Lie–Jordan Banach algebras and Quantum Reduction

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Introduction

1. The algebraic structure of mechanics

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3. Quantum Reduction
   - Reduction of C*–algebras
   - Reduction of Lie–Jordan algebras

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Joint work with F. Falceto, A. Ibort and G. Marmo.

Motivations

- Understand the quantization procedure and the classical limit of quantum mechanics

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- Understand the quantization procedure and the classical limit of quantum mechanics
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- Motivational examples: the gravitational field may be regarded as a constrained particle moving on the infinite-dimensional space of metrics (superspace); the configuration space of a Yang-Mills theory is that of a particle moving on the infinite-dimensional space $\mathcal{A}/G$,where $\mathcal{A}$ is the space of smooth connections on some bundle and G is the corresponding group of local gauge transformations.

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Quantum mechanics is the space of self-adjoint operators $\mathcal{L} = \mathcal{B}_{sa}$ on some Hilbert space $\mathcal{H}$ (closed under operator multiplication) with
$a *_s b = \frac{1}{2}(ab + ba)$ anticommutator (Jordan product)
$a *_a b = \frac{i\lambda}{2}(ab - ba)$ scaled commutator (Lie bracket).

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These axioms define the so-called Lie–Jordan algebra.

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In the case of Quantum Mechanics $\hbar\lambda = \frac{1}{4}$.
One can always define an associative product on $\mathcal{L} \otimes \mathbb{C} = \mathcal{L} + i\mathcal{L}$ by putting

$$a * b = a *_s b - i\sqrt{\hbar}\; a *_a b$$

but this product lacks direct physical meaning, as the product of two observable operators fails to be observable.
The observables are closed under $*_s$ and $*_a$.
The symmetric product $*_s$ leads to spectral calculus and the antisymmetric $*_a$ expresses the dual role of observables: as observable and as generators of dynamics.

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The symmetric product $*_s$ leads to spectral calculus and the antisymmetric $*_a$ expresses the dual role of observables: as observable and as generators of dynamics.
In the following $*_s$ will be denoted by $\circ$ and $*_a$ by $[\cdot,\cdot]$.

Space of states and Jordan–Banach algebras

The state space $\mathcal{S}$ of the quantum system does not determine univocally the $\mathbb{C}^*$ structure of the algebra of observables but only its Jordan–Banach real algebra.

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It follows from results of Kadison (Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc. 1951) that the self-adjoint part of a $\mathbb{C}^*$–algebra is isometrically isomorphic, as an ordered normed linear space, to the space of all w*-continuous affine functions on the state space.

In view of this, characterizing the self-adjoint part of a $\mathbb{C}^*$–algebra is equivalent to characterizing the state space of a $\mathbb{C}^*$–algebra.

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Then the question of when a given Jordan–Banach algebra is the real part of a $\mathbb{C}^*$–algebra raises. A. Connes and Alfsen & Shultz gave different answers:

A. Connes, *Charactérisation des espaces vectoriels ordonnés sous-jacent aux algébres de von Neumann*, Ann. Inst. Fourier (Grenoble) **24** (1974), 121. E. M. Alfsen, F. W. Shultz, *On Orientation and Dynamics in Operator Algebras. Part I*, Commun. Math. Phys. **194** (1998) 87.

Jordan–Banach algebras and Order Derivations

A Jordan–Banach algebra is a Jordan algebra $(\mathcal{L}, \circ)$ with a complete norm $\|\cdot\|$ such that $\forall\, a, b \in \mathcal{L}$:

I) $\|a \circ b\| \leq \|a\|\,\|b\|$

II) $\|a^2\| = \|a\|^2$

III) $\|a^2\| \leq \|a \circ a + b \circ b\|$

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A unital JB–algebra $\mathcal{L}$ is a complete order unit space with respect to the positive cone

$$\mathcal{L}^+ = \{\, a^2 \mid a \in \mathcal{L}\,\}.$$

Definition

A bounded linear operator δ on a JB–algebra $\mathcal{L}$ is called an order derivation if $\exp^{t\delta}(\mathcal{L}^+) \subset \mathcal{L}^+$, $\forall\, t \in \mathbb{R}$.

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Definition

An order derivation δ on a unital JB–algebra $\mathcal{L}$ is self-adjoint if there exists $a \in \mathcal{L}$ such that $\delta = \delta_a$ and is skew-adjoint if $\delta(1) = 0$.

Definition (Dynamical Correspondence)

A dynamical correspondence on a unital JB–algebra $\mathcal{L}$ is a linear map $\psi\colon a \to \psi_a$ from $\mathcal{L}$ into the set of skew order derivations on $\mathcal{L}$ s.t.

I) $\exists \kappa \in \mathbb{R}$ such that $\kappa\left[\psi_a, \psi_b\right] = -\left[\delta_a, \delta_b\right], \quad \forall\, a, b \in \mathcal{L}$, and

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Let $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}\,, \circ)$ be a LJB–algebra. Then $[a,\cdot]_{\mathcal{L}}$ is an order derivation on $\mathcal{L}$ $\forall\, a \in \mathcal{L}$.

Theorem

Let $\mathcal{L}$ be a unital JB–algebra. There exists a dynamical correspondence ψ on $\mathcal{L}$ if and only if $\mathcal{L}$ is a LJB–algebra with Lie product $[\cdot,\cdot]_{\mathcal{L}}$ such that

$$[a, b]_{\mathcal{L}} = \psi_a b$$

Dynamical Correspondence and Lie–Jordan Banach algebras

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Corollary

A unital JB–algebra $\mathcal{L}$ is Jordan isomorphic to the self-adjoint part of a $\mathbb{C}^*$–algebra if and only if it is a LJB–algebra.

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Let $\mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$. $\mathcal{D}$ is a subalgebra of $\mathcal{F}$ and is the largest non-unital $\mathbb{C}^*$–algebra in $\bigcap_{\omega \in \mathcal{S}_D} \ker \omega$.

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$$\mathcal{O} = \{\, F \in \mathcal{F} \mid FH \in \mathcal{D} \text{ and } HF \in \mathcal{D}, \quad \forall\, H \in \mathcal{D} \,\}$$

i.e. the largest set for which $\mathcal{D}$ is a bilateral ideal corresponds to the Lie normalizer of $\mathcal{D}$.
It follows that the maximal $\mathbb{C}^*$–algebra of physical observables determined by the constraints $\mathcal{C}$ is

$$\widetilde{\mathcal{F}} = \mathcal{O}/\mathcal{D}.$$

Reduction of Lie–Jordan algebras

Let $\mathcal{L}$ be a LJB–algebra $(\mathcal{L}, \circ, [\cdot,\cdot]_{\mathcal{L}})$.
Consider a closed Jordan ideal $\mathcal{J}$:

$$\forall a \in \mathcal{L},\ \forall x \in \mathcal{J}, \quad x \circ a \in \mathcal{J}$$

Then the quotient space

$$\widetilde{\mathcal{L}} = \mathcal{L}/\mathcal{J}$$

inherits a canonical LJB–algebra structure with respect to the quotient norm

$$\|\widetilde{a}\| = \inf_{b \in \mathcal{J}} \|a + b\|$$

Equivalence between the reduction of LJB– and $\mathbb{C}^*$– algebras

Lemma

Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of a LJB $\mathcal{V}$. Then $\mathcal{Z}^{\mathbb{C}} = \mathcal{Z} \oplus i\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}^{\mathbb{C}} = \mathcal{I} \oplus i\mathcal{I}$ if and only if $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$, i.e. $\mathcal{Z} = \mathcal{N}_{\mathcal{I}}$.

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Lemma

Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of $\mathcal{V}$. Then $\mathcal{I}$ is a Lie–Jordan ideal of $\mathcal{Z}$ if and only if $\mathcal{I}^{\mathbb{C}} = \mathcal{I} \oplus i\mathcal{I}$ is an associative bilateral ideal of $\mathcal{Z}^{\mathbb{C}} = \mathcal{Z} \oplus i\mathcal{Z}$.

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Theorem

Let $\mathcal{F} = \mathcal{L} \oplus i\mathcal{L}$ be the field algebra of the quantum system and $\mathcal{C}$ a real constraint set. Let $\mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$, $\mathcal{O} = \mathcal{D}_W$ and $\widetilde{\mathcal{F}} = \mathcal{O}/\mathcal{D} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}$ be the reduced field algebra. Then:

$$\widetilde{\mathcal{L}} = \mathcal{N}_{\mathcal{J}}/\mathcal{J},$$

with $\mathcal{N}_{\mathcal{J}}$ and $\mathcal{J}$ being the s.a. part of $\mathcal{O}$ and $\mathcal{D}$ respectively, i.e. $\mathcal{O} = \mathcal{N}_{\mathcal{J}} \oplus i\mathcal{N}_{\mathcal{J}}$ and $\mathcal{D} = \mathcal{J} \oplus i\mathcal{J}$.

Equivalence between the reduction of LJB– and $\mathbb{C}^*$– algebras

$$
\begin{tikzcd}
\mathcal{L} \arrow[d, "\mathcal{J}\ \text{J. ideal},\ \mathcal{N}_{\mathcal{J}}"'] & \mathcal{F} = \mathcal{L} \oplus i\mathcal{L} \arrow[l] \arrow[d, "\mathcal{D}\ \text{ass. ideal},\ \mathcal{O} = \mathcal{D}_W = \mathcal{N}_{\mathcal{J}} \oplus i\,\mathcal{N}_{\mathcal{J}}"] \\
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\end{tikzcd}
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Is there a more general reduction?

A more general way to reduce a Lie–Jordan algebra $\mathcal{L}$ is to consider a linear subspace $B \subset \mathcal{L}$ and quotient it with respect to another subspace $S \subset \mathcal{L}$ s.t. $S \cap B \neq \{0\}$.
Which are the conditions to be imposed on B and S such that ${}^{B}\!/_{B \cap S}$ is a Lie–Jordan algebra?

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Which are the conditions to be imposed on B and S such that ${}^{B}\!/_{B \cap S}$ is a Lie–Jordan algebra?
The new products $\circ'$ and $[\cdot,\cdot]'$ on ${}^{B}\!/_{B \cap S}$ are

$$a' \circ' b' \equiv \widetilde{a} \circ \widetilde{b} + S$$
$$[a', b']' \equiv [\widetilde{a}, \widetilde{b}] + S,$$

where $a' = \widetilde{a} + S$ and $b' = \widetilde{b} + S$ with $\widetilde{a}, \widetilde{b} \in B$. For the definitions to make sense we must require:

$$B \circ B \subset B + S \qquad [B, B] \subset B + S$$
$$B \circ B \cap S \subset S \qquad [B, B \cap S] \subset S$$

But they do not satisfy in general the compatibility conditions.

Is there a more general reduction?

Observe that the linear space $B + S$ can be decomposed as a direct sum:

$$B + S = B \cap S \oplus \Sigma \oplus \Gamma,$$

with $\Sigma \subset B$, $\Sigma \cap S = \{0\}$ and $\Gamma \subset S$, $\Gamma \cap B = \{0\}$. From this decomposition we can also write

$$B = B \cap S \oplus \Sigma,$$

and then

$$B + S = B \oplus \Gamma.$$

A new framework for Quantum Anomalies?

Theorem

Let $(\mathcal{L}, \circ, [\cdot,\cdot])$ be a Lie–Jordan algebra and B and S two subsets such that $B \cap S \neq \{0\}$. Then $({}^{B}/_{B \cap S}, \circ', [\cdot,\cdot]')$ is a Lie–Jordan algebra if and only if there exist a subset $\Gamma \subset S$ such that $B + S = B \oplus \Gamma$ and the following conditions are satisfied:

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$$\begin{array}{ll} B \circ B \subset B \oplus \Gamma & [B, B] \subset B \oplus \Gamma \\ B \circ B \cap S \subset S & [B, B \cap S] \subset S \\ B \circ \Gamma \subset S & [B, \Gamma] \subset S \end{array} \tag{1}$$

If we imagine that S represents a symmetry which is not preserved at the quantum level (quantum anomaly), then conditions (1) tell us that twe can still construct a good algebra of observables by satisfying those properties.

Super Lie–Jordan Banach algebras

Graded Lie–Jordan Banach algebra $\mathcal{L} = \bigoplus_g \mathcal{L}_g$ together with two bilinear operations preserving the grading:

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and

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- $[a, b] = -(-1)^{|a||b|}[b, a]$
  $[a, [b, c]] + [c, [a, b]] + (-1)^{|a||b|}[b, [c, a]] = 0$ (Lie superalgebra)

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- $[a, b \circ c] = [a, b] \circ c + (-1)^{|a||b|} b \circ [a, c]$ (superderivation)

Super Lie–Jordan Banach algebras

Graded Lie–Jordan Banach algebra $\mathcal{L} = \bigoplus_g \mathcal{L}_g$ together with two bilinear operations preserving the grading:

$$\circ : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

and

$$[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

such that:

- $a \circ b = (-1)^{|a||b|} b \circ a$ (supercommutative superalgebra)
- $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$ (weak associativity)
- $[a, b] = -(-1)^{|a||b|}[b, a]$
  $[a, [b, c]] + [c, [a, b]] + (-1)^{|a||b|}[b, [c, a]] = 0$ (Lie superalgebra)
- $[a, b \circ c] = [a, b] \circ c + (-1)^{|a||b|} b \circ [a, c]$ (superderivation)
- $(a \circ b) \circ c - a \circ (b \circ c) = k\ [b, [c, a]]$, for some $k \in \mathbb{R}^+$ (associator identity)

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$$[\alpha, X] = \alpha(X) = [X, \alpha] \quad [X, Y] = 0 = [\alpha, \beta].$$
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We then extend it to all of $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ as an odd derivation. It is a $\mathbb{Z}$–graded SLJB–algebra with elements of $\mathfrak{g}$ having degree -1 and elements of $\mathfrak{g}^*$ having degree $+1$.

Action of a group on a Super Lie–Jordan Banach algebra

Let G be a Lie group acting on a SLJB–algebra $\mathcal{L}$, that is it exists a map:

$$\hat{g}\colon G \to \mathrm{Aut}(\mathcal{L})$$

which assigns to each element g of the group, an automorphism of the algebra $U(g)$.

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It is also possible to define the action of the Lie algebra $\mathfrak{g}$ on the LJB–algebra $\mathcal{L}$. Let $a \in \mathcal{L}$ and $\xi \in \mathfrak{g}$, then

$$\xi(a) = \frac{d}{ds} U(\exp^{s\xi})(a) \mid_{s=0}$$

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This action is a derivation of the algebra, that is:

$$\xi(a \circ b) = \xi(a) \circ b + a \circ \xi(b)$$

and then by the previous theorem on Jordan derivations there exists $J \in \mathcal{L}$ such that $\forall a \in \mathcal{L}$,

$$\xi(a) = [J, a].$$

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Example:

$\mathcal{C} = \mathcal{L} \otimes \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ is a $\mathbb{Z}$–graded SLJB–algebra:

$$\mathcal{C} = \bigoplus_n \mathcal{C}^n = \bigoplus_{i-j=n} \mathcal{C}^{i,j} = \bigoplus_{i-j=n} \Lambda^i(\mathfrak{g}^*) \otimes \Lambda^j(\mathfrak{g}) \otimes \mathcal{L}$$

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Although the bigrading is preserved by the exterior product, the Lie bracket does not preserve it, infact

$$[\mathcal{C}^{i,j}, \mathcal{C}^{k,l}] \subset \mathcal{C}^{i+k,j+l} \oplus \mathcal{C}^{i+k-1,j+l-1}$$

but the total degree is preserved.

Supercharge and BRST Reduction

A superderivation of degree k is a linear map $D\colon \mathcal{C}^n \to \mathcal{C}^{n+k}$ s.t.

$$D(a \circ b) = (Da) \circ b + (-1)^{k|a|} a \circ (Db), \quad D[a, b] = [Da, b] + (-1)^{k|a|} [a, Db].$$

The map $a \to [Q, a]$ for some $Q \in \mathcal{C}^k$ is a superderivation.

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The total differential $D = [Q, \cdot]$, where $Q \in \mathcal{C}^1$ is given explicitly by:

$$Q = J_i \theta^i - \frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k \wedge X_i$$

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where X_i (antighosts) is a basis of $\mathfrak{g}$ with $[X_i, X_j] = c^k_{ij} X_k$ and θ^i (ghosts) is a basis for $\mathfrak{g}^*$.It is also verified that $[Q,Q] = 0$ which is equivalent to $D^2 = 0$ and then we have defined our graded complex $(\mathcal{C}, D)$.

Theorem (BRST reduction)

The zero-th class cohomology of the graded complex $(\mathcal{C}, D)$ is given by

$$H^0_D(\mathcal{C}) = \mathcal{N}_{\mathcal{J}}/\mathcal{J}$$