m \sigma_2$.

$\Rightarrow$ the scale of supersymmetry breaking in the massive case is of the order of m, and the contribution of the potential is controlled by the factor $|\sin\alpha| \leq 1$.

$\Rightarrow$ For the close-to-the-normal incidence ($m = \delta k_y \sim 0$), the potential barrier remains almost perfectly transparent for any $V(x)$.

Conclusion: Klein tunneling in carbon nanostructures is explained by the exotic SUSY with the first-order Hamiltonian and zero-order supercharge operators.

• *Another* exotic supersymmetric structure can be revealed in the *first-order finite-gap systems*.

Consider one-gap self-isospectral second-order Lamé system with the period $2\mathbf{K}$,

$$\mathcal{H} = diag(H(x-\tau), H(x+\tau)), \quad H(x) = -\frac{d^2}{dx^2} + 2k^2\mathrm{sn}^2(x|k) - k^2.$$

Isospectral (displaced) subsystems $H(x+\tau)$ and $H(x-\tau)$ are related by an intertwining operator, $\mathcal{D}(x;\tau)H(x+\tau) = H(x-\tau)\mathcal{D}(x;\tau)$, where

$$\mathcal{D}(x;\tau) = F(x;\tau)\,\frac{d}{dx}\,\frac{1}{F(x;\tau)},$$

$F(x;\tau) = \exp(xz(\tau))\Theta(x-\tau)/\Theta(x+\tau)$, $\tau \neq n\mathbf{K}$, $z(\tau) = \mathrm{Z}(2\tau + i\mathbf{K}')\ + i\frac{\pi}{2\mathbf{K}}$, describes eigenstates of $H(x+\tau)$ from the lower *prohibited* band with $E < 0$.

As a consequence, $\mathcal{H}$ possesses seven *local* integrals of motion:

- σ_3 (zero order),
- $S_a, \quad a = 1, 2,$ (first order),
- $Q_a, \quad a = 1, 2,$ (second order),
- $L_a, \quad a = 1, 2,$ (third order, diagonal Lax operators).

All the six local integrals S_a, Q_a and L_a are constructed in terms of the intertwining operator.

$\mathcal{H}$ has also six nonlocal integrals $\mathcal{R}\sigma_a$, $\mathcal{T}\sigma_a$, $a = 1, 2$, and $\mathcal{R}\mathcal{T}\sigma_3$, $\mathcal{R}\mathcal{T}$, where $\mathcal{R}$ and $\mathcal{T}$ are the operators of reflections in x and τ.

$$S_1 = \begin{pmatrix} 0 & \mathcal{D}^\dagger(x;\tau) \\ \mathcal{D}(x;\tau) & 0 \end{pmatrix}, \qquad S_2 = i\sigma_3 S_1\,.$$

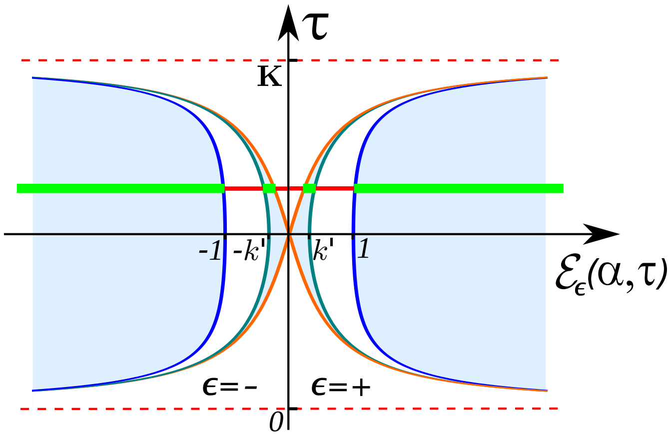


Figure 3: Spectrum of the periodic Bogoliubov-de Gennes system $H_{BdG} = S_1$.

• S_1 plays a role of the Bogoliubov-de Gennes Hamiltonian, that describes kink-antikink, or kink solutions (crystalline phases) of the Gross-Neveu model

$$\mathcal{L}_{GN} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi + \frac{1}{2}g^2(\bar{\psi}\psi)^2.$$

• When $\tau \neq (\frac{1}{2} + n)\mathrm{K}$, ($m_0 \neq 0$), there are 4 allowed bands in the spectrum of S_1; for $\tau = (\frac{1}{2} + n)\mathrm{K}$, ($m_0 = 0$), the middle gap disappears, and only three allowed bands are left.

• $\mathcal{R}\sigma_1$ is an integral for the first order Hamiltonian $H_{BdG} = S_1$, which may be identified as a grading operator Γ, $\Gamma^2 = 1$.

