FROM THE EQUATIONS OF MOTION TO THE CANONICAL COMMUTATION RELATIONS

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Abstract

The problem of whether or not the equations of motion of a quantum system determine the commutation relations was posed by E.P.Wigner in 1950. A similar problem (known as *"The Inverse Problem in the Calculus of Variations"*) was posed in a classical setting as back as in 1887 by H.Helmoltz and has received great attention also in recent times. The aim of this paper is to discuss how these two apparently unrelated problems can actually be discussed in a somewhat unified framework. After reviewing briefly the Inverse Problem and the existence of alternative structures for classical systems, we discuss the geometric structures that are intrinsically present in Quantum Mechanics, starting from finite-level systems and then moving to a more general setting by using the Weyl-Wigner approach, showing how this approach can accomodate in an almost natural way the existence of alternative structures in Quantum Mechanics as well.

Keywords: Classical and Quantum Alternative Structures; Wigner problem; Quantization; Geometric Quantum Mechanics

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1 Introduction and Motivations

1.1 Introductory Considerations

Back in 1950, E.P.Wigner [229] (see also Refs.[25, 150, 196]) raised the problem of whether the equations of motion determine or not the quantum commutation relations. A few papers [199, 235] followed immediately, and the same problem was considered by S.Schweber [211] in the framework of Quantum Field Theory. It also originated the interest for parastatistics [79, 91, 92]. Physicists were apparently motivated in this research by the search of a way out of the apparently uncontrollable divergences that were plaguing Relativistic Quantum Field Theory.

As reported by F.Dyson [63], also Feynman addressed the same problem, looking for commutation relations not associated with Lagrangian descriptions. One would have also avoided in this way [35] the introduction of gauge potentials. In the classical setting the problem, known as the *"Inverse Problem in the Calculus of Variations"* [186], was stated and clearly formulated already by H.Helmoltz [99]. An example of a system admitting of two alternative Hamiltonian descriptions had already been given by J.L.Lagrange [118] when dealing with linear problems.

With the advent of Relativity. T.Levi-Civita [127] considered a similar problem when looking for a Lagrangian description of massless particles in General Relativity. P.Bergmann also noticed, in his famous book on Relativity [21], that, when the Lagrangian function is itself a constant of the motion, as it happens, e.g., for geodesic motions in General Relativity, then any function of the Lagrangian can be shown to provide, under very mild assumptions, a possible alternative Lagrangian description of the same dynamical system.

Other motivations for interest in the same problem arose from the so-called "no-interaction theorem" [10, 46, 162] concerning the covariant canonical description of relativistic interacting particles [9]. Here too alternative Lagrangian descriptions were sought that could allow to evade the theorem [47]. The so-called "quadratic Hamiltonian theorem" [48] was also considered in the same spirit.

A complete mathematical investigation of the inverse problem was initiated by J.Douglas [58] (who was also one of the first Field medalists) back in 1941. Many investigators considered in particular the problem with reference to the Nöther theorem [1] connecting symmetries and constants of the motion [186].

A first differential-geometric formulation of the problem appeared in the mid-Seventies [151]. A few years later, R.M.Santilli [208] initiated a systematic presentation of the problem for both particles and fields.

The Inverse Problem arises quite naturally if one starts from the "experimentalist's" point of view [167] that the trajectories (think of the observations in a bubble-chamber experiment) are the first raw data that are provided by the direct observation of a dynamical evolution. It is therefore natural to start from the trajectories to build up a vector field and, afterwards, to look for Lagrangian and/or Hamiltonian descriptions. A first attempt in this direction had been made by E.K.Kasner [111] already in 1913.

As the "raw data" are usually given on some configuration space, the first problem one is faced with are the ambiguities that are present when trying to go from a second-order differential equation on a configuration space to a first-order one (i.e. a vector field) on a larger carrier space. This problem was analyzed in detail in Ref.[167].

To clearly identify and formulate the problem, it is very useful to consider linear dynamical systems first, and to investigate the existence of Hamiltonian descriptions from the point of view of Poisson brackets.

In this context, writing the equations of motion in Hamiltonian form, i.e.:

$$\left| \begin{array}{c} \frac{dq^i}{dt} \\ \frac{dp_i}{dt} \end{array} \right| = \left| \begin{array}{cc} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{array} \right| \left| \begin{array}{c} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{array} \right| \tag{1.1}$$

or, in collective coordinates:

$$\dot{\xi}^i = \Lambda^{ij}\frac{\partial H}{\partial \xi^j} = A^i{}_j \xi^j \tag{1.2}$$

amounts to looking for a decomposition [83] of the matrix representing the (linear) dynamics, say A, into the product of a skew-symmetric matrix Λ, which stays for the Poisson tensor and defines the Poisson brackets and, if it is non-singular, the symplectic structure, and of a symmetric matrix H which represents the Hamiltonian, i.e.:

$$A = \Lambda \cdot H \tag{1.3}$$

Out of all possible such decompositions we obtain all the alternative quadratic Hamiltonian descriptions for a given dynamical system. It is easy to realize (see below, Chapt.3) that all symmetries for A, once applied to the factorization, will take from one factorization to another one unless they correspond to canonical transformations.

When going from a linear vector space to a generic differentiable manifold, matrices should be replaced by tensor fields and, when "moving from a point" to a neighboring one, we will have to take into account also differential relations (partial differential equations will arise in addition to algebraic relations).

One may trace the existence of alternative Lagrangian and/or Hamiltonian descriptions to the existence of a large group of symmetries for the dynamics, some of them being non-canonical symmetries.

The most obvious transformation taking one Lagrangian into another one is a scale transformation. For instance, we might scale the mass in a Lagrangian containing only a kinetic term, or we could do that, thanks to the equivalence principle [21], for a massive particle moving in a gravitational field.

When moving to the quantum descriptions, it becomes already clear that the scaling of the Lagrangian will give rise to a selection of the "allowed" periodic motion within a Bohr-Sommerfeld quantization scheme which will depend on the scale. This is not surprising, as the Lagrangian times the period is measured in units of Planck's constant.

This observation shows that we should not expect that the quantum description of a dynamical evolution would trivially exhibit properties similar to the classical ones.

On the other hand, there is a strong belief that Classical Mechanics should be a suitable limit of Quantum Mechanics. To quote from Dirac's book [56]:

"Classical mechanics must be a limiting case of quantum mechanics. We should thus expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics and, from an understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well known results in classical mechanics."

This, along with the existence of alternative Hamiltonian descriptions for solitonic equations [180], strongly suggests that a proper formulation of bi-Hamiltonian descriptions should exist for quantum dynamical systems as well.

Here one can be more or less demanding. For instance, one may require that known situations of bi-Hamiltonian descriptions of specific classical dynamical systems be fully recovered in the quantum framework. As we shall see, these requirements may have far-reaching consequences in the acceptable formulations of Quantum Mechanics.

For instance, one of the fundamental principles of Quantum Mechanics as formulated by Dirac [56] is the existence of a superposition rule for wave functions in order to deal with interference phenomena. This is usually translated into the requirement [56] that the carrier space should be a vector space.

On the other hand, the approach in terms of C^*-algebras shows clearly that the Hilbert space we arrive at with the GNS construction [95] depends on the initial state we choose, which is obviously "prepared", so-to-speak, "in the laboratory".

A spin-off of this construction is also the need for a clear distinction between the "abstract" C^*-algebra and its specific realizations in terms of operators acting on the Hilbert space that results from the GNS construction.

Considering next more closely the Dirac prescription of replacing Poisson brackets with commutator brackets, one finds that, while in the classical case all possible Poisson brackets generate derivations for the pointwise product of functions on the carrier space (i.e. the classical observables), in the quantum setting another result by Dirac (see Chapt. IV of Ref.[56]) shows that the associative product of operators identifies completely (up to a scale factor) the associated Lie algebra structure (the commutator brackets). In some sense, therefore, the associative product and the Lie product strongly determine each other in the quantum case.

Many of these issues will be closely scrutinized in the present Report, which has been organized in the following way.

The remainder of this Chapter and Chapt.2 serve to, so-to-speak, "set the stage" for the analysis of the following Chapters, discussing, to begin with, how the Schrödinger equation can be recast in the form of a Hamiltonian system, both in the finite and the infinite-dimensional case, and how alternative Hamiltonian descriptions of the same quantum system can be generated. As bi-Hamiltonian systems are usually associated with complete integrability [50, 54, 134], Chapt.2 reviews some general problems concerning complete (Liouville) integrability and related invariant structures. In Chapt.3 we discuss the existence of alternative structures at the classical level starting, as anticipated in these introductory notes, with a discussion of the case of linear vector fields. Chapt.4 moves to the quantum setting. Also in order to set the problem within a framework similar to that of the classical case, and to take into account the fact that pure states in Quantum Mechanics are a manifold rather than a vector space, we begin with a discussion of how geometric (tensorial) structures that are somehow hidden by the linear vector space structure of the Hilbert space emerge nonetheless as fundamental structures. We emphasize there how the proper carrier space for quantum dynamical system is instead the (no more linear) complex projective space associated with the Hilbert space. We conclude by discussing here too possible bi-Hamiltonian descriptions of quantum systems and with a brief account of the extensions of the concepts developed along the Chapter to the infinite-dimensional case. In Chapt.5 we discuss the Wigner-Weyl approach to Quantum Mechanics, beginning with a review of the Weyl map, illustrated also with a good number of examples, we continue with the Wigner map, the Moyal product, Quantum Mechanics in phase space and we discuss also the quantum-classical transition. In the following Chapt.6 we discuss how one can induce either on the same space or on spaces that are diffeomorphically related alternative linear structures, i.e. linear structures on the same carrier space that are however not linearly related. We discuss how alternative linear structures can offer a way of "reformulating", in a sense explained in the text, the von Neumann uniqueness theorem [222], as well as their rôle in Statistical Mechanics. Chapt.7 contains some further generalizations and our concluding remarks.

In order to make the paper more readable, some technical matters have been discussed in details in the Appendices, that expert readers can of course skip reading.

1.2 The Schrödinger Equation as a (Classical) Dynamical System

1.2.1 The Finite-Dimensional case

We begin by considering the Schrödinger equation:

$$\frac{d}{dt}\psi(t) = -\frac{i}{\hbar}H\psi(t)\,; \quad \psi(0) = \psi \tag{1.4}$$

on a *finite-dimensional* (complex) Hilbert space $\mathcal{H}$, deferring the discussion of some infinite-dimensional examples to the end of this Chapter. Hence, for the time being: $\mathcal{H} \approx \mathbb{C}^n$ for some n, As $\mathcal{H}$ is a vector space, there is a natural identification of the tangent space at any point $\psi \in \mathcal{H}$ with $\mathcal{H}$ itself: $T_\psi \mathcal{H} \approx \mathcal{H}$. In other words, *vectors in a Hilbert space play*[1] *a double rôle*, as "points" in the space and as tangent vectors at a given point. Which rôle they play should be (hopefully) clear from the context. More generally, we have the identification: $T\mathcal{H} \approx \mathcal{H} \times \mathcal{H}$, with $T\mathcal{H}$ the tangent bundle of $\mathcal{H}$.

As in the case of differentiable manifolds, $\psi = \psi(t), \psi(0) = \psi$ will define a curve in $\mathcal{H}$, and hence the quantity $(d\psi(t)/dt)|_{t=0}$ will define the tangent vector at the curve at $\psi \in \mathcal{H}$. A smooth assignment of tangent vectors at every point $\psi \in \mathcal{H}$ will define then a *vector field*, i.e. a smooth (and global) section of $T\mathcal{H}$:

$$\Gamma : \mathcal{H} \to T\mathcal{H};\ \psi \mapsto (\psi, \phi), \psi \in \mathcal{H},\ \phi \in T_\psi \mathcal{H} \approx \mathcal{H} \tag{1.5}$$

where the second argument may depend in a smooth way on ψ and with the tangent bundle projection:

$$\pi : (\psi, \phi) \mapsto \psi \tag{1.6}$$

such that: $\pi \circ \Gamma = Id_{\mathcal{H}}$. We will employ the notation: $\Gamma(\psi)$ for the vector field evaluated at the point ψ with tangent vector at ψ given by Eqn.(1.5). The latter defines a flow on $\mathcal{H}$ determined by the differential equation:

$$\frac{d}{dt}\psi(t) = \phi(\psi(t)),\ \psi(0) = \psi \tag{1.7}$$

Every vector field will define a derivation on the algebra of functions just as in the case of real manifolds. Specifically, if: $\phi = (d\psi(t)/dt)|_{t=0},\ \psi(0) = \psi$ and: $f : \mathcal{H} \mapsto \mathbb{R}$ is a function, then, in intrinsic terms:

$$(\mathcal{L}_\Gamma(f))(\psi) = \frac{d}{dt} f(\psi(t))|_{t=0} \tag{1.8}$$

will define the Lie derivative along Γ on the algebra of functions.

In local coordinates, choosing, e.g., an orthonormal ($O.N.$ from now on) basis $\{e_i\}_i^n$ ($n = \dim \mathcal{H}$), vectors (and tangent vectors) will be represented by n-tuples of complex numbers ($\psi = \left(\psi^1, ..., \psi^n\right), \psi^j =: \langle e_j|\psi\rangle$ and so on), and[2]:

$$(\mathcal{L}_\Gamma(f))(\psi) = \phi^i(\psi)\frac{\partial f}{\partial \psi^i}(\psi) \tag{1.9}$$

Notice that, in the infinite-dimensional case (for a separable and infinite-dimensional Hilbert space), "functions" will become *functionals*, and ordinary derivatives will have to be replaced by properly defined functional derivatives.

Constant as well as *linear* (with respect to the linear structure identified by the vector space) vector fields will play a role in what follows. The former are characterized by: $\phi = const.$ in the second argument of Eqn.(1.5), and give rise to the one-parameter group:

$$\mathbb{R} \ni t \mapsto \psi(t) = \psi + t\phi \tag{1.10}$$

[1] As in any linear vector space.

[2] As ψ_j is complex: $\psi_j = q_j + ip_j,\ q_j, p_j \in \mathbb{R}$, the derivative here has to be understood simply as: $\partial/\partial\psi_j = \partial/\partial q_j - i\partial/\partial p_j$.

The latter are characterized instead by $\phi(\psi)$ being a linear and homogeneous function of ψ, i.e.: $\phi = A\psi$ for some linear operator A. Eqn.(1.7) integrates in this case to[3]:

$$\psi(t) = \exp\{tA\}\psi \tag{1.11}$$

Of particular interest is the *dilation vector field* Δ:

$$\Delta : \psi \mapsto (\psi, \psi) \tag{1.12}$$

which corresponds to: $A = Id_{\mathcal{H}}$. In this case Eqns.(1.7) and (1.11) become:

$$\frac{d}{dt}\psi(t) = \psi \Rightarrow \psi(t) = e^t\psi \tag{1.13}$$

Eqn.(1.12) exhibits clearly the fact that the dilation field leads to an identification of $\mathcal{H}$ with the fiber $T_\psi\mathcal{H}$. The latter carrying a natural linear structure, Eqn.(1.12) provides a tensorial characterization of the linear structure of the base space $\mathcal{H}$ by means of the vector field Δ. For more details, see, e.g., Ref.[52].

With every linear operator[4] $\mathbb{A}$ there is therefore associated the linear vector field:

$$\mathbb{X}_{\mathbb{A}} : \mathcal{H} \to T\mathcal{H};\ \psi \to (\psi, \mathbb{A}\psi) \tag{1.14}$$

In local coordinates, this vector field can be written as:

$$\mathbb{X}_{\mathbb{A}} =: A^i{}_j\psi^j\frac{\partial}{\partial\psi^i} \tag{1.15}$$

and is of course entirely defined by the representative matrix: $\mathbb{A} = \left\|A^i{}_j\right\|$ of the linear operator. In particular, then:

$$\Delta = \psi^i\frac{\partial}{\partial\psi^i} \tag{1.16}$$

Notice however that, while linear operators form an associative algebra, vector fields do not : they form instead only a Lie algebra. An associative algebra can be recovered by using the same matrix $\mathbb{A}$ to define instead the $(1,1)$ tensor[5]:

$$\mathbb{T}_{\mathbb{A}} =: A^i{}_j d\psi^j \otimes \frac{\partial}{\partial\psi^i} \tag{1.17}$$

Then it is easy to check that the vector field $\mathbb{X}_{\mathbb{A}}$ is recovered from $\mathbb{T}_{\mathbb{A}}$ and the dilation field as:

$$\mathbb{X}_{\mathbb{A}} = \mathbb{T}_{\mathbb{A}}(\Delta) \tag{1.18}$$

[3]in the finite-dimensional case there are of course no problems in exponentiating a linear operator.

[4]Not considering questions of domain, which are of no relevance in the finite-dimensional case.

[5]Notice that, while $\mathbb{X}_{\mathbb{A}}$ depends on the choice of the origin of the coordinates, $\mathbb{T}_{\mathbb{A}}$ does not, i.e. it has an affine character.

Coming back to the Schrödinger equation, the linear operator H will define a linear vector field that we will denote[6] for short as Γ_H:

$$\Gamma_H : \mathcal{H} \to T\mathcal{H}; \quad \Gamma_H : \psi \mapsto (\psi, -(i/\hbar) H\psi) \tag{1.19}$$

and then:

$$\mathcal{L}_{\Gamma_H}\psi \equiv \frac{d}{dt}\psi = -\frac{i}{\hbar}H\psi \tag{1.20}$$

In this sense, the Schrödinger equation (1.4) can be viewed as a classical evolution equation on a complex vector space.

At variance with the infinite-dimensional case, every linear vector field is complete in finite dimensions. Then, if in addition we require conservation of probability, Wigner's theorem [227] states that the associated one-parameter group has to be unitary[7] and, by Stone-von Neumann's theorem [201], H has to be essentially self-adjoint, i.e. it will be symmetric with a unique self-adjoint extension. In the sequel we will refer always to the latter, and will simply say that H is self-adjoint. In the finite-dimensional case no distinctions between Hermitian, symmetric and self-adjoint operators [201] need to be made, of course.

Let now:

$$h : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \tag{1.21}$$

be a Hermitian structure on $\mathcal{H}$, i.e. let:

$$h(\phi, \psi) =: \langle \phi | \psi \rangle \tag{1.22}$$

define an Hermitian scalar product on $\mathcal{H}$ with the usual properties, namely;

- $h(\phi, \psi) = \overline{h(\psi, \phi)}$
- $h(\phi, \phi) \geq 0, \;\; h(\phi, \phi) = 0 \leftrightarrow \phi = 0$
- $h(\lambda\phi, \psi) = \overline{\lambda} h(\phi, \psi), \;\; h(\phi, \lambda\psi) = \lambda h(\phi, \psi)$

Remark 1 *If h is viewed more properly as a $(0,2)$ tensor field, then ϕ and ψ in Eqn.(1.22) have to be viewed as tangent vectors at a point in $\mathcal{H}$, and a more complete (albeit a bit more cumbersome) notation should be:*

$$h(\varphi)\left(\Gamma_\phi(\varphi), \Gamma_\psi(\varphi)\right) = \langle \phi | \psi \rangle \tag{1.23}$$

[6]We use here the notation Γ_H instead of $\mathbb{X}_H$ as a reminder of the fact that we had to include the "extra" factor $(-i/\hbar)$ in its definition.

[7]To be a bit more precise, *pure states* in Quantum Mechanics are described by elements of the projective Hilbert space $P\mathcal{H}$ (for instance, one-dimensional projectors of the form: $P_\psi = |\psi\rangle\langle\psi| / \langle\psi|\psi\rangle, \; |\psi\rangle \in \mathcal{H}$. The Hermitian structure on $\mathcal{H}$ induces a binary product: $\langle ., . \rangle$ on $P\mathcal{H}$ via: $\langle P_\psi, P_\phi \rangle =: Tr\{P_\psi P_\phi\} = |\langle\phi|\psi\rangle|^2 / (\langle\phi|\phi\rangle \langle\psi|\psi\rangle)$ and yields a transition probability. Wigner's theorem states then that any bijective map on $P\mathcal{H}$ preserving transition probabilities can be realized as a unitary or anti-unitary transformation on the original Hilbert space.

where $h(\varphi)$ stands for h evaluated at point $\varphi \in \mathcal{H}$. As the r.h.s. of this equation does not depend on φ, this implies : $\mathcal{L}_{\Gamma_H} \langle\phi|\psi\rangle \equiv \mathcal{L}_{\Gamma_H}(h(\phi,\psi)) = 0$ and, using Eqn.(1.4):

$$\begin{aligned} 0 &= \mathcal{L}_{\Gamma_H}(h(\phi,\psi)) = (\mathcal{L}_{\Gamma_H} h)(\phi,\psi) + h(\mathcal{L}_{\Gamma_H}\phi,\psi) + h(\phi,\mathcal{L}_{\Gamma_H}\psi) = \\ &= (\mathcal{L}_{\Gamma_H} h)(\phi,\psi) + \frac{i}{\hbar}\{\langle H\phi|\psi\rangle - \langle\phi|H\psi\rangle\} \end{aligned} \tag{1.24}$$

which implies in turn, as H is self-adjoint, that:

$$\mathcal{L}_{\Gamma_H} h = 0 \tag{1.25}$$

i.e. that the Hermitian structure be invariant under the (unitary) flow of Γ_H (and viceversa), or, stated equivalently, that Γ_H be a Killing vector field for the Hermitian structure. If instead the Hermitian structure is not invariant, then H will fail to be self-adjoint w.r.t. the given Hermitian structure.

Remark 2 *A family of privileged (actually global) charts for $\mathcal{H}$, all unitarily related to each other, is provided by the choice of any O.N. basis $\{|k\rangle\}_i^n$, $\langle h|k\rangle = \delta_{hk}$. In any such basis: $h(\phi,\psi) =: \langle\phi|\psi\rangle = h_{ij}\overline{\phi^i}\psi^j$ with: $h_{ij} = \delta_{ij}$, and all the above statements (in particular Eqn.(1.25)) are self-evident. However, the statements of the previous Remark have a tensorial meaning. As such, they will remain true also under (possible) non-linear changes of coordinates.*

Remark 3 *We can decompose the Hermitian structure into real and imaginary parts as:*

$$h(.,.) = g(.,.) + i\omega(.,.) \tag{1.26}$$

where:

$$g(\phi,\psi) = \frac{1}{2}[\langle\phi|\psi\rangle + \langle\psi|\phi\rangle] \tag{1.27}$$

and:

$$\omega(\phi,\psi) = \frac{1}{2i}[\langle\phi|\psi\rangle - \langle\psi|\phi\rangle] \tag{1.28}$$

According to Eqn.(1.23) we may consider h as an Hermitian tensor. It is clear that both g and ω are $(0,2)$ tensors, and that g is symmetric, while ω is skew-symmetric, hence a two-form. Eqn.(1.25) implies then that both tensors are (separately) invariant under Γ_H. Notice that: $\omega(\phi,i\psi) = g(\phi,\psi)$. Hence, non-degeneracy of h entails separately that of ω and of g.

Remark 4 *The non-degenerate two-form ω will be represented, in any one of the privileged charts, by a constant (and unitarily invariant) matrix. Hence it will be closed:*

$$d\omega = 0 \tag{1.29}$$

But, again, we stress that an equation like Eqn.(1.29) has a tensorial meaning. Hence, ω will be a symplectic form, while g will be a (non-degenerate and constant in any privileged chart) metric tensor.

Let now Γ_H be a vector field of the form (1.19). Then, a little algebra shows that:

$$\left(i_{\Gamma_H}\omega\right)(\psi) = \omega\left(-\frac{i}{\hbar}H\phi, \psi\right) = \frac{1}{2\hbar}\left[\langle H\phi|\psi\rangle + \langle\psi|H\phi\rangle\right] \tag{1.30}$$

On the other hand, if we define the quadratic function:

$$f_H(\phi) = \frac{1}{2\hbar}\langle\phi|H\phi\rangle \tag{1.31}$$

we can define its differential as the one-form:

$$df_H(\phi) = \frac{1}{2}\left[\langle .|H\phi\rangle + \langle\phi|H.\rangle\right] = \frac{1}{2}\left[\langle .|H\phi\rangle + \langle H\phi|.\rangle\right] \tag{1.32}$$

the last passage following from H being self-adjoint. Therefore: $\left(i_{\Gamma_H}\omega\right)(\psi) = df_H(\phi)(\psi)\,\forall\psi$, and hence:

$$i_{\Gamma_H}\omega = df_H \tag{1.33}$$

i.e. Γ_H is *Hamiltonian* w.r.t. the symplectic structure with the quadratic Hamiltonian f_H.

As a further remark, we recall that $\mathcal{H}$ is endowed with a natural complex structure J defined simply by :

$$J : \phi \to i\phi \tag{1.34}$$

Then: $J^2 = -\mathbb{I}$ (the identity on $\mathcal{H}$) and:

$$\omega(\phi, J\psi) = g(\phi, \psi) \tag{1.35}$$

Therefore the complex structure J is *compatible*[160] with the pair (g, ω) and we can reconstruct the Hermitian structure as:

$$h(\phi, \psi) = \omega(\phi, J\psi) + i\omega(\phi, \psi) \tag{1.36}$$

or equivalently, as:

$$h(\phi, \psi) = g(\phi, \psi) - ig(\phi, J\psi) \tag{1.37}$$

Notice also that:

$$\omega(J\phi, J\psi) = \omega(\phi, \psi) \tag{1.38}$$

as well as:

$$g(J\phi, J\psi) = g(\phi, \psi) \tag{1.39}$$

We can summarize what has been proved up to now by saying that $\mathcal{H}$ is a Kähler manifold[40, 41, 224], and that h is the associated Hermitian metric, while g is the Riemannian metric and ω the fundamental two-form. As ω is closed, g is also [224] a Kähler metric.

Choosing[8] an $O.N.$ basis $\{|k\rangle\}_1^n$, $\langle h|k\rangle = \delta_{hk}$, the Hermitian product can be written as:

$$h(\phi, \psi) = \delta_{ij}\overline{\phi}^i\psi^j \tag{1.40}$$

[8]Of course the best choice would be a basis in which the Hamiltonian is diagonal.

where: $|\phi\rangle = \phi^k |k\rangle$, and similarly for ψ.

Writing: $\phi = \phi_1 + i\phi_2$, $\phi_{1,2} \in \mathbb{R}^n$, we can *realify* [4, 82] $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ via:

$$\mathbb{C}^n \ni \phi \to \begin{vmatrix} \phi_1 \\ \phi_2 \end{vmatrix} \tag{1.41}$$

In this way:

$$g(\phi, \psi) = \mathrm{Re}\left\{\delta_{ij}\overline{\phi}^i \psi^j\right\} = \begin{vmatrix} \phi_1 & \phi_2 \end{vmatrix} G \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix} \tag{1.42}$$

where G is the matrix:

$$G = \mathbb{I}_{2n} \equiv \begin{vmatrix} \mathbb{I}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbb{I}_n \end{vmatrix} \tag{1.43}$$

the $\mathbb{I}$'s being the identity matrices. Quite similarly, we find that ω has the representative matrix Ω given by:

$$\Omega = \begin{vmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{vmatrix} \tag{1.44}$$

in $\mathbb{R}^{2n}$, and J is represented by the matrix:

$$J = \begin{vmatrix} \mathbf{0}_n & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbf{0}_n \end{vmatrix} = -\Omega = \Omega^{-1} \tag{1.45}$$

consistently with Eqn.(1.35) which implies,in terms of the representative matrices:

$$J = \Omega^{-1} G \tag{1.46}$$

Notice, however, that while G and Ω are representatives of $(0, 2)$ tensors, J *is the representative of a* $(1, 1)$ *tensor.* Explicitly, denoting with $\left\|\Omega^{ij}\right\|$ the inverse of Ω (i.e. a $(2, 0)$ tensor):

$$\Omega^{ij}\Omega_{jk} = \delta^i{}_k \tag{1.47}$$

then:

$$J^i{}_j = \Omega^{ik} G_{kj} \tag{1.48}$$

Let us turn now to the Schrödinger equation (1.4). Written in components, it reads[9]:

$$\frac{d}{dt}\psi^h = -\frac{i}{\hbar}\langle h|H|k\rangle \psi^k \tag{1.49}$$

Writing then, as before, $\psi = \psi_1 + i\psi_2, \psi_{1,2} \in \mathbb{R}^n$ and introducing the real column vector:

$$\begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix} \in \mathbb{R}^{2n} \tag{1.50}$$

[9]It is clear that the matrix elements of the Hamiltonian have to be viewed as those of a $(1, 1)$ tensor.

we find (separating real and imaginary parts) the equation:

$$\frac{d}{dt}\left|\begin{array}{c}\psi_1\\ \psi_2\end{array}\right| = A\left|\begin{array}{c}\psi_1\\ \psi_2\end{array}\right| \tag{1.51}$$

where A is the skew-symmetric matrix:

$$A =: \frac{1}{\hbar}\left|\begin{array}{cc}\operatorname{Im} H & \operatorname{Re} H\\ -\operatorname{Re} H & \operatorname{Im} H\end{array}\right| \tag{1.52}$$

and $\operatorname{Im} H$ and $\operatorname{Re} H$ are the $n \times n$ matrices:

$$(\operatorname{Im} H)^h{}_k = \operatorname{Im} \langle h|H|k\rangle\,, \;\; (\operatorname{Re} H)^h{}_k = \operatorname{Re} \langle h|H|k\rangle \tag{1.53}$$

Just as before, $\operatorname{Im} H$ will be skew-symmetric and $\operatorname{Re} H$ symmetric.

Remark 5 *If we write the representative matrix of the Hamiltonian as: $H = \operatorname{Re} H + i \operatorname{Im} H$, then the "realified" version of it is [4] the symmetric matrix:*

$$^R H = \left|\begin{array}{cc}\operatorname{Re} H & -\operatorname{Im} H\\ \operatorname{Im} H & \operatorname{Re} H\end{array}\right| \tag{1.54}$$

Then it is easy to check that:

$$A = -J \circ (^R H/\hbar) \tag{1.55}$$

This completes the identification of the Schrödinger equation as a real dynamical system on a real space of dimension $2n$.

Taking a further time derivative, we obtain:

$$\frac{d^2}{dt^2}\left|\begin{array}{c}\psi_1\\ \psi_2\end{array}\right| = A^2\left|\begin{array}{c}\psi_1\\ \psi_2\end{array}\right| \tag{1.56}$$

and a simple calculation shows that:

$$A^2 = -\left(\frac{^R H}{\hbar}\right)^2 \tag{1.57}$$

Actually this result follows simply from the fact that the complex structure and the realified form of H commute, i.e.:

$$J \circ^R H =^R H \circ J \tag{1.58}$$

and from: $J^2 = -\mathbb{I}$.

As already remarked, things simplify if the basis in $\mathbb{C}^n$ is chosen as the basis of the eigenvectors of H itself: $H|k\rangle = E_k|k\rangle$. Then it is immediate to see that:

$$A = \frac{1}{\hbar}\begin{vmatrix} \mathbf{0} & H \\ -H & \mathbf{0} \end{vmatrix} \tag{1.59}$$

where H is now the diagonal $n \times n$ matrix:

$$H = diag\left\{E_1, ..., E_n\right\} \tag{1.60}$$

Then we obtain the equations of motion:

$$\frac{d}{dt}\psi_1 = H\psi_2 \ , \quad \frac{d}{dt}\psi_2 = -H\psi_1 \tag{1.61}$$

or:

$$\frac{d^2}{dt^2}\psi_i + \left(\frac{H}{\hbar}\right)^2 \psi_i = 0 \ , \ i = 1, 2 \tag{1.62}$$

Explicitly:

$$\frac{d^2}{dt^2}\psi_i^k + \left(\frac{E_k}{\hbar}\right)^2 \psi_i^k = 0 \ , k = 1, ..., n, \ i = 1, 2 \tag{1.63}$$

i.e. in this basis each one of the components of the real vectors ψ_1 and ψ_2 behaves as a simple harmonic oscillator with frequency $\nu_k = E_k/\hbar$.

1.2.2 Alternative Schrodinger and Heisenberg descriptions via modified Hermitian structures

Let now K be a (strictly) *positive* linear operator on $\mathcal{H}$, and consider the bilinear (sesquilinear) functional:

$$\langle\phi|K\psi\rangle \equiv h\left(\phi, K\psi\right) , \ \phi, \psi \in T\mathcal{H} \tag{1.64}$$

It is immediate to check that this functional enjoys all the three properties listed after Eqn.(1.22). Hence it defines a *new* Hermitian structure that we will denote as $h_K\left(.,.\right)$ or as: $\langle.|.\rangle_K$:

$$h\left(\phi, K\psi\right) =: h_K\left(\phi, \psi\right) =: \langle\phi|\psi\rangle_K \tag{1.65}$$

It is easy to show now that, as a consequence of the Hermiticity of H:

$$\mathcal{L}_{\Gamma_H}\left(h_K\left(\phi, \psi\right)\right) = \frac{i}{\hbar} h\left(\phi, \left[H, K\right]\psi\right) \tag{1.66}$$

Invariance of the new Hermitian structure w.r.t. the dynamics requires then that K be a "constant of the motion" for H:

$$\left[H, K\right] = 0 \tag{1.67}$$

h_K will now be given explicitly as: $h_K(\phi,\psi) = (h_K)_{ij}\overline{\phi}^i\psi^j$, $(h_K)_{ij} = \langle i|K|j\rangle = S_{ij} + iA_{ij}$, with S, A $n \times n$ real matrices. Hermiticity implies then: $\widetilde{S} = S$ and $\widetilde{A} = -A$, i.e. that S be symmetric and A skew-symmetric. Proceeding as before, it is not difficult to see that the new metric tensor, symplectic form and complex structure g_k ,ω_k and J_K would be represented in the previous basis by the matrices:

$$G_K = \begin{vmatrix} S & A \\ -A & S \end{vmatrix}, \quad \Omega_K = \begin{vmatrix} A & S \\ -S & A \end{vmatrix} \tag{1.68}$$

with J_K being given again by Eqn.(1.46).

The above results have been derived by considering "time" (i.e. Hamiltonian) evolution of vectors in the Hilbert space, i.e. in the framework of the Schrödinger picture.

It is not hard to show that similar results can be achieved in the context of the Heisenberg picture. Indeed, the new scalar product (1.65) induces a new associative product among linear operators, namely[10]:

$$A, B \rightarrow A \underset{(K)}{\cdot} B =: AKB \tag{1.69}$$

and a new commutator:

$$[A, B]_{(K)} =: A \underset{(K)}{\cdot} B - B \underset{(K)}{\cdot} A = AKB - BKA \tag{1.70}$$

that will fulfill the Jacobi identity in view of the associativity of the product (1.69).

Now, if we want to represent the same dynamics in terms of the new commutator bracket, we will have to define a new Hamiltonian H' such that:

$$i\hbar\frac{dA}{dt} = [H', A]_{(K)} = [H, A] \tag{1.71}$$

As A is generic,this requires: $H'K = KH' = H$, and hence:

$$H' = HK^{-1} \tag{1.72}$$

as well as:

$$[H, K] = 0 \tag{1.73}$$

as before. Notice that this will ensure that "time" evolution will be a derivation on the new product algebra, i.e. that:

$$\frac{d}{dt}\left(A \underset{(K)}{\cdot} B\right) = \frac{dA}{dt} \underset{(K)}{\cdot} B + A \underset{(K)}{\cdot} \frac{dB}{dt} \tag{1.74}$$

for all A, B.

Let us summarize at this point what we have found starting from the Schrödinger equation (1.4):

[10]See also Ref. [206] for the Abelian case.

- Eqn.(1.4) defines a real, linear Hamiltonian vector field on the realification of the complex (and finite-dimensional, for the time being) Hilbert space $\mathcal{H}$.
- On this space, Eqn.(1.4) defines a Killing vector field for the Euclidean metric tensor associated with the real part of the Hermitian scalar product.
- Eqn.(1.4) decomposes into n non-interacting harmonic oscillators with proper frequencies $E_k/\hbar$ and is therefore [54, 134] (see also next Chapter) a completely integrable system. Finally:
- Eqn.(1.4) preserves alternative Hermitian structures associated with positive linear operators K which commute with H. Therefore, Γ_H is also Killing for the new metric tensor and Hamiltonian for the new symplectic structure.

1.2.3 From Finite to Infinite Dimensions

We turn now to the infinite-dimensional case, concentrating on a quantum system described, in the Schrödinger picture, on the Hilbert space $\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{C}\right)$, $d \geq 1$, of complex, square-integrable[11] functions. Defining real variables q and p via:

$$\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{C}\right) \ni \psi\left(\mathbf{r}, t\right) =: q\left(\mathbf{r}, t\right) + ip\left(\mathbf{r}, t\right), \; \mathbf{r} \in \mathbb{R}^d \tag{1.75}$$

q and p will be functions in $\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{R}\right)$[12].

With a Schrödinger operator of the form:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + U\left(\mathbf{r}\right) \tag{1.76}$$

(with $U\left(\mathbf{r}\right)$ a potential), the (time-dependent) Schrödinger equation will be:

$$i\hbar\frac{d\psi}{dt} = \mathcal{H}\psi \tag{1.77}$$

In a natural way, we will have to deal here with (real) functionals instead of functions. We will consider functionals such that the functional differential δF of any one of them, $F = F\left[q, p\right]$ ($\int d\mathbf{r}... =: \int d^d r...$):

$$\delta F = \int d\mathbf{r} \left\{ \frac{\delta F}{\delta q\left(\mathbf{r}\right)}\delta q\left(\mathbf{r}\right) + \frac{\delta F}{\delta p\left(\mathbf{r}\right)}\delta p\left(\mathbf{r}\right) \right\} \tag{1.78}$$

is well defined, and this will require both the "differentials" (i.e. the variations) δq and δp and the functional derivatives $\delta F/\delta q$ and $\delta F/\delta p$ to be (real) square-integrable functions.

[11] With respect to the Lebesgue measure.

[12] One can also identify [180] $\mathcal{L}_2(\mathbb{R}^d, \mathbb{C})$ with the cotangent bundle of $\mathcal{L}_2(\mathbb{R}^d, \mathbb{R})$.

Defining a Hamiltonian functional $H_1[q,p]$ as:

$$H_1[q,p] = \frac{1}{2}\int d\mathbf{r}\left\{\frac{\hbar^2}{2m}\left[(\nabla q)^2 + (\nabla p)^2\right] + U(\mathbf{r})\left(q^2+p^2\right)\right\} \tag{1.79}$$

or (integrating by parts):

$$H_1[q,p] = \frac{1}{2}\left\{\langle q, \mathcal{H}q\rangle + \langle p, \mathcal{H}p\rangle\right\} \tag{1.80}$$

with $\langle .,. \rangle$ denoting the (real) scalar product in $\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{R}\right)$, we have, taking functional derivatives:

$$\frac{\delta H_1}{\delta q(\mathbf{r})} = \mathcal{H}q(\mathbf{r}), \quad \frac{\delta H_1}{\delta p(\mathbf{r})} = \mathcal{H}p(\mathbf{r}) \tag{1.81}$$

and the Schrödinger equation (1.77) can be rewritten as the (infinite-dimensional) Hamiltonian system:

$$\frac{d}{dt}\begin{vmatrix} p \\ q \end{vmatrix} = \frac{1}{\hbar}J\begin{vmatrix} \frac{\delta H_1}{\delta p} \\ \frac{\delta H_1}{\delta q} \end{vmatrix} \tag{1.82}$$

where:

$$J = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \tag{1.83}$$

As:

$$J\begin{vmatrix} p \\ q \end{vmatrix} = \begin{vmatrix} -q \\ p \end{vmatrix} \tag{1.84}$$

the tensor J is the realified [4] version of the standard complex structure J_0 on $\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{C}\right)$ defined by:

$$J_0 : \psi \to i\psi \tag{1.85}$$

Explicitly:

$$J = \int d\mathbf{r}\left(\delta p(\mathbf{r}) \otimes \frac{\delta}{\delta q(\mathbf{r})} - \delta q(\mathbf{r}) \otimes \frac{\delta}{\delta p(\mathbf{r})}\right) \tag{1.86}$$

The Schrödinger equation (1.82) can be rewritten as:

$$\frac{d}{dt}\begin{vmatrix} p \\ q \end{vmatrix} = \begin{vmatrix} \{p,H\}_1 \\ \{q,H\}_1 \end{vmatrix} \tag{1.87}$$

where the Poisson bracket $\{.,.\}_1$ and the associated Poisson tensor $\Lambda_1(.,.)$ are defined, for any two functionals $F[q,p]$ and $G[q,p]$, as:

$$\Lambda_1(\delta F, \delta G) =: \{F,G\}_1 = \frac{1}{\hbar}\int d\mathbf{r}\left\{\frac{\delta F}{\delta q(\mathbf{r})}\frac{\delta G}{\delta p(\mathbf{r})} - \frac{\delta F}{\delta p(\mathbf{r})}\frac{\delta G}{\delta q(\mathbf{r})}\right\} \tag{1.88}$$

or:

$$\{F,G\}_1 = \frac{1}{\hbar}\int d\mathbf{r}\left\{\begin{vmatrix} \delta F/\delta p & \delta F/\delta q \end{vmatrix} J \begin{vmatrix} \delta G/\delta p \\ \delta G/\delta q \end{vmatrix}\right\} \tag{1.89}$$

The corresponding symplectic structure[13] ω_1 is given by:

$$\omega_1 = \hbar \int d\mathbf{r} \left(\delta q \wedge \delta p \right) \tag{1.90}$$

or:

$$\omega_1 = \hbar \int d\mathbf{r} \left| \begin{array}{cc} \delta p & \delta q \end{array} \right| \otimes J \left| \begin{array}{c} \delta p \\ \delta q \end{array} \right| \tag{1.91}$$

and the composition of the symplectic and the complex structures gives rise [160, 180] to the metric tensor:

$$g =: J \circ \omega_1 = \hbar \int d\mathbf{r} \left(\delta p \left(\mathbf{r} \right) \otimes \delta p \left(\mathbf{r} \right) + \delta q \left(\mathbf{r} \right) \otimes \delta q \left(\mathbf{r} \right) \right) \tag{1.92}$$

Given any functional $F = F\left[q, p\right]$, the Hamiltonian vector field X_F associated with F via:

$$i_{X_F} \omega_1 = \delta F \tag{1.93}$$

is easily seen to be:

$$X_F = \frac{1}{\hbar} \int d\mathbf{r} \left\{ \frac{\delta F}{\delta p\left(\mathbf{r}\right)} \frac{\delta}{\delta q\left(\mathbf{r}\right)} - \frac{\delta F}{\delta q\left(\mathbf{r}\right)} \frac{\delta}{\delta p\left(\mathbf{r}\right)} \right\} \tag{1.94}$$

In particular:

$$X_{H_1} = \frac{1}{\hbar} \int d\mathbf{r} \left\{ \mathcal{H}p\left(\mathbf{r}\right) \frac{\delta}{\delta q\left(\mathbf{r}\right)} - \mathcal{H}q\left(\mathbf{r}\right) \frac{\delta}{\delta p\left(\mathbf{r}\right)} \right\} \tag{1.95}$$

The Poisson bracket (1.89) can then be written also as:

$$\left\{F, G\right\}_1 = \omega_1 \left(X_G, X_F \right) \tag{1.96}$$

Digression.

Things acquire a more familiar (and manageable) form if we introduce a (real) complete orthonormal set of functions[14]:

$$\left\{\psi_n\left(\mathbf{r}\right)\right\}_1^{\infty}; \left\langle \psi_n, \psi_m \right\rangle = \delta_{nm}; \sum_n \int d\mathbf{r} \psi_n\left(\mathbf{r}\right) \psi_n\left(\mathbf{r}'\right) = \delta\left(\mathbf{r} - \mathbf{r}'\right) \tag{1.97}$$

in $\mathcal{L}_2\left(\mathbb{R}^d, \mathbb{R}\right)$. Then, defining:

$$\delta q\left(\mathbf{r}\right) = \sum_n \psi_n\left(\mathbf{r}\right) dq_n, \quad dq_n =: \left\langle \psi_n, \delta q \right\rangle \tag{1.98}$$

and similarly for δp, the functional differential (1.78) becomes:

$$\delta F = \sum_n \left\{ \frac{\partial F}{\partial q_n} dq_n + \frac{\partial F}{\partial p_n} dp_n \right\} \tag{1.99}$$

[13] A non-degenerate two-form which is closed, being constant in the (global) (q, p) chart.

[14] They could be, e.g., the eigenfunctions of a d-dimensional isotropic harmonic oscillator.

where:

$$\frac{\partial F}{\partial q_n} =: \left\langle \psi_n, \frac{\delta F}{\delta q} \right\rangle \tag{1.100}$$

(and similarly for $\partial F/\partial p_n$)[15]. In other words:

$$\delta F = \left\langle \frac{\delta F}{\delta q}, \delta q \right\rangle + \left\langle \frac{\delta F}{\delta p}, \delta p \right\rangle \tag{1.101}$$

Proceeding in a similar way, it is easy to check that the Poisson tensor (3.196), the symplectic form (1.90) and the Hamiltonian vector field (1.94) associated with F can be written in this basis as:

$$\Lambda_1 = \frac{1}{\hbar} \sum_n \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n} \tag{1.102}$$

$$\omega_1 = \hbar \sum_n dq_n \wedge dp_n \tag{1.103}$$

and:

$$X_F = \frac{1}{\hbar} \sum_n \left\{ \frac{\partial F}{\partial p_n} \frac{\partial}{\partial q_n} - \frac{\partial F}{\partial q_n} \frac{\partial}{\partial p_n} \right\} \tag{1.104}$$

1.2.4 Alternative Hamiltonian Descriptions

Let's assume now the Schrödinger operator (1.76) to be *positive*[16] or, more generally, invertible, and let, for simplicity, the ψ_n's be the associated eigenfunctions:

$$\mathcal{H}\psi_n = E_n \psi_n, \ E_n > 0 \forall n \tag{1.105}$$

Then, defining [180] a new Poisson tensor and Poisson bracket as:

$$\Lambda_0 (\delta F, \delta G) =: \{F, G\}_0 = \frac{1}{\hbar} \int d\mathbf{r} \left\{ \frac{\delta F}{\delta q (\mathbf{r})} \mathcal{H} \frac{\delta G}{\delta p (\mathbf{r})} - \frac{\delta F}{\delta p (\mathbf{r})} \mathcal{H} \frac{\delta G}{\delta q (\mathbf{r})} \right\} \tag{1.106}$$

the same Schrödinger equation can be written also as:

$$\frac{d}{dt} \begin{vmatrix} p \\ q \end{vmatrix} = \frac{1}{\hbar} \begin{vmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{vmatrix} \begin{vmatrix} \frac{\delta H_0}{\delta p} \\ \frac{\delta H_0}{\delta q} \end{vmatrix} \tag{1.107}$$

or:

$$\frac{d}{dt} \begin{vmatrix} p (\mathbf{r}) \\ q (\mathbf{r}) \end{vmatrix} = \begin{vmatrix} \{p (\mathbf{r}), H_0\}_0 \\ \{q (\mathbf{r}), H_0\}_0 \end{vmatrix} \tag{1.108}$$

where:

$$H_0 [q, p] = \frac{1}{2} \int d\mathbf{r} \left(q^2 + p^2 \right) \tag{1.109}$$

---

[15] Note that, under the stated assumptions, the series on the r.h.s. of Eqn.(1.99) will be convergent.

[16] It could be, e.g., the Schrödinger operator for the isotropic harmonic oscillator: $\mathcal{H} = -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r})$ with: $U (\mathbf{r}) = m\omega^2 \mathbf{r}^2/2$.

is a sort of "universal" Hamiltonian functional.

In the basis of the eigenfunctions of $\mathcal{H}$ the Poisson bracket (1.106) can be written as:

$$\{F,G\}_0 = \frac{1}{\hbar}\sum_n E_n \left\{ \frac{\partial F}{\partial q_n}\frac{\partial G}{\partial p_n} - \frac{\partial F}{\partial p_n}\frac{\partial G}{\partial q_n} \right\} \tag{1.110}$$

and the associated symplectic form will be given by:

$$\omega_0 = \hbar \sum_n E_n^{-1} dq_n \wedge dp_n \tag{1.111}$$

or, in a basis-free notation:

$$\omega_0 = \hbar \int d\mathbf{r} \left(\mathcal{H}^{-1} \delta q \wedge \delta p \right) \tag{1.112}$$

Moreover, the Hamiltonian vector field associated, via ω_0 now, with the functional $F = F[q,p]$ is given by:

$$X_F = \frac{1}{\hbar}\sum_n \epsilon_n \left\{ \frac{\partial F}{\partial p_n}\frac{\partial}{\partial q_n} - \frac{\partial F}{\partial q_n}\frac{\partial}{\partial p_n} \right\} \tag{1.113}$$

or, in basis-independent form:

$$X_F = \frac{1}{\hbar}\int d\mathbf{r} \left\{ \mathcal{H}\frac{\delta F}{\delta p(\mathbf{r})}\frac{\delta}{\delta q(\mathbf{r})} - \mathcal{H}\frac{\delta F}{\delta q(\mathbf{r})}\frac{\delta}{\delta p(\mathbf{r})} \right\} \tag{1.114}$$

In particular:

$$X_{H_0} = \frac{1}{\hbar}\int d\mathbf{r} \left\{ \mathcal{H}p(\mathbf{r})\frac{\delta}{\delta q(\mathbf{r})} - \mathcal{H}q(\mathbf{r})\frac{\delta}{\delta p(\mathbf{r})} \right\} \tag{1.115}$$

which coincides with the Hamiltonian vector field (1.95).

Remark 6 *One could have also rewritten ω_0 as:*

$$\omega_0 = \hbar \int d\mathbf{r} \left(\delta q \wedge \mathcal{H}^{-1} \delta p \right) \tag{1.116}$$

but the two forms of course coincide, in view of the fact that $\mathcal{H}$ is self-adjoint.

What has been proved up to here is that the *same* vector field, namely:

$$\Gamma = \frac{1}{\hbar}\int d\mathbf{r} \left\{ \mathcal{H}p(\mathbf{r})\frac{\delta}{\delta q(\mathbf{r})} - \mathcal{H}q(\mathbf{r})\frac{\delta}{\delta p(\mathbf{r})} \right\} \tag{1.117}$$

is Hamiltonian w.r.t. two different Poisson brackets[17] ($\{.,.\}_1$ and $\{.,.\}_0$) and Hamiltonian functionals (H_1 and H_0), i.e. that it is *bi-Hamiltonian.* It turns

[17] I.e.: $\Gamma = \{\mathcal{H}_1, .\}_1 = \{\mathcal{H}_0, .\}_0$.

out [54, 134] that this, together with the compatibility condition, can lead to complete integrability.

The procedure can actually be iterated, leading to the conclusion [180] that the Schrödinger equation admits of infinitely many alternative Hamiltonian descriptions, with Hamiltonians:

$$H_n\left[q,p\right]=\frac{1}{2}\left\{\langle q,\mathcal{H}^n q\rangle+\langle p,\mathcal{H}^n p\rangle\right\},\quad n\geq 1; H_1[q,p]=H[q,p] \tag{1.118}$$

with associated symplectic forms:

$$\omega_n=\hbar\int d\mathbf{r}\left(\mathcal{H}^{n-1}\delta q\wedge\delta p\right) \tag{1.119}$$

and Poisson tensors:

$$\Lambda_n\left(\delta F,\delta G\right)=\left\{F,G\right\}_n=\frac{1}{\hbar}\int d\mathbf{r}\left\{\frac{\delta F}{\delta q\left(\mathbf{r}\right)}\mathcal{H}^{1-n}\frac{\delta G}{\delta p\left(\mathbf{r}\right)}-\frac{\delta F}{\delta p\left(\mathbf{r}\right)}\mathcal{H}^{1-n}\frac{\delta G}{\delta q\left(\mathbf{r}\right)}\right\} \tag{1.120}$$

such that:

$$i_\Gamma\omega_n=\delta H_n\ \forall n \tag{1.121}$$

where Γ is the vector field (1.117) and that the Hamiltonian functionals H_n are pairwise in involution w.r.t. *all* the Poisson brackets, i.e.:

$$\left\{H_n,H_m\right\}_k=0\quad\forall n,m,k \tag{1.122}$$

In other words, the Schrödinger equation admits of infinitely many constants of the motion pairwise in involution, which is another hallmark [54, 134] of complete integrability. Having established this, as well as the fact that the Schrödinger equation admits of infinitely many Hamiltonian descriptions, and that it can be considered as an infinite-dimensional Hamiltonian system on some infinite-dimensional space, it will be appropriate to devote the next Chapter to the study of completely-integrable dynamical systems and of their alternative Hamiltonian descriptions.

2 Completely Integrable Systems and Bi-Hamiltonian Descriptions

2.1 Liouville Integrability and Linearization

In order to avoid reducing the generality of our treatment, and for future reference, when the carrier space of a quantum system may be a manifold (like the complex projective Hilbert space (see below Sect.4.2.2)) instead of a vector space, we will work here in the framework of symplectic manifolds and Hamiltonian systems. So, let $(\mathcal{M}, \omega)$ be a symplectic manifold ($\dim \mathcal{M}=2n$ for some n and ω a symplectic form). A dynamical system, i.e. a vector field $\Gamma \in T\mathcal{M}$ is *ω-Hamiltonian* or, for short,*Hamiltonian* iff:

$$i_\Gamma \omega = d\mathcal{H} \tag{2.1}$$

for some $\mathcal{H} \in \mathcal{F}(\mathcal{M})$. A Hamiltonian dynamical system is said to be *completely integrable* if it has n constants of the motion $f_1, ..., f_n$ that are:

i) functionally independent:

$$df_1 \wedge ... \wedge df_n \neq 0 \tag{2.2}$$

and:

ii) pairwise in involution, i.e.:

$$\{f_i, f_j\} = 0 \ \ \forall i, j \tag{2.3}$$

where $\{.,.\}$ is the Poisson bracket associated with the symplectic form ω. The *Arnold-Liouville theorem*[4] states then that the level sets:

$$\mathbb{M}_{\mathbf{c}} = f^{-1}(\mathbf{c}), \ \ \mathbf{c} \in \mathbb{R}^n, \ \dim \mathbb{M}_{\mathbf{c}} = n \tag{2.4}$$

provide a foliation of $\mathcal{M}$ whose leaves are invariant manifolds for the Hamiltonian flow (2.1). Moreover, if the leaves of the foliation (2.4) are compact and connected, then they are diffeomorphic to n-dimensional tori, i.e.:

$$\mathbb{M}_{\mathbf{c}} \approx \mathbb{T}^n = \underbrace{\mathbb{S}^1 \times ... \times \mathbb{S}^1}_{n\ times} = \left\{\phi \equiv \left(\phi^1, ..., \phi^n\right) \mod 2\pi\right\} \tag{2.5}$$

and one can find a set of frequencies: $\nu \equiv (\nu_1, ..., \nu_n)$, $\nu = \nu(f)$ such that the Hamiltonian flow on the torus is given by[18]:

$$\frac{d\phi_i}{dt} = \nu_i \Rightarrow \phi_i(t) = \phi_i(0) + \nu_i t \tag{2.6}$$

and Hamilton's equations of motion are integrable by quadratures.

[18]Such motions are called *quasi-periodic* or *conditionally periodic*.

Let's summarize briefly how this leads to the well-known construction of *action-angle* variables.

Calling X_i the Hamiltonian vector field associated with f_i, $i = 1, ..., n$, Eqn.(2.3) leads at once to:

$$\{f_i, f_j\} \equiv \mathcal{L}_{X_j} f_i \equiv \omega(X_j, X_i) = 0 \tag{2.7}$$

Moreover, as:

$$i_{[X,Y]} = \mathcal{L}_X \cdot i_Y - i_Y \cdot \mathcal{L}_X \tag{2.8}$$

we obtain[19]:

$$i_{[X_i,X_j]}\omega = \mathcal{L}_X \cdot (i_{X_j}\omega) - i_{X_j} \cdot (\mathcal{L}_{X_i}\omega) \equiv d(\mathcal{L}_{X_i} f_j) = 0 \tag{2.9}$$

the final result following from Eqn.(2.7). Therefore, the X_i's commute pairwise. Moreover, it follows again from Eqn.(2.7) that the invariant leaves (2.4) of the foliation are Lagrangian submanifolds. Defining the immersion: $i_{\mathbf{c}} : \mathbb{M}_{\mathbf{c}} \hookrightarrow \mathcal{M}$, we have therefore:

$$i_{\mathbf{c}}^* \omega = 0 \tag{2.10}$$

Therefore, if we denote by θ the Cartan one-form ($\omega = -d\theta$), its pull-back $i_{\mathbf{c}}^*\theta$ will be *closed*:

$$di_{\mathbf{c}}^*\theta = i_{\mathbf{c}}^* d\theta = 0 \tag{2.11}$$

It need not be exact, though, as the invariant tori are not contractible. Cycles on the torus need not be boundaries, and therefore the integral of $i_{\mathbf{c}}^*\theta$ along a one-dimensional cycle need not vanish. We can select a *basis* $(\gamma_1, ..., \gamma_n)$ of loops, i.e. n one-dimensional cycles each one of which winds around the torus exactly once and none of which is homologous [2] to any other one (nor to the trivial loop), and define the *action variables* I_i as:

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} i_{\mathbf{c}}^*\theta, \quad i = 1, ..., n \tag{2.12}$$

Of course: $I_i = I_i(f)$ depends only on the homology class [2] of γ_i and, provided the jacobian of the transformation does not vanish or, equivalently:

$$dI_1 \wedge dI_2 \wedge ... \wedge dI_n \neq 0 \tag{2.13}$$

invariant tori can be uniquely labelled by the set $I = (I_1, ..., I_n)$ of the values of the action variables. Defining then:

$$S = S(I, q) = \int_{q_0}^{q} i_{\mathbf{c}}^*\theta \tag{2.14}$$

the integral being along a path γ on the invariant torus labelled by I joining a fiducial point q_0 to the point q, S will depend only on the homology class of

[19] As X_j is Hamiltonian, $\mathcal{L}_{X_j}\omega = 0$.

paths from q_0 to q to which γ belongs[20]. Switching to a different homology class multiplying γ by, say, a loop γ_i in the basis will change S by a fixed amount:

$$S \to S + \Delta S_i; \quad \Delta S_i = 2\pi I_i \tag{2.15}$$

We can then use S as the generator of a time-independent canonical transformation:

$$(q,p) \to (\phi, I) \tag{2.16}$$

with the I's playing the rôle of the new momenta, via[21]:

$$p_i = \frac{\partial S}{\partial q^i}, \quad \phi^i = \frac{\partial S}{\partial I_i} \tag{2.17}$$

and with the new Hamiltonian: $\mathcal{K} = \mathcal{H}$. Now, as n is the maximum allowed number of independent constants of the motion pairwise in involution[22], either the Hamiltonian is one of the f_i's or is a function thereof: $\mathcal{H} = \mathcal{H}(f)$ and therefore it is ultimately a function of the action variables alone. Hamilton's equations become then:

$$\frac{d}{dt} I_i = 0, \quad \frac{d}{dt}\phi^i = \nu^i; \quad \nu^i =: \frac{\partial \mathcal{H}}{\partial I_i} = \nu^i(I) \tag{2.18}$$

and we recover Eqn.(2.6). In the new coordinates the dynamical vector field will be given by:

$$\Gamma = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \phi^i} \tag{2.19}$$

and the symplectic structure will be:

$$\omega = \sum_{i=1}^{n} d\phi^i \wedge dI_i \tag{2.20}$$

We notice that in these coordinates the dynamics is nilpotent of index two,i.e.:

$$\frac{d\phi^i}{dt} = \nu^i; \ \frac{d\nu^i}{dt} = 0 \tag{2.21}$$

Moreover, in these coordinates the system is linear and associated with a nilpotent matrix. It should be remarked that the transformation (2.16) is not linear. Therefore, even if the system is linear in the (q,p) coordinates, the transformation need not be isospectral, i.e. it may take us from a semisimple matrix to a nilpotent one.

[20] This approach goes back to a paper [64] by A.Einstein of 1917.

[21] The ambiguity expressed by Eqn.(2.15) tells us that the ϕ's are actually defined "modulo" 2π, i.e. that they are indeed angles.

[22] If ω is non-degenerate, n is the maximum allowed dimension for an isotropic subspace.

2.2 From Invariant Structures to Integrability

In the case of Eqn.(2.18), if we are in the so-called *non-resonant* case, i.e. if:

$$d\nu_1 \wedge d\nu_2 \wedge ... \wedge d\nu_n \neq 0 \tag{2.22}$$

we can choose the ν_i's as new momenta (the transformation will be in general *not* canonical, however!). In the new coordinates the dynamical system will be completely *separated* into n independent systems, while the Hamiltonian and symplectic structure will become respectively:

$$H = \frac{1}{2}\sum_{i=1}^{n}\nu_i^2 \tag{2.23}$$

and:

$$\omega = \sum_{i=1}^{n} d\phi^i \wedge d\nu_i \tag{2.24}$$

Separability of a dynamical system into a family of non-interacting subsystems appears therefore to be intimately connected with integrability[23]. It is also well-known that a way to achieve (if possible) integrability via separability occurs in the Hamilton-Jacobi theory [16, 17, 18, 107, 158], a subject that we will not discuss here, though. Notice also that, in general, the two notions of separability do not in general coincide.

In this Subsection we will discuss a way to achieve separability (and eventually integrability) with the aid of additional invariant structures [53, 54]. We will not make reference, for the time being, to symplectic structures and the like. What we are going to say generalizes to vector fields, and hence also to non-linear situations, the familiar block-diagonal form of matrices.

Let then $\mathcal{M}$ be a smooth manifold and let $\Gamma \in \mathfrak{X}(\mathcal{M})$ be a vector field. Γ will be said to be *separable into dynamics of lower dimension* on an open set $\mathcal{U} \subseteq \mathcal{M}$ if a holonomic frame $\left\{e_{(i,k)}\right\}$ can be found for the tangent bundle $T\mathcal{U}$, with dual forms $\left\{\theta^{(i,k)}\right\}$, such that:

$$\mathcal{L}_{e_{(i,k)}}\left\langle \theta^{(j,h)} | \Gamma \right\rangle \neq 0 \Leftrightarrow i = j \tag{2.25}$$

This implies of course, in local coordinates, that we can choose coordinates $x^{(i,k)}$ ($e_{(i,k)} = \partial/\partial x^{(i,k)}$) in such a way that:

$$\Gamma = \Gamma^{(i,k)}\frac{\partial}{\partial x^{(i,k)}} \tag{2.26}$$

and:

$$\Gamma^{(i,k)} = \Gamma^{(i,k)}\left(x^i\right); \quad x^i =: \left(x^{(i,1)}, x^{(i,2)}, ..., x^{(i,k)}, ...\right) \tag{2.27}$$

[23]See also Refs.[72, 73] for a similar discussion in the Lagrangian context.

Finally, the vector field Γ will be said to be *separable* if we can choose $\mathcal{U} = \mathcal{M}$ or, at least, $\mathcal{U}$ to be an open dense set in $\mathcal{M}$.

Let us review briefly how one can achieve separation of the dynamics in the presence of an *invariant diagonalizable* $(1,1)$ *tensor field* $T \in \mathcal{F}_1^1(\mathcal{M})$ *with at least two distinct eigenvalues and vanishing Nijenhuis torsion.*

Recall[24] that, given a $(1,1)$ tensor T, the *Nijenhuis torsion* [78, 152, 194] associated with T is the $(0,2)$ tensor $\mathcal{N}_T$ defined by:

$$\mathcal{N}_T(\alpha, X, Y) =: \langle \alpha | \mathcal{H}_T(X,Y) \rangle ; \ \alpha \in \mathcal{X}^*(\mathcal{M}), \ X, Y \in \mathcal{X}(\mathcal{M}) \tag{2.28}$$

where:

$$\mathcal{X}(\mathcal{M}) \ni \mathcal{H}_T(X,Y) =: [TX, TY] + T^2[X,Y] - T[TX,Y] - T[X,TY] \tag{2.29}$$

Let's remark that, if T is diagonalizable:

$$Te_i = \lambda_i e_i \tag{2.30}$$

the eigenvectors e_i are (locally at least) a basis of vector fields[25], and we will denote as S_{λ_i} the eigenspace of the eigenvalue λ_i. The e_i' being a basis implies:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k; \ c_{ij}^k = -c_{ji}^k \tag{2.31}$$

for some set of "structure constants" (actually in principle functions) c_{ij}^k. The dual cobasis $\{\theta^i\}$, defined as usual via:

$$\left\langle \theta^i | e_j \right\rangle = \delta_j^i \tag{2.32}$$

will be also a basis of eigenforms:

$$\widetilde{T}\theta^i = \lambda_i \theta^i \tag{2.33}$$

where $\widetilde{T}$ denotes the transpose action of T on forms ($\langle \theta | TX \rangle =: \left\langle \widetilde{T}\theta | X \right\rangle$). Using then the identity [41]:

$$d\theta(X,Y) = \mathcal{L}_X(\theta(Y)) - \mathcal{L}_Y(\theta(X)) - \langle \theta | [X,Y] \rangle \tag{2.34}$$

it is easy to prove that:

$$d\theta^k(e_i, e_j) = -c_{ij}^k \tag{2.35}$$

i.e. that:

$$d\theta^k = -\frac{1}{2}\sum_{ij} c_{ij}^k \theta^i \wedge \theta^j \tag{2.36}$$

[24]More properties of Nijenhuis torsions and tensors are briefly reviewed in App.A.

[25]In fact, they are not only a vector space, but have in addition the structure of an $\mathcal{F}(\mathcal{M})$-module.

Contracting the Nijenhuis torsion with the eigenvectors one finds, with some algebra:

$$\mathcal{H}_T\left(e_i, e_j\right) = \left(T - \lambda_i\right)\left(T - \lambda_j\right)\left[e_i, e_j\right] + \left(\lambda_i - \lambda_j\right)\left\{\left(\mathcal{L}_{e_i}\lambda_j\right)e_j + \left(\mathcal{L}_{e_j}\lambda_i\right)e_i\right\} \tag{2.37}$$

Let us remark first that:

$$\left(T - \lambda_i\right)\left(T - \lambda_j\right)\left[e_i, e_j\right] = \sum_k \left(\lambda_k - \lambda_i\right)\left(\lambda_k - \lambda_j\right)c_{ij}^k e_k \tag{2.38}$$

has *no* components in $S_{\lambda_i} \oplus S_{\lambda_j}$. If the Nijenhuis torsion vanishes[26], then the condition $\mathcal{H}_T\left(e_i, e_j\right) = 0$ separates into:

$$\left(T - \lambda_i\right)\left(T - \lambda_j\right)\left[e_i, e_j\right] = 0 \tag{2.39}$$

and:

$$\left(\lambda_i - \lambda_j\right)\mathcal{L}_{e_i}\lambda_j \equiv \left(\lambda_i - \lambda_j\right)d\lambda_i\left(e_j\right) = 0 \tag{2.40}$$

Contracting the first of the above equations with θ^k we obtain:

$$\left(\lambda_k - \lambda_i\right)\left(\lambda_k - \lambda_j\right)\left\langle\theta^k\right|\left[e_i, e_j\right]\rangle = 0 \tag{2.41}$$

which implies: $\left\langle\theta^k\right|\left[e_i, e_j\right]\rangle = 0$ for $\lambda_k \neq \lambda_i, \lambda_j$, i.e.:

$$\left[e_i, e_j\right] \in S_{\lambda_i} \oplus S_{\lambda_j} \tag{2.42}$$

and hence:

$$c_{ij}^k = 0 \text{ when } \lambda_k \neq \lambda_i, \lambda_j \tag{2.43}$$

At this point we can somehow sharpen the analysis and make it a bit more precise. If the eigenspaces are not one-dimensional (i.e. the eigenvalues of T have degeneracy), denoting by $\left\{e_{(i,r)}\right\}, r = 1, 2, ..., d_i,$ d_i being the dimension of the i-th eigenspace, a basis of eigenvectors in S_{λ_i}, it is not difficult to prove that Eqn.(2.41) generalizes to:

$$\left(\lambda_k - \lambda_i\right)\left(\lambda_k - \lambda_j\right)\left\langle\theta^{(k,r)}\right|\left[e_{(i,p)}, e_{(j,q)}\right]\right\rangle = 0 \;\forall r, p, q \tag{2.44}$$

which holds in particular for $i = j$, thus leading to the conclusion that:

$$\left[e_{(i,p)}, e_{(i,q)}\right] \in S_{\lambda_i} \tag{2.45}$$

i.e. that *if T is diagonalizable and has vanishing Nijenhuis torsion, the eigenvectors belonging to every eigenspace are an involutive distribution*. As such, the distribution will be integrable by Frobenius' theorem [167], and we can speak (locally at least) of *eigenmanifolds*.

We can also reach the same conclusion in dual form as follows. Eqn.(2.43) implies that in Eqn.(2.36) at least one of the one-forms on the r.h.s. must be in the (dual) eigenspace of the eigenvalue λ_k. To be more specific, if we denote by

[26] i.e. T is (see App.A) a Nijenhuis tensor

$\theta^{(k,r)}, r = 1, 2, ...$ the eigenforms belonging to the eigenvalue λ_k and by $c^{(k,r)}_{(i,s)(j,p)}$ the "structure constants", Eqns.(2.36) and (2.43) imply:

$$d\theta^{(k,r)} = -\sum_{(i,p),s} c^{(k,s)}_{(i,p)(k,s)} \theta^{(i,p)} \wedge \theta^{(k,s)} \tag{2.46}$$

But this is equivalent to the statement that:

$$d\theta^{(k,r)} \bigwedge_s \theta^{(k,s)} = 0 \tag{2.47}$$

which is again [54] a statement of Frobenius' theorem.

The main conclusion is then that, under the stated assumptions, one can always find a *holonomic* frame (and coframe) that diagonalizes T in the form:

$$T = \sum_i \lambda_i e_i \otimes \theta^i \tag{2.48}$$

Let us turn now to the consequences of the invariance of T under the dynamics. First of all, an invariant $(1,1)$ tensor T will generate an algebra $\mathcal{A}$ of vector fields all commuting with Γ given by:

$$\mathcal{A} = \left\{\Gamma, T\Gamma, T^2\Gamma, .., T^k\Gamma, ...\right\} \tag{2.49}$$

If $\mathcal{L}_\Gamma T = 0$, it can be proved [54] that:

$$\left[T^k\Gamma, T^{k+h}\Gamma\right] = \sum_{\substack{\alpha+\beta+\gamma=2k+h-2 \\ \alpha,\beta\geq 0,\gamma\geq k}} T^\alpha \mathcal{H}_T \left(T^\beta\Gamma, T^\gamma\Gamma\right) \tag{2.50}$$

hence, if $\mathcal{H}_T = 0$, $\mathcal{A}$ will be an abelian algebra of vectors fields all commuting with Γ, i.e. an abelian algebra of symmetries [167].

Consider next the eigenvalue equation for T. Let e and θ be an eigenvector and an eigenform belonging to the same eigenvalue λ:

$$Te = \lambda e, \quad \widetilde{T}\theta = \lambda\theta \tag{2.51}$$

We can assume, without loss of generality: $\langle\theta|e\rangle = 1$.

If T is invariant under the dynamics, $\mathcal{L}_\Gamma T = 0$, then:

$$T\left(\mathcal{L}_\Gamma e\right) = \mathcal{L}_\Gamma \left(Te\right) = \mathcal{L}_\Gamma \left(\lambda e\right) = \left(\mathcal{L}_\Gamma \lambda\right) e + \lambda \left(\mathcal{L}_\Gamma e\right) \tag{2.52}$$

On the other hand:

$$\left\langle \mathcal{L}_\Gamma e|\widetilde{T}\theta \right\rangle = \left\langle T\left(\mathcal{L}_\Gamma e\right)|\theta\right\rangle = \mathcal{L}_\Gamma\lambda + \lambda \left\langle \mathcal{L}_\Gamma e|\theta\right\rangle \equiv \mathcal{L}_\Gamma\lambda + \left\langle \mathcal{L}_\Gamma e|\widetilde{T}\theta \right\rangle \tag{2.53}$$

and hence:

$$\mathcal{L}_\Gamma \lambda \equiv i_\Gamma d\lambda = 0 \tag{2.54}$$

i.e., *if T is invariant under the dynamics, so are the eigenvalues of T* .

Notice that, by Cartan's identity [167]:

$$\mathcal{L}_{e_i}\theta^j = \left\langle e_i|d\theta^j\right\rangle + d\left\langle e_i|\theta^j\right\rangle \tag{2.55}$$

and hence, if the (co)basis is holonomic, $d\theta^j = 0$ (together with $\left\langle e_i|\theta^j\right\rangle = \delta_i^j$) leads to:

$$\mathcal{L}_{e_i}\theta^j = 0 \ \forall i, j \tag{2.56}$$

Then, for $i \neq j$ we obtain:

$$\begin{aligned}
(\lambda_i - \lambda_j)\,\mathcal{L}_{e_i}\left\langle \Gamma|\theta^j\right\rangle &= \lambda_i\left\langle \mathcal{L}_{e_i}\Gamma|\theta^j\right\rangle - \left\langle \mathcal{L}_{e_i}\Gamma|\widetilde{T}\theta^j\right\rangle = \qquad (2.57)\\
&= \lambda_i\left\langle \mathcal{L}_{e_i}\Gamma|\theta^j\right\rangle - \left\langle T\left(\mathcal{L}_{e_i}\Gamma\right)|\theta^j\right\rangle = \\
&= \lambda_i\left\langle \mathcal{L}_{e_i}\Gamma|\theta^j\right\rangle - \left(\mathcal{L}_{i\Gamma}\lambda_i\right)\left\langle e_i|\theta^j\right\rangle - \lambda_i\left\langle \mathcal{L}_{e_i}\Gamma|\theta^j\right\rangle = 0
\end{aligned}$$

Hence:

$$\mathcal{L}_{e_i}\left\langle \Gamma|\theta^j\right\rangle = 0, \quad i \neq j \tag{2.58}$$

and (cfr. Eqn.(2.25)) this proves *separability* of Γ. To be more explicit, we can write T as:

$$T = \sum_{i=1}^{n}\lambda_i\sum_{k=1}^{d_i}\frac{\partial}{\partial x^{(i,k)}} \otimes dx^{(i,k)} \tag{2.59}$$

where n is the number of distinct eigenvalues and d_i is the degeneracy of the i-th eigenvalue. Finally, Γ will be of the form already given in Eqns.(2.26) and (2.27). On the eigenspaces of T that are one-dimensional integrability of Γ will be then essentially trivial, and this case will not be considered further.

Proceeding further we obtain from Eqn.(2.40):

$$0 = (\lambda_i - \lambda_j)\left\langle e_i|d\lambda_j\right\rangle = \left\langle Te_i|d\lambda_j\right\rangle - \left\langle e_i|\lambda_j d\lambda_j\right\rangle = \left\langle e_i|\widetilde{T}d\lambda_j\right\rangle - \left\langle e_i|\lambda_j d\lambda_j\right\rangle \tag{2.60}$$

and hence:

$$\widetilde{T}d\lambda_j = \lambda_j d\lambda_j \tag{2.61}$$

i.e. $d\lambda_j$ is an eigenform belonging to the eigenvalue λ_j. Let us now assume the eigenvalues of Γ to be *doubly* degenerate and functionally independent. This implies: $\dim(\mathcal{M}) = 2n$ and:

$$d\lambda_1 \wedge d\lambda_2 \wedge \ldots \wedge d\lambda_n \neq 0 \tag{2.62}$$

Then the $d\lambda_i$'s can be taken as half of the cobasis, and we can write T as:

$$T = \sum_{i=1}^{n}\lambda_i\left(e_i \otimes \theta^i + e_{n+i} \otimes d\lambda_i\right) \tag{2.63}$$

With this choice, Eqn.(2.54) tells us that Γ has no components "along" the $d\lambda_i$'s, and that it is therefore of the form:

$$\Gamma = \sum_{i=1}^{n}\Gamma^i e_i \tag{2.64}$$

Proceeding further, closure of the θ^i's allow us to write: $\theta^i = d\phi^i$, and hence: $e_i = \partial/\partial\phi^i$ for $i = 1, ..., n$. The ϕ^i's are in general only locally defined (while the λ_i's are globally defined), and can be allowed to be angles. Hence we can rewrite T as:

$$T = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial\phi^i} \otimes d\phi^i + \frac{\partial}{\partial\lambda^i} \otimes d\lambda^i \right) \tag{2.65}$$

and, in view of Eqn.(2.64), Γ will be of the form:

$$\sum_{i=1}^{n} \Gamma^i \left(\lambda^i, \phi^i\right) \frac{\partial}{\partial\phi^i} \tag{2.66}$$

The associated equations of motion will be:

$$\begin{array}{l} \frac{d}{dt}\phi^i = \Gamma^i \\ \\ \frac{d}{dt}\lambda^i = 0 \end{array} \quad ; \quad i = 1, .., n \tag{2.67}$$

Now, it is easy to show that the dynamical system (2.67) can be made Hamiltonian with respect to a large family of symplectic structures. Indeed, let's assume that no one of the Γ^i's vanishes identically[27]. Then, with any set of (smooth) functions $g_i = g_i\left(\lambda^i\right)$ we can associate the symplectic form:

$$\omega = \sum_{i=1}^{n} f_i\left(\lambda^i, \phi^i\right) d\phi^i \wedge d\lambda^i \tag{2.68}$$

where:

$$f_i\left(\lambda^i, \phi^i\right) =: \frac{g_i\left(\lambda^i\right)}{\Gamma^i\left(\lambda^i, \phi^i\right)} \tag{2.69}$$

and Γ will be Hamiltonian:

$$i_\Gamma \omega = d\mathcal{H} \tag{2.70}$$

with:

$$d\mathcal{H} = \sum_{i=1}^{n} g_i\left(\lambda^i\right) d\lambda^i \tag{2.71}$$

Therefore, under the assumption that there exists a $(1,1)$ diagonalizable tensor field T invariant under the dynamics, with vanishing Nijenhuis torsion and at most doubly degenerate and functionally independent eigenvalues, what has been proved up to now is that *the dynamical vector field* Γ *is separable, integrable and, on the eigenspaces of doubly degenerate eigenvalues, Hamiltonian.* ■

The equation $\mathcal{L}_\Gamma T = 0$ expresses the invariance of the tensor T in intrinsic terms. It may be useful to write down the same condition in the language of

[27] If they have isolated zeros, the closed set of the zeros, which is an invariant subset, can be excluded from the manifold. The case in which some component of Γ vanishes has been discussed in Ref.[54].

coordinates. If $\left(x^i, ..., x^m\right)$, $m = (\dim(\mathcal{M}))$ are local coordinates, and T and Γ are given by:

$$T = T^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}; \quad \Gamma = \Gamma^i \frac{\partial}{\partial x^i} \tag{2.72}$$

then:

$$\mathcal{L}_\Gamma T = \left\{ \mathcal{L}_\Gamma T^i{}_j - \frac{\partial \Gamma^i}{\partial x^k} T^k{}_j + T^i{}_k \frac{\partial \Gamma^k}{\partial x^j} \right\} dx^j \otimes \frac{\partial}{\partial x^i} \tag{2.73}$$

and hence invariance under Γ implies the *matrix* equation:

$$\mathcal{L}_\Gamma T =: \frac{d}{dt} T = [C, T] \tag{2.74}$$

where, with abuse of notation, we have denoted by T the $m \times m$ matrix: $T = \left\| T^i{}_j \right\|$and:

$$C = \left\| C^i{}_j \right\|; \quad C^i{}_j =: \frac{\partial \Gamma^i}{\partial x^j} \tag{2.75}$$

while $[.,.]$ denotes the usual commutator among matrices. Whenever two matrices C and T satisfy Eqn.(2.74) they are said to form a *Lax pair* [122, 123, 124, 152, 221][28].

Whenever we may define a map μ from $\mathcal{M}$ to a space of matrices such that the dynamics is μ-related to a dynamics on the matrix space of the form of Eq.(2.74), we say that the original dynamics can be given a "Lax form". This is what might be called also a "Heisenberg" form, and has many general properties. For instance, the evolution of T ruled by the "Hamiltonian" C is clearly isospectral.

Whenever it is possible to find a map from our carrier space to a space of linear operators such that the dynamics on the carrier space may be casted into the Heisenberg form we will say that our dynamics may be put into the Lax form. As a matter of fact, by using the momentum map associated with the symplectic action of the unitary group on the Hilbert space or on the complex projective space (see below, Sect.4.2), we may relate the Schrödinger picture with the Heisenberg picture on the space of observables.

2.3 From Liouville Integrability to Invariant Structures

Reversing somehow our path, let's start by considering a dynamical system Γ that is Hamiltonian and completely integrable "a' la" Liouville. Hence: $\dim(\mathcal{M}) = n$. Introducing action-angle variables $\left(I_1, .., I_n; \phi^1, .., \phi^n\right)$ in the neighborhood of an Arnold- Liouville torus $\mathbb{T}^n$, we will have:

$$dI_1 \wedge dI_2 \wedge ... \wedge dI_n \neq 0 \tag{2.76}$$

[28] We should notice that Eq.(2.74) depends on the coordinate system we are using, and therefore has no intrinsic meaning.

and the condition that the Hamiltonian $\mathcal{H}$ be a function of the action variables alone can be written as:

$$d\mathcal{H} \wedge dI_1 \wedge ... \wedge dI_n = 0 \tag{2.77}$$

The symplectic form can be written as:

$$\Omega = \sum_k d\phi^k \wedge dI_k \tag{2.78}$$

and the vector field Γ in action-angle variables will be given by:

$$\Gamma = \sum_k \omega^k \frac{\partial}{\partial \phi^k}; \quad \omega^k =: \frac{\partial \mathcal{H}}{\partial I_k} \tag{2.79}$$

Assume first that the Hamiltonian is separable:

$$\mathcal{H} = \sum_k \mathcal{H}_k \left(I_k\right) \tag{2.80}$$

Then the class of $(1,1)$ tensor fields defined by:

$$T = \sum_k \lambda_k \left(I_k\right) \left\{ dI_k \otimes \frac{\partial}{\partial I_k} + d\phi^k \otimes \frac{\partial}{\partial \phi^k} \right\} \tag{2.81}$$

with the λ_k's arbitrary functions with nowhere vanishing differential has all the required properties. Indeed:

- It is invariant under the dynamics;
- It has doubly degenerate eigenvalues and:
- It has vanishing Nijenhuis torsion.

This last property can be checked directly by testing Eqn.(2.29) on: $(X, Y) = (\partial/\partial I_h, \partial/\partial I_k), (\partial/\partial I_h, \partial/\partial \phi^k)$ and $(\partial/\partial \phi^h, \partial/\partial \phi^k)$ ■.

A second case in which an invariant $(1,1)$ tensor can be constructed is the "non-resonant" case, i.e. when the Hamiltonian has a non-vanishing Hessian:

$$\det \left\| \frac{\partial^2 \mathcal{H}}{\partial I_h \partial I_k} \right\| \neq 0 \tag{2.82}$$

This means, of course:

$$d\omega^1 \wedge d\omega^2 \wedge ... \wedge d\omega^n \neq 0 \tag{2.83}$$

Solving then for the I's as functions of the ω's, we can use the ω's as new coordinates and introduce[29] a new symplectic structure:

$$\widetilde{\Omega} = \sum_k d\omega^k \wedge d\phi^k = \sum_{hk} \frac{\partial^2 \mathcal{H}}{\partial I_h \partial I_k} dI_h \wedge d\phi^k \tag{2.84}$$

[29] This change of variables need not be a canonical transformation.

and Γ will be Hamiltonian with the separable Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\sum_{k}\left(\omega^{k}\right)^{2} \tag{2.85}$$

The class of $(1,1)$ tensor fields will be given now by:

$$T = \sum_{k}\lambda_{k}\left(\omega^{k}\right)\left\{d\omega^{k}\otimes\frac{\partial}{\partial\omega^{k}} + d\phi^{k}\otimes\frac{\partial}{\partial\phi^{k}}\right\} \tag{2.86}$$

Complete integrability is also known to be related to the existence of *recursion operators* [53, 120, 179, 239]. A brief account of the latter is given in Appendix B.

3 Alternative Structures for Classical Systems

3.1 Preliminaries. A cursory look at the Inverse Problem in a classical context

After having examined briefly in the previous Chapter the problem of the integrability of a classical dynamical system, and before turning to the main topic of this review, i.e. *quantum* systems[30], we restate here in a very cursory way what is known in the literature as the "Inverse Problem of Classical Dynamics".

Let then Γ be a vector field on a (smooth) manifold $\mathcal{M}$. In a nutshell, the Inverse Problem (IP) can be formulated in (at least[31]) three different, and often related, contexts, namely:

- $IP1$: Lagrangian context [98, 99, 186]. Let then $\mathcal{M}$ be the tangent bundle of a smooth manifold Q, i.e. $\mathcal{M} = TQ$ equipped with tangent bundle coordinates $\left(q^i, v^i\right)$ such that $\Gamma \in \mathcal{X}(TQ)$ is a second-order vector field [184], i.e.:

  $$\Gamma = v^i \frac{\partial}{\partial q^i} + F^i(q, v) \frac{\partial}{\partial v^i} \tag{3.1}$$

  The Lagrangian IP amounts then to the following: find all the smooth functions $\mathcal{L} = \mathcal{L}(q, v) \in \mathcal{F}(TQ)$ such that:

  $$\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} F^j = \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\partial^2 \mathcal{L}}{\partial v^i \partial q^j} v^j, \quad i = 1, ..., n = \dim Q \tag{3.2}$$

  It follows that if the *Lagrangian* $\mathcal{L}$ is regular, i.e.:

  $$\det \left\| \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \right\| \neq 0 \tag{3.3}$$

  then the Euler-Lagrange equations can be put in normal form and, via a Legendre transformation [4, 167] one can go over to a Hamiltonian description of the dynamical system on the cotangent bundle T^*Q. We will not discuss this setting of the IP any further, and refer for a full account of it to the literature [184].

- $IP2$: Hamiltonian context. Let instead $\mathcal{M} = T^*Q$ for some smooth manifold Q and $\Gamma \in \mathcal{X}(T^*Q)$. The Hamiltonian IP amounts then to finding all pairs $(\omega, \mathcal{H})$ with ω a symplectic form (a closed and non-degenerate two-form) and $\mathcal{H} \in \mathcal{F}(T^*Q)$ such that:

  $$i_\Gamma \omega = d\mathcal{H} \tag{3.4}$$

  At a local level, the problem reduces to finding all the closed and non-degenerate two-forms ω such that:

  $$\mathcal{L}_\Gamma \omega = 0 \tag{3.5}$$

[30] What we mean exactly by a "quantum" system will be specified in the next Chapter.

[31] We will not consider here the Hamilton-Jacobi form of Classical Dynamics, but see [158]

with $\mathcal{L}_\Gamma$ denoting the Lie derivative w.r.t. Γ, which is a system of coupled $PDE's$ in $\begin{pmatrix} n \\ 2 \end{pmatrix} = 2n^2 - n$ unknowns[32]. As a simple example, in a neighborhood $U \subseteq M$ in which $\Gamma \neq 0$ and defines a flow-box (the "straightening-up-of-the-flux" theorem [4] holds) we can find coordinates $(x_0, x_1, ..., x_{2n-1})$ such that $\Gamma = \partial/\partial x_0$ and hence the problem has infinite solutions:

$$\omega = dx_0 \wedge df + a_{ij} df^i \wedge df^j \tag{3.6}$$

with: $a_{ij} = -a_{ji} \in \mathbb{R}$, $\det \|a_{ij}\| \neq 0$ and: $\partial f/\partial x_0 = \partial f^i/\partial x_0 = 0$, $dx_0 \wedge df \wedge df^1 \wedge ... \wedge df^{2n-2} \neq 0$, and any such f will be an acceptable Hamiltonian ($i_\Gamma \omega = df$).

- $IP3$: Poisson context [35, 60]. $\mathcal{M}$ is assumed here to be a Poisson manifold [167]. In local coordinates x^i, $i = 1, ..., \dim \mathcal{M}$, and the IP in this context amounts to finding all pairs $(\{.,\}, \mathcal{H})$ with $\{.,\}$ a (possibly degenerate[33]) Poisson bracket and $\mathcal{H} \in \mathcal{F}\{\mathcal{M}\}$ such that:

$$\left\{x^i, \mathcal{H}\right\} = \frac{dx^i}{dt}; \quad \left\{\left\{x^i, \mathcal{H}\right\}, \mathcal{H}\right\} = F^i\left(x, \left\{x^i, \mathcal{H}\right\}\right) \tag{3.7}$$

3.2 The Hamiltonian Inverse Problem for linear vector fields

In view of the fact that what we are interested in this paper is a theory that is usually casted in a linear setting, i.e. Quantum Mechanics on Hilbert spaces, we will review here[83] the Inverse Problem in the Hamiltonian context for linear vector fields, and we will assume: $\mathcal{M} = \mathbb{R}^{2n}$ for some n. In the appropriate coordinates, a *linear* vector field is then a vector field of the form:

$$\Gamma = G^i{}_j x^j \frac{\partial}{\partial x^i}, \quad G^i{}_j \in \mathbb{R} \tag{3.8}$$

and the matrix $\left\|G^i{}_j\right\|$ (which represents a $(1,1)$-type tensor field) will be non-degenerate iff the origin is an isolated fixed point of Γ.

A Digression on: "Extracting the linear part" of a vector field. In general, let $\mathcal{M}$ be a smooth manifold and $\Gamma \in \mathfrak{X}(\mathcal{M})$ be a vector field with an isolated fixed point at $m_0 \in \mathcal{M}$: $\Gamma(m_0) = 0$. Considering then, for an arbitrary vector field $Y \in \mathfrak{X}(\mathcal{M})$ and function $f \in \mathcal{F}(\mathcal{M})$ the quantity $\mathcal{L}_Y(\mathcal{L}_\Gamma f)(m_0)$, it is not hard to see that it is linear in Y *and*, by virtue of $\Gamma(m_0) = 0$, in df. it defines then a $(1,1)$ tensor[34] T_Γ at m_0:

$$\mathcal{L}_Y(\mathcal{L}_\Gamma f)(m_0) = T_\Gamma(df, Y)(m_0) \tag{3.9}$$

[32] Notice that, in this as well as in the previous case, $\mathcal{M}$ has obviously to be an *even*-dimensional manifold.

[33] Which will be certainly the case if $\mathcal{M}$ is *odd*-dimensional.

[34] *Not* a tensor field, in general.

Then, the *linear part* of Γ at m_0, Γ_0, will be defined as:

$$\Gamma_0 = T_\Gamma\left(\Delta\right) \tag{3.10}$$

with Δ the Liouville field.

Indeed, in the domain of a chart $\left(x^1, ..., x^n\right)$ $(n = \dim \mathcal{M})$ with the origin at m_0 and: $\Gamma = \Gamma^i \partial/\partial x^i, Y = Y^i \partial/\partial x^i$: $\mathcal{L}_Y\left(\mathcal{L}_\Gamma f\right) = Y^i \partial\left(\Gamma^j \partial f/\partial x^j\right)/\partial x^i = Y^i\left(\partial\Gamma^j/\partial x^i\right)\left(\partial f/\partial x^j\right) + Y^i\Gamma^j\left(\partial^2 f/\partial x^i \partial x^j\right) T$. But the second term vanishes at $m_0 = 0$, and hence:

$$T_\Gamma = T^j{}_i dx^i \otimes \frac{\partial}{\partial x^j};\ T^j{}_i = \frac{\partial \Gamma^j}{\partial x^i}|_{m_0} \tag{3.11}$$

and:

$$\Gamma_0 = T_\Gamma\left(\Delta\right) = \left(\frac{\partial \Gamma^j}{\partial x^i}|_{m_0}\right) x^i \frac{\partial}{\partial x^j} \tag{3.12}$$

This is of course what one would have guessed on much more elementary grounds. The advantage of the definition (3.9) is that it provides a tensorial characterization of the linear part of a vector field at a critical point.

In a shorthand notation we can write Γ as:

$$\Gamma = \left(\widetilde{\mathbb{G}x}, \partial/\partial x\right) \tag{3.13}$$

where: $\left(\mathbb{G}x\right)^i = G^i{}_j x^j$, "$\sim$" stands for the transpose:

$$\frac{\partial}{\partial x} = \left|\begin{array}{c} \partial/\partial x^1 \\ . \\ . \\ . \\ \partial/\partial x^{2n} \end{array}\right| \tag{3.14}$$

and: $(a, b) =: a^i b_i$.

A symplectic form can be written as:

$$\omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j \tag{3.15}$$

and the matrix: $\Omega = \|\Omega_{ij}\|$ will be (pointwise) skew-symmetric and non-degenerate. ω will be said to be a *constant* symplectic form iff the Ω_{ij}'s are constant. If:

$$\Omega = \left|\begin{array}{cc} \mathbf{0}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0}_{n\times n} \end{array}\right| \tag{3.16}$$

ω will be said to be in the *canonical (or Darboux) form.* If Γ is linear and Hamiltonian w.r.t. a constant symplectic form, then the Hamiltonian is forced to be a quadratic function, i.e.:

$$\mathcal{H} = \frac{1}{2}H_{ij}x^i x^j, \quad H_{ij} \in \mathbb{R} \tag{3.17}$$

Remark 7 *The above is clearly a coordinate-dependent definition of a quadratic function. A coordinate (and dimension)-free characterization of quadratic functions, and one that is more suitable in the case of (infinite-dimensional) Hilbert spaces, can be given as follows. A mapping:* $\mathcal{H} : \mathbb{V} \to \mathbb{V}'$ *with* $\mathbb{V}, \mathbb{V}'$ *vector spaces (over a field* $\mathbb{K}$*, with* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) is quadratic(a quadratic function if* $\mathbb{V}' = \mathbb{R}$ *or* $\mathbb{C}$*) if:*

- $$\mathcal{H}(\lambda x) = \lambda^2 \mathcal{H}(x), \quad \forall x \in \mathbb{V}, \ \lambda \in \mathbb{K} \tag{3.18}$$

  and:

- $$b(x, y) =: \mathcal{H}(x + y) - \mathcal{H}(x) - \mathcal{H}(y) \tag{3.19}$$

  is a bilinear mapping for all $x, y \in \mathbb{V}$.

Remark 8 *Notice that, while* $\mathbb{G}$ *is a* $(1,1)$*-type tensor (it "maps vectors to vectors")* Ω *and* $\mathbb{H} = \|H_{ij}\|$ *are* $(0,2)$*-type tensors (they "map vectors to covectors" (and viceversa in both cases)). This difference manifests itself in the transformation under a general change of coordinates. If:* $x^i = T^i{}_j y^{\prime j}$*, then:*

$$\mathbb{G} \to \mathbb{G}' = \mathbb{T}^{-1} \mathbb{G} \mathbb{T} \tag{3.20}$$

while ($\widetilde{\mathbb{T}}$ *standing for the transpose of* $\mathbb{T}$*):*

$$\Omega \to \Omega' = \widetilde{\mathbb{T}} \Omega \mathbb{T}, \quad \mathbb{H} \to \ \mathbb{H}' = \widetilde{\mathbb{T}} \mathbb{H} \mathbb{T} \tag{3.21}$$

(the difference is not apparent when $\mathbb{T}^{-1} = \widetilde{\mathbb{T}}$*, i.e.* $\mathbb{T}$ *is an orthogonal transformation,* $\mathbb{T} \in \mathbb{O}(2n)$*).*

Restricting from now on to linear vector fields and constant symplectic structures, and omitting the superscripts and suffixes "0", if Λ is the Poisson tensor ($\{x^i, x^j\} = \Lambda^{ij}$, $\Lambda^{ij}\Omega_{jk} = \delta^i{}_k$), then if $\Gamma = G^i{}_j x^j \partial/\partial x^i$ is Hamiltonian w.r.t. $\omega = (1/2)\,\Omega_{ij} dx^i \wedge dx^j$, this implies:

$$\Omega G = -H \tag{3.22}$$

and, equivalently:

$$G = -\Lambda H \tag{3.23}$$

Hence: *Looking for a Hamiltonian description w.r.t. a constant symplectic structure for a linear vector field* Γ *is therefore equivalent to looking for the decomposition of the representative matrix* G *into the product of an* ***invertible*** *skew-symmetric matrix* Λ *and a symmetric matrix* H*. The former will provide a (non-degenerate) Poisson structure, the latter a Hamiltonian adapted to the given Poisson structure*[35].

[35] Λ will be a $(2,0)$-type tensor, and under a general linear change of coordinates (see above) will transform as: $\Lambda \to \Lambda' = T^{-1}\Lambda\widetilde{(T^{-1})}$.

At this point we can make contact with the discussion of Ch.1, where we dealt with linear Hamiltonian vector fields on a finite-dimensional Hilbert space. There it was shown that the Hermitian structure gives rise to both a metric tensor and a symplectic form, and that the two are compatible in the sense that they are connected to one another by a third structure, the complex structure J. Here too we can reconstruct a (compatible[36]) complex structure starting from the tensors Λ and H, at least in the case when H is positive-definite. If such is the case, we can find, as already discussed elsewhere, a system of coordinates in which the vector field Γ is given explicitly as a sum of independent harmonic oscillators with proper frequencies $\nu_1, .., \nu_n$ (possibly not all distinct):

$$\Gamma = \sum_{i=1}^{n} \nu_i \Gamma_i^{(0)};\ \Gamma_i^{(0)} = x_{i+n} \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_{i+n}},\ i = 1, ..., n \tag{3.24}$$

i.e.:

$$G = \begin{vmatrix} \mathbf{0}_{n\times n} & \nu \\ -\nu & \mathbf{0}_{n\times n} \end{vmatrix} \tag{3.25}$$

where: $\nu = diag\,(\nu_1, .., \nu_n)$, with the standard Poisson tensor:

$$\Lambda = \frac{1}{2} \Lambda_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{3.26}$$

whose representative matrix will be:

$$\Lambda = |\Lambda_{ij}| = \begin{vmatrix} \mathbf{0}_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & \mathbf{0}_{n\times n} \end{vmatrix} \tag{3.27}$$

and Hamiltonian: $H = (1/2) \sum_i \nu_i \left(x_i^2 + x_{i+n}^2 \right)$. It is now clear that the vector field:

$$\Gamma^{(0)} = \sum_{i=1}^{n} \Gamma_i^{(0)} \tag{3.28}$$

will be Hamiltonian with a new Hamiltonian: $H' = (1/2) \sum_i \left(x_i^2 + x_{i+n}^2 \right)$ and that, in terms of the representative matrices:

$$(\Lambda H')^2 = -\mathbb{I} \tag{3.29}$$

i.e. that the $(1, 1)$ tensor $\Lambda H'$ (whose representative matrix will coincide with the matrix (3.27)) will provide the required complex structure.

Some (necessary) consequences of Γ being Hamiltonian have been drawn in Ref.[83], namely:

1. As $\widetilde{G} = H\Lambda = \Lambda \left(\Lambda^{-1} H \Lambda \right)$, $\widetilde{G}$ is a representative of a vector field which is Hamiltonian w.r.t. the same Poisson structure with Hamiltonian: $-\Lambda^{-1} H \Lambda$. Indeed, in the basis in which Λ has the standard form, i.e.:

$$\Lambda = \begin{vmatrix} 0_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0_{n\times n} \end{vmatrix} \tag{3.30}$$

[36]See Ref.[160] and the following Ch.4.

$\Lambda^{-1} = -\Lambda$. Hence: $-\Lambda^{-1}H\Lambda = \Lambda H\Lambda$, which is symmetric. Notice, however, that in general $\widetilde{G}$ and G will not commute, nor will then the associated vector fields.

2. $G^3 = -\Lambda H\Lambda H\Lambda H = \Lambda(H\Lambda)H(-\Lambda H) = \Lambda\left(\widetilde{G}HG\right)$. More generally, G^{2k+1} can be written as:

$$G^{2k+1} = -\underbrace{\Lambda H...\Lambda H}_{2k+1} = -\Lambda\underbrace{H\Lambda...H\Lambda}_{k}H\underbrace{\Lambda H...\Lambda H}_{k} \tag{3.31}$$

i.e.:

$$G^{2k+1} = -(-)^k\Lambda\left(\widetilde{G}^kHG^k\right) \tag{3.32}$$

Hence: G^{2k+1} *will represent a Hamiltonian vector field* Γ_k *with the Hamiltonian:*

$$\mathcal{H}_k = \frac{1}{2}(-)^k\left(\widetilde{G}^kHG^k\right)_{ij}x^ix^j;\ \mathcal{H}_0 = \mathcal{H} \tag{3.33}$$

w.r.t. the **same** *Poisson structure. As the correspondence between matrices and linear vector fields is a Lie algebra homomorphism, all these Hamiltonian vector fields will commute pairwise. As the correspondence between linear vector fields and Hamiltonian functions is a Lie algebra antihomomorphism*[37]*, in the linear case* $\mathcal{H}_k$ *will be a constant of the motion for* $\Gamma_{k'}$ $\forall k,k'$*, and they will be pairwise in involution*[38]*.*

Remark 9 *If G is generic (and Hamiltonian), we will generate in this way also a maximal set of (i.e. n) constants of the motion pairwise in involution, and* Γ *will be completely integrable a' la Liouville.*

iii) As: $\widetilde{G} = H\Lambda = \Lambda^{-1}(\Lambda H)\Lambda = \Lambda^{-1}(-G)\Lambda \Rightarrow TrG = 0$ it follows that:

$$TrG^{2k+1} = 0 \ \ \forall k \tag{3.34}$$

Notes.

a) That this is a necessary condition for the representative matrix of a Hamiltonian vector field is pretty obvious. Indeed, for any vector field Γ on a symplectic $2n$-dimensional manifold, the divergence of Γ is defined by:

$$\mathcal{L}_\Gamma\omega^n =: (div\Gamma)\,\omega^n \tag{3.35}$$

where ω is the symplectic form and ω^n the symplectic volume. If the flow associated with Γ is Hamiltonian, it must be volume-preserving (Liouville's

[37] The Lie algebra on functions being defined by the Poisson bracket. Recall that: $\{f,g\} = i_{X_g}i_{X_f}\omega = L_{X_g}f = -L_{X_f}g$, with X_f, X_g the associated Hamiltonian vector fields, and that, for any two vector fields X and Y: $i_{[X,Y]} = L_Xi_Y - i_XL_Y$. Therefore: $i_{[X_f,X_g]}\omega = -d\{f,g\}$.

[38] Notice that, in general (see the previous footnote): $i_{[X_f,X_g]}\omega = -d\{f,g\}$ and that, therefore: $[X_f,X_g] = 0$ only implies in general: $\{f,g\} = const.$For linear vector fields, however, both f and g will be quadratic functions. The Poisson bracket $\{f,g\}$ will be quadratic as well, and it will be constant iff it vanishes.

theorem [4]), and this implies: $div\Gamma = 0$. But it is easy to prove that, for a linear vector field and for a constant symplectic structure: $div\Gamma = TrG$.

b) The vanishing of the trace of odd powers of G implies that the characteristic polynomial $P(\lambda)$ will contain only *even* powers of λ (i.e. $P(\lambda)$ will be actually a polynomial in λ^2 of degree n). Real roots will appear then in pairs $(\lambda, -\lambda)$ and (the coefficients of $P(\lambda)$ being real) complex roots will appear in quadruples $\left(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda}\right)$.

c) If T is an invertible matrix:

$$T^{-1}GT = -T^{-1}(\Lambda H)T = -\left(T^{-1}\Lambda\widetilde{(T^{-1})}\right)\left(\widetilde{T}HT\right) \tag{3.36}$$

Then, if T is in the *commutant* of G $([T, G] = 0)$ we find a new Hamiltonian description $(H' = \widetilde{T}HT)$ with a new Poisson structure $(\Lambda' = T^{-1}\Lambda\widetilde{(T^{-1})})$ provided: $T^{-1}\Lambda\widetilde{(T^{-1})} \neq \Lambda$. This implies that T **be not** a canonical transformation. *Any "non-canonical" matrix T in the commutant of G will provide a new Hamiltonian description for the same vector field.*

Powers of G are of course in the commutant of G. From: $\Omega G = -H$ we obtain (H being symmetric and Ω skew-symmetric): $\widetilde{G}\Omega = H$ and hence:

$$\widetilde{G}\Omega = -\Omega G \tag{3.37}$$

It is then easy to prove that, in general:

$$\widetilde{G}^h\Omega = (-)^h\Omega G^h \tag{3.38}$$

Indeed, this holds for $h = 1$. By induction: $\widetilde{G}^{h+1}\Omega = (-)^h\widetilde{G}\Omega G^h = (-)^h(-\Omega G)G^h = (-)^{h+1}\Omega G^{h+1}$.

As (Ω being skew-symmetric):

$$\Omega G^h = -\widetilde{\left(\widetilde{G}^h\Omega\right)} \tag{3.39}$$

this result implies:

$$\widetilde{G}^h\Omega = (-)^{h+1}\widetilde{\left(\widetilde{G}^h\Omega\right)} \tag{3.40}$$

and hence $\widetilde{G}^h\Omega$ will be *symmetric* for h odd (and, indeed, for $h = 2k+1$, $\widetilde{G}^{2k+1}\Omega = -\mathcal{H}_k$) and *skew*-symmetric for even $h = 2k$. Moreover:

$$\widetilde{G}(\widetilde{G}^{2k}\Omega) = -(\widetilde{G}^{2k}\Omega)G \tag{3.41}$$

i.e. $\widetilde{G}^{2k}\Omega$ will be an admissible skew-symmetric factor in the decomposition of G.

A slightly different way [135] to exploit even powers of G to generate alternative Hamiltonian descriptions is as follows (basically, we are reverting from a finite to an infinitesimal description). Let, e.g., $\Gamma_{(2)}$ be the linear vector field associated with G^2, i.e.:

$$\Gamma_{(2)} = \left(G^2\right)^i{}_j x^j \frac{\partial}{\partial x^i} \tag{3.42}$$

If the Poisson structure is given by:

$$\Lambda = \frac{1}{2}\Lambda^{hk}\frac{\partial}{\partial x^h} \wedge \frac{\partial}{\partial x^k} \tag{3.43}$$

then:

$$\mathcal{L}_{\Gamma_{(2)}}\Lambda = -\left(G^2\Lambda\right)^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = -\left(\Lambda H \Lambda H \Lambda\right)^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{3.44}$$

Notice that $G^2\Lambda = \Lambda H \Lambda H \Lambda$ is manifestly skew-symmetric. Therefore $\mathcal{L}_{\Gamma_{(2)}}\Lambda$, if it does not vanish, defines a new Poisson structure:

$$\Lambda_{(2)} = \frac{1}{2}\left(\Lambda H \Lambda H \Lambda\right)^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{3.45}$$

and Poisson brackets:

$$\{f, g\}_{(2)} = \left(\Lambda H \Lambda H \Lambda\right)^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{3.46}$$

The new Poisson structure will be non-degenerate iff both Λ and H are invertible, i.e., as $G = -\Lambda H$, iff G is invertible. Requiring then that there exists a new Hamiltonian $\mathcal{H}_{(2)}$ s.t. Γ is again Hamiltonian w.r.t. the new Poisson structure, i.e.:

$$\left\{x^i, \mathcal{H}_{(2)}\right\} = \left(\Lambda_{(2)}\right)^{ij}\frac{\partial \mathcal{H}_{(2)}}{\partial x^j} = G^i{}_j x^j \tag{3.47}$$

together with $G = -\Lambda H$ leads to:

$$\mathcal{H}_{(2)} = \frac{1}{2}H_{(2)ij}x^i x^j;\ H_{(2)} = \left(\Lambda H \Lambda\right)^{-1} \tag{3.48}$$

If G is not invertible, then one can proceed by exponentiation [83, 135].

Example 10 *We have seen in Ch.*1 *how the dynamics of a quantum system separates into that of a set of non-interacting harmonic oscillators. All finite-level quantum systems can be written as a family of harmonic oscillators with frequencies related to the eigenvalues of the Hamiltonian. It is therefore appropriate to consider here again the harmonic oscillator. For this system the above procedure (i.e. taking Lie derