Algebraic Theory of Entanglement

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Outline

Introduction

- In studies of foundations of quantum theory, it is of interest to study mixed states and their origins.
- Focus has been on separable states and entropy created by partial tracing.
- But this method is not so good for identical particles as we will show.
- A much more universal construction is based on restrictions of states to subalgebras and the GNS construction.
- This talk will explain this approach

WHAT THEY DO

Separable state

Consider a bipartite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.
A vector state

$$|\psi\rangle = \sum_{i,j} \psi_{ij} |i\rangle \otimes |j\rangle$$

is said to be *separable* if it can be brought to the form

$$|\psi\rangle = |v\rangle \otimes |w\rangle .$$

Otherwise, it is said to be *entangled*.

Singular value decomposition (SVD)

A : $m \times n$ complex matrix. A can always be written in the form

$$A = UDV^\dagger$$

- U : $m \times m$, unitary (columns of U are eigenvectors of $AA^\dagger$).
- D : $m \times n$, diagonal, positive (eigenvalues of $\sqrt{A^\dagger A}$.)
- V : $n \times n$ unitary (columns of V are eigenvectors of $A^\dagger A$).

Schmidt decomposition

$$|\psi\rangle = \sum_{i,j} A_{ij}|i\rangle \otimes |j\rangle = \sum_{i,j} (UDV^\dagger)_{ij}|i\rangle \otimes |j\rangle$$

$$= \sum_{i,j,k,l} U_{ik} \underbrace{D_{kl}}_{=\lambda_k \delta_{kl}} V^\dagger_{lj}|i\rangle \otimes |j\rangle = \sum_k \lambda_k \left(\sum_i U_{ik}|i\rangle\right) \otimes \left(\sum_j \bar{V}_{jk}|j\rangle\right)$$

$$= \sum_k \lambda_k |k\rangle_A \otimes |k\rangle_B$$

Schmidt rank

The Schmidt rank is the number of nonzero $\lambda_k' s$.

- $|\psi\rangle$ separable precisely when Schmidt rank = 1.
- Reduced density matrix: $\rho_A = \text{Tr}_B \rho$.
- von Neumann entropy: $S(\rho) = -\text{Tr}\rho \log \rho$. We have $S(\rho_A) = S(\rho_B)$.
- $|\psi\rangle$ separable precisely when $S(\rho_A) = 0$.

WHAT WE DO

Our main motivation

There are certain situations where the use of partial trace may not be the “best thing to do”. An example of this is provided by the study of entanglement for systems of indistinguishable particles, where the notion of separability is more subtle.

Idea

Partial trace = Restriction

Thus consider two distinguishable particles A and B in a pure state $|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B$. Partial trace means restricting its density matrix to observables of subsystem A . They are of the form $K_A \otimes \mathbb{1}_B$. But for identical fermions....

Identical Fermions

For identical fermions a two particle state is a linear combination of vector states of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\phi\rangle)$$

and observables are all symmetric combinations $K \otimes L + L \otimes K$. Partial tracing has no physical meaning. How do we study the mixture created by observing only the single particle observables? $K \otimes \mathbb{1} + \mathbb{1} \otimes K$ or perhaps $L \otimes L$? We turn now to this problem.

Outline

C*-algebras

C*-algebras

Observables in quantum field theory come from C^*-algebras.
All finite-dimensional matrix algebras are C^*.

Representations of C^*-algebras

Given a state or density matrix on such an algebra, there is a way to recover the Hilbert space due to Gelfand, Naimark and Segal. We will explain it below. It is used widely in

- Quantum field theory.
- Statistical physics.
- Noncommutative geometry.

The GNS construction

- Let $\mathcal{A}$ be the C^*-algebra of observables and $\omega : \mathcal{A} \to \mathbb{C}$ a state. That is $\omega(\alpha)$ is a complex number, $\omega(\alpha^*\alpha) \geq 0$, and $\omega(\mathbb{1}_{\mathcal{A}}) = 1$.
- Regard $\mathcal{A}$ as a vector space: $\alpha \to |\alpha\rangle$.
- Introduce a scalar product: $\langle\alpha|\beta\rangle = \omega(\alpha^*\beta)$
- This space can have a subspace $\mathcal{N}$ of vectors of 0 norm: $\mathcal{N} = \{\alpha \in \mathcal{A} \mid \langle\alpha|\alpha\rangle = 0\}$. $\mathcal{N}$ is a left ideal: $\beta\mathcal{N} = \mathcal{N}$. Proved easily using Schwartz inequality.
- The Hilbert space is: $\mathcal{H}_\omega = \mathcal{A}/\mathcal{N}$. An element of this space is the equivalence class $[\alpha] = \alpha + \mathcal{N}$ for α in $\mathcal{A}$.
- The representation π_ω of $\mathcal{A}$ on this space is given by $\pi_\omega(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$

von Neumann entropy and the GNS construction

- Fact: $\mathcal{H}_\omega$ irreducible precisely when ω pure.
- The state ω corresponds to the density matrix $\rho_\omega = |[\mathbb{1}]\rangle\langle[\mathbb{1}]|$.
- $\mathcal{H}_\omega = \oplus_j \mathcal{H}_j$, where each $\mathcal{H}_j$ carries an irreducible representation.
- If P_j's are projectors from $\mathcal{H}_\omega$ to $\mathcal{H}_j$, set $|[\mathbb{1}_j]\rangle = \frac{P_j|[\mathbb{1}]\rangle}{\|P_j|[\mathbb{1}]\rangle\|}$.
- Define $\mu_j = \|P_j|[\mathbb{1}]\rangle\|$. Then , $\sum_j \mu_j^2 = 1$. Finally,

$\rho_\omega = \sum_j \mu_j^2 |[\mathbb{1}_j]\rangle\langle[\mathbb{1}_j]|$. This state is mixed if its rank exceeds 1. It has entropy

$$S = -\sum_j \mu_j^2 \log \mu_j^2 .$$

Outline

A simple example

- $\mathcal{A} = M_2(\mathbb{C})$.
- $\omega_\lambda : \mathcal{A} \to \mathbb{C}, \quad \omega_\lambda(\alpha) = \lambda\alpha_{11} + (1-\lambda)\alpha_{22}, \quad 0 \leq \lambda \leq 1.$
- The null space $\mathcal{N}_{\omega_\lambda}$ is generated by those α such that

$$\omega_\lambda(\alpha^*\alpha) = 0.$$

- Explicit form of this condition:

$$\lambda(|\alpha_{11}|^2 + |\alpha_{21}|^2) + (1-\lambda)(|\alpha_{12}|^2 + |\alpha_{22}|^2) = 0. \qquad (1)$$

- Each value of λ gives a GNS-representation. There are two inequivalent cases: $0 < \lambda < 1$ and $\lambda = 0$ (or 1).
- Notation:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ etc..}$$

Case 1: $0 < \lambda < 1$.

- For $0 < \lambda < 1$ the only solution to (1) is $\alpha = 0$ $\Rightarrow \mathcal{N}_{\omega_\lambda} = \{0\}$.
- GNS-space given by $\mathcal{H}_{\omega_\lambda} \cong \mathbb{C}^4$.
- The e_{ij} act on this space as: $\pi_{\omega_\lambda}(e_{ij})|[e_{kl}]\rangle = \delta_{jk}|[e_{il}]\rangle$. For instance, the matrix of $\pi_{\omega_\lambda}(e_{11})$ is:

$$\pi_{\omega_\lambda}(e_{11}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- $\mathcal{H}_{\omega_\lambda} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$, with $\mathcal{H}^{(l)}$ $(l = 1, 2)$ invariant and spanned by $\{|[e_{kl}]\rangle\}_{k=1,2}$.

Case 1: $0 < \lambda < 1$.

- Decomposing ω_λ into pure states ($\mathbb{1}_{\mathcal{A}} = \mathbb{1}_2$):

$$|[\mathbb{1}_{\mathcal{A}}]\rangle \;=|[e_{11}]\rangle + |[e_{22}]\rangle$$

- The norms of the two components are $\sqrt{\lambda}$ and $\sqrt{1-\lambda}$, so

$$|[\mathbb{1}_{\mathcal{A}}]\rangle \;=\; \sqrt{\lambda}|[\chi_1]\rangle + \sqrt{1-\lambda}|[\chi_2]\rangle, \quad \langle[\chi_i]|\chi_j]\rangle = \delta_{ij}.$$

- It follows that

$$\rho_{\omega_\lambda} \;=\; \lambda|[\chi_1]\rangle\langle[\chi_1]| + (1-\lambda)|[\chi_2]\rangle\langle[\chi_2]|,$$

  so that ω_λ is not pure.
- It has von Neumann entropy

$$S(\omega_\lambda) = -\lambda \log \lambda - (1-\lambda)\log(1-\lambda).$$

Case 2: $\lambda = 0$.

- If we choose $\lambda = 0$, from (1) we see that $\mathcal{N}_{\omega_\lambda} \cong \mathbb{C}^2$, since it is spanned by elements of the form

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix},$$

that is, by linear combinations of $|e_{11}\rangle$ and $|e_{21}\rangle$.

- Accordingly, the GNS-space $\mathcal{H}_{\omega_\lambda} = \hat{\mathcal{A}}/\mathcal{N}_{\omega_\lambda} \cong \mathbb{C}^2$ is generated by $|[e_{12}]\rangle$ and $|[e_{22}]\rangle$. In this case the representation of $\mathcal{A}$ is irreducible and given by 2×2 matrices $\pi_{\omega_\lambda}(e_{ij})$:

$$\pi_{\omega_\lambda}(e_{ij})|[e_{k2}]\rangle = \delta_{jk}|[e_{i2}]\rangle.$$

- The state ω_λ is pure with zero entropy.
- A similar situation is found for $\lambda = 1$.

Outline

IDENTICAL PARTICLES: FERMIONS

One-particle space

- Single-particle space:

$$\mathcal{H} \simeq \mathbb{C}^d.$$

- Symmetry group:

$$U(d).$$

- Algebra of observables: $\mathcal{A}$, given by a $*$-representation of $\mathbb{C}U(d)$ for the group algebra:

$$\widehat{\alpha} = \int_{U(d)} d\mu(g)\alpha(g)U^{(1)}(g),$$

IDENTICAL PARTICLES: FERMIONS

Many-particle space

Consider now a fermion system whose single-particle space is given by $\mathcal{H} = \mathbb{C}^d$:

- Many particle space $\mathcal{F}$ is the "Fock" space:

$$\mathcal{F} = \bigoplus_{k=0}^{d} \mathcal{H}^{(k)}, \text{ where } \mathcal{H}^{(k)} \equiv \Lambda^k \mathcal{H}.$$

- $\Lambda^0 \mathcal{H} = \mathbb{C}$, generated by the "vacuum" $|\Omega\rangle$.
- $\Lambda^1 \mathcal{H}$: 1-particle space, $\Lambda^2 \mathcal{H}$: 2-particle space, and so on..
- dim $\Lambda^k \mathcal{H} = \binom{d}{k}$, so dim $\mathcal{F} = 2^d$. This reflects the fact that $\mathcal{F} \cong \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, an isomorphism often used in statistical physics models.

$\{|e_1\rangle, |e_2\rangle, \ldots |e_d\rangle\}$: Orthonormal basis for $\mathcal{H}$. Given $v_i \in \mathcal{H}$ $(i = 1, \ldots, k)$, put

$$|v_1 \wedge \ldots \wedge v_k\rangle = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_1 \otimes \cdots \otimes v_k.$$

Then:

- $\{|e_{i_1} \wedge \ldots \wedge e_{i_k}\rangle\}_{1 \leq i_1 < \cdots < i_k \leq d}$: Orthonormal basis for $\mathcal{H}^{(k)}$.
- A self-adjoint operator A acting on $\mathcal{H}^{(1)} \equiv \mathcal{H}$ can be made to act on $\mathcal{H}^{(k)}$ by defining $d\Gamma^k(A)$ follows:

$$d\Gamma^k(A) = A \otimes \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d + \mathbb{1}_d \otimes A \otimes \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d + \cdots + \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d \otimes A$$

- This operator preserves the symmetry of the states on which it acts, as well as the commutation relations of the self-adjoint operators acting on $\mathcal{H}$, namely:

$$d\Gamma^k([A, B]) = [d\Gamma^k(A), d\Gamma^k(B)]$$

- $d\Gamma(A) = \sum_k d\Gamma^k(A)$ acts on the whole Fock space $\mathcal{F}$ (it is the "second quantized" form of A).
- At the group level, we may consider exponentials of such operators, of the form $e^{id\Gamma(A)}$. These are unitary operators acting on $\mathcal{F}$. Let $U = e^{iA}$ be a unitary operator acting on $\mathcal{H}$. The global version of $d\Gamma$ is given by:

$$\Gamma^k U = U \otimes \cdots \otimes U.$$

- We then have, with $\Gamma(U) = \sum_k \Gamma^k(U)$,

$$\Gamma(e^{iA}) = e^{id\Gamma(A)}.$$

- Following the previous remarks, we see that operators of the form
$$\widehat{\alpha}^k = \int_{U(d)} d\mu(g)\alpha(g) g\otimes\cdots\otimes g \quad (\ k\text{-fold product},\ \ g \in U(d)\ ),$$
act properly on $\mathcal{H}^{(k)}$.
- All of this can be conveniently expressed in terms of a *coproduct*: approach based on Hopf algebras (can easily include braid-group statistics).
- Here, the construction of the observable algebra corresponds to the following simple choice for the coproduct Δ:
$$\Delta(g) = g\otimes g, \quad g \in U(d), \tag{2}$$
linearly extended to all of $\mathbb{C}U(d)$. This choice fixes the form of $\widehat{\alpha}^k$.

Δ is a homomorphism from the single-particle algebra to the two-particle Hilbert space.

So it makes sense to identify its image with observations of single-particle observables.

An example with fermions

- Consider the 2-fermion space $\mathcal{H}^{(2)}$ for the case $d = 3$.
- Let the action of $U(3)$ on $\mathcal{H} = \mathbb{C}^3$ be given by the defining representation. Call it $U^{(1)}$. (Hence $U^{(1)}(g) = g$).
- We have $U^{(1)}(g)|e_i\rangle = \sum_{j=1}^{3} D(g)_{ij}|e_j\rangle$, for a fixed ONB $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$.
- The action of $\mathbb{C}U(3)$ is then given by the 3-dimensional conjugate representation ($3 \otimes 3 = 6 \oplus \bar{3}$).
- $\bar{3}$ is the antisymmetric $3 \otimes_A 3$.
- The basis vectors of $\bar{3}$ are $|f^k\rangle := \varepsilon^{ijk}|e_i \wedge e_j\rangle$ ($k = 1, 2, 3$).

- Action of $\mathbb{C}U(3)$ on $\mathcal{H}^{(1)}$: $\widehat{\alpha} = \int_{U(3)} d\mu(g)\alpha(g)D^{(3)}(g)$.
- Basis: the 8 Gell-Mann matrices $\{T_i\}_i$ plus $\mathbb{1}_3$.
- On $\bar{3}$, they become $\widetilde{T}_j = -\overline{T}_j$ (use Δ and restrict to $\bar{3}$).
- This amounts to consider only the action of the operators

$$\widehat{\alpha}^2 = \int_{U(3)} d\mu(g)\alpha(g)D^{(3)}(g) \otimes D^{(3)}(g)$$

  on the (invariant) subspace generated by the antisymmetric vectors $|f^k\rangle = \varepsilon_{ijk}|e_i \wedge e_j\rangle$ $(k = 1, 2, 3)$.
- Thus, one-particle observables acting on the two-particle (fermionic) sector are generated by the matrices $\{\widetilde{T}_j \equiv -\overline{T}_j\}_{j=1,\ldots 8}$ of $\bar{3}$, plus the identity.

Summarizing:

The algebra of operators $\mathcal{A}$ acting on the 2-particle sector $\mathcal{H}^{(2)} = \Lambda^2\mathbb{C}^3$ is the matrix algebra generated by $\{\widetilde{T}_1, \ldots, \widetilde{T}_8, \mathbb{1}_3\}$.

If we now assume that we only have access to the observables pertaining to the states $|e_1\rangle$ and $|e_2\rangle$, then the relevant algebra of operators will be a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, namely the one generated by $\{\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3, \mathbb{1}_2, \mathbb{1}_3\}$.

In general we expect that a general (2-particle) pure state, given by a state vector

$$|\psi\rangle = \sum_{k=1}^{3} \psi_k |f^k\rangle = \psi_1 |e_2 \wedge e_3\rangle + \psi_2 |e_3 \wedge e_1\rangle + \psi_3 |e_1 \wedge e_2\rangle$$

may become *mixed* when restricted to $\mathcal{A}_0$.

In order to detect this “entanglement” we perform the GNS construction when the state $|\psi\rangle\langle\psi|$ is restricted to $\mathcal{A}_0$.

We put $\omega_{\psi,o} = \omega_\psi \mid_{\mathcal{A}_0}$. The GNS construction furnishes a representation $\pi : \mathcal{A}_0 \to B(\mathcal{H}_{\omega_{\psi,o}})$, as explained before.

In general, $\mathcal{H}_{\omega_{\psi,o}}$ will split as a sum of irreducibles of $\mathcal{A}_0$. This reducibility reflects the fact that, when restricted to $\mathcal{A}_0$, the original state ω_ψ might become *mixed*. The entropy of this state can then be computed from the decomposition of $\mathcal{H}_{\omega_{\psi,o}}$ into irreducibles.

Example:

$$
\begin{aligned}
|\psi\rangle &= \cos\theta|f^1\rangle + \sin\theta|f^3\rangle \\
&\equiv \cos\theta|e_2 \wedge e_3\rangle + \sin\theta|e_1 \wedge e_2\rangle
\end{aligned}
$$

Null states

The number of *null states* in the GNS construction based on $\omega_{\psi,o}$ depends on the specific value of θ:

- For $\theta = 0$, there will be 3 null states.
- For $0 < \theta < \pi/2$ we find 2 null states.
- For $\theta = \pi/2$ we will get 4 null states.

von Neumann entropy of $\omega_{\psi,0}$:

$$S(\theta) = \cos^2\theta \log \frac{1}{\cos^2\theta} + \sin^2\theta \log \frac{1}{\sin^2\theta}$$

Dimensions

$$\mathcal{H}^{\theta}_{GNS} \cong \begin{cases} \mathbb{C}^2, & \theta = 0 \\ \mathbb{C}^3 \cong \mathbb{C}^2 \oplus \mathbb{C}, & \theta \in (0, \pi/2) \\ \mathbb{C}, & \theta = \pi/2. \end{cases}$$

Partial trace vs. GNS

- In the previous example (two fermions with single-particle space $\mathbb{C}^3$), we obtain a θ-dependent entropy, whereas partial trace entropy always gives log 2 (since Slater rank 1, independently of θ).
- If $\mathbb{C}^4$ describes single-particles, there are pure states of Slater rank 1 with zero for GNS entropy, log 2 for partial trace entropy (next example).
  The former is more reasonable, the state being least entangled.

Two Fermions, $\mathcal{H}^{(1)} = \mathbb{C}^4$.

Using creation/annihilation operators

- Fermionic creation/annihilation operators: $a_\sigma^{(\dagger)}, \quad b_\sigma^{(\dagger)}$.
- · $a \leftrightarrow$ *'left'*, $\quad b \leftrightarrow$:*'right'*.
  · $\sigma = 1, 2 \leftrightarrow$ spin up/down.
- Basis for $\mathcal{H}^{(2)}$: $\quad a_1^\dagger a_2^\dagger|\Omega\rangle,\ b_1^\dagger b_2^\dagger|\Omega\rangle,\ a_\sigma^\dagger b_{\sigma'}^\dagger|\Omega\rangle\ (\sigma, \sigma' \in \{1, 2\})$.

Algebras

- $\mathcal{A} = M_6(\mathbb{C})$ $(4 \otimes 4 = 10 \oplus 6)$.
- $\mathcal{A}_0$: *one-particle observables, left location.*

State

$$|\psi_\theta\rangle = \left(\cos\theta a_1^\dagger b_2^\dagger + \sin\theta a_2^\dagger b_1^\dagger\right)|\Omega\rangle.$$

Basis for $\mathcal{A}_0$:

$\mathcal{A}_0$ is the six-dimensional algebra generated by

$$\mathbb{1}_{\mathcal{A}}, \quad T_1 := \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad T_2 := -\frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1),$$
$$T_3 := \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \; n_{12} := (a_1^\dagger a_1 a_2^\dagger a_2), \; N_a := (a_1^\dagger a_1 + a_2^\dagger a_2).$$

Entropy

$$S(\theta) = -\cos^2\theta \log\cos^2\theta - \sin^2\theta \log\sin^2\theta.$$

Dimensions

$$\mathcal{H}_\theta \cong \begin{cases} \mathbb{C}^2, & \theta = 0, \pi/2 \\ \mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2, & \theta \in (0, \pi/2). \end{cases}$$

Remarks

- The significant aspect of this example is the fact that for the values of θ for which the Slater rank of $|\psi_\theta\rangle$ is one ($\theta = 0$ and $\frac{\pi}{2}$), we obtain exactly zero for the entropy.
- In previous treatments of entanglement for identical particles, the minimum value for the von Neumann entropy of the reduced density matrix (obtained by partial trace) has been found to be $\log 2$.
- This has been a source for controversy.
- Different entanglement criteria: non-identical particles, bosons, fermions..

An example with bosons.

- $\mathcal{H}^{(1)} = \mathbb{C}^3$, with orthonormal basis$\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$.
- The two-boson space $\mathcal{H}^{(2)}$: symmetrized vectors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ $(3 \otimes 3 = 6 \oplus \bar{3})$.
- $\mathcal{A}_0$: subalgebra of one-particle observables pertaining *only* to the one-particle vectors $|e_1\rangle$ and $|e_2\rangle$.
- $|\psi_{(\theta,\phi)}\rangle =$ $\sin\theta\cos\phi|e_1 \vee e_2\rangle + \sin\theta\sin\phi|e_1 \vee e_3\rangle + \cos\theta|e_3 \vee e_3\rangle$.

Dimensions

The 6 $SU(3)$ representation in $3 \otimes 3 = 6 \oplus \bar{3}$ decomposes as

$$6 = 3 \oplus 2 \oplus 1$$

with respect to $\mathcal{A}_0$. From this we can read off the decomposition of $\mathcal{H}_{(\theta,\phi)}$ into irreducibles, depending on the coefficients of $|\psi\rangle$.

Entropy

$$S(\theta,\phi) = -\sin^2\theta[\cos^2\phi\log(\sin\theta\cos\phi)^2 + \sin^2\phi\log(\sin\theta\sin\phi)^2] \\ - \cos^2\theta\log(\cos\theta)^2.$$

$S(\theta, \phi)$

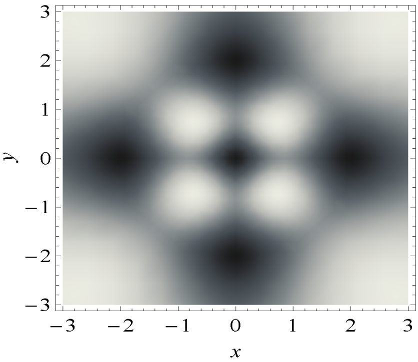

Figure: The two-boson entropy as a function of x and y which represent the (θ, ϕ)-sphere via stereographic projection. Darker regions correspond to lower values of the entropy.

CONCLUSIONS

- GNS-based approach: generalizes partial trace.
- Applications to entanglement of indistinguishable particles.
- Applications to quantum phase transitions?
- Applications to black hole physics?

$3 \otimes 3 = 6 \oplus \bar{3}$ in detail:

- $|u\rangle \otimes |v\rangle$ in $3 \otimes 3$, with $|u\rangle = \sum_{i=1}^{3} u^i |e_i\rangle$ and $|v\rangle = \sum_{j=1}^{3} v^j |e_j\rangle$.
- Decomposition into irreducible components:

$$\sum_{i,j=1}^{3} u^i v^j |e_i\rangle \otimes |e_j\rangle = \sum_{i \le j} \frac{1}{2}(u^i v^j + u^j v^i)(|e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle) + \sum_{i<j} \frac{1}{\sqrt{2}}(u^i v^j - u^j v^i)|e_i \wedge e_j\rangle.$$

- First term: symmetric, 6-dimensional irreducible representation.
- Second term: antisymmetric, 3-dimensional *complex* irreducible representation, with basis vectors $|e^k\rangle := \varepsilon^{ijk}|e_i \wedge e_j\rangle$ ($k = 1, 2, 3$), as stated above.

$$T_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, T_5 = \begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}, T_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} \\ 0 & \frac{i}{2} & 0 \end{pmatrix}, T_8 = \begin{pmatrix} \frac{\sqrt{3}}{6} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{6} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{3} \end{pmatrix}, \mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$