• For $\tau \neq (\frac{1}{2}+n)\mathbf{K}$, nontrivial *odd* integrals are $\mathcal{L}_1 = L_1$ and $\mathcal{L}_2 = i\mathcal{R}\sigma_1\mathcal{L}_1$, $\{\mathcal{R}\sigma_1, \mathcal{L}_a\} = 0$, where $iL_1 = diag(\mathcal{P}(x+\tau), \mathcal{P}(x-\tau))$ is a third order Lax integral for Lamé self-isospectral system $\mathcal{H}$,

$$\mathcal{P}(x) = d^3/dx^3 + (1+k^2-3k^2\mathrm{sn}^2 x)d/dx - 3k^2 \mathrm{sn}\, x\, \mathrm{cn}\, x\, \mathrm{dn}\, x.$$

Nonlinear SUSY algebra:

$$[\mathcal{R}\sigma_1, \mathcal{L}_a] = -2i\epsilon_{ab}\mathcal{L}_b, \qquad \{\mathcal{L}_a, \mathcal{L}_b\} = 2\delta_{ab}\hat{P}(S_1, \tau),$$

where the six order spectral polynomial is
$\hat{P}(S_1,\tau) = (S_1^2 - \varepsilon(\tau))(S_1^2 - \varepsilon(\tau) - k'^2)(S_1^2 - \varepsilon(\tau) - 1)$, $\varepsilon(\tau) = \frac{\mathrm{cn}^2 2\tau}{\mathrm{sn}^2 2\tau}$,
whose roots correspond to the energies of the six edge states of $H_{BdG} = S_1$.

For $\tau = (\frac{1}{2} + n)\mathbf{K}$ the system possesses other two integrals of motion, a local, second order integral $\mathcal{Q}_1$ and nonlocal $\mathcal{Q}_2 = i\mathcal{R}\sigma_1\mathcal{Q}_1$, which are odd with respect to $\mathcal{R}\sigma_1$, $\{\mathcal{R}\sigma_1, \mathcal{Q}_a\} = 0$,

$$\mathcal{Q}_1 = i \left(\begin{array}{cc} 0 & \mathcal{Y}^\dagger(x) \\ -\mathcal{Y}(x) & 0 \end{array} \right),$$

where $\mathcal{Y}(x) = \left(1/\mathrm{sn}\, x_- \frac{d}{dx} \mathrm{sn}\, x_-\right) \left(\mathrm{cn}\, x_+ \frac{d}{dx} 1/\mathrm{cn}\, x_+\right)$, $x_\pm = x \pm \frac{1}{2}\mathbf{K}$.
Integrals $\mathcal{L}_a$ are not independent anymore, $\mathcal{L}_a = -S_1\mathcal{Q}_a$.

$$[\mathcal{R}\sigma_1, \mathcal{Q}_a] = -2i\epsilon_{ab}\mathcal{Q}_b\,, \qquad \{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}\hat{P}_{\mathcal{Q}}(S_1)\,,$$

where $\hat{P}_{\mathcal{Q}}(S_1) = (S_1^2 - k'^2)(S_1^2 - 1)$.

• $\mathcal{L}_a$ are the supercharges for H_{BdG} which annihilate all the six edge eigenstates of $H_{BdG} = S_1$ in the case of $\tau \neq (\frac{1}{2} + n)\mathbf{K}$. Anticommutator of supercharges is the sixth order spectral polynomial in the Hamiltonian S_1.

• For $\tau = (\frac{1}{2} + n)\mathbf{K}$, supercharges $\mathcal{Q}_a$ annihilate all the four edge states of S_1; $\mathcal{L}_a = -S_1\mathcal{Q}_a$ annihilate the two zero energy eigenstates in the middle of the central band. Anticommutator of the supercharges is the fourth order spectral polynomial in the Hamiltonian S_1.

• The cases $\tau \neq (\frac{1}{2}+n)\mathbf{K}$ and $\tau = (\frac{1}{2}+n)\mathbf{K}$ correspond to $m_0 \neq 0$ and $m_0 = 0$ in the Gross-Neveu model Lagrangian

$$\mathcal{L}_{GN} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi + \frac{1}{2}g^2(\bar{\psi}\psi)^2.$$

At zero value of the bare mass, $m_0 = 0$, discrete chiral symmetry is restored there.

! $\Rightarrow$ A restoration of the discrete chiral symmetry at zero value of the bare mass in GN model, when the kink-antikink crystalline condensate transforms into the kink crystal, is accompanied by structural changes of the exotic supersymmetry.

• The origin of the hidden supersymmetric structures in some systems via a nonlocal Foldy-Wouthuysen transformation: V. Jakubsky, L.-M. Nieto, MP, 2010, PLB, 692, 51

• Exotic supersymmetry of finite-gap reflectionless Pöschl-Teller system in the light of Aharonov-Bohm effect and AdS/CFT holography: F. Correa, V. Jakubsky, MP, 2009, Ann. Phys. 324, 1078

• Exotic SUSY admits extension to *PT symmetric quantum systems* where it sheds a new light on peculiar properties of the complexified Scarf II potential (which appears in quantum field theory in curved spacetimes, soliton theory in nonlinear integrable systems and also in the physics of optical solitons) [F. Correa, MP, 2012, Ann. Phys. 327, 1761]

THANK YOU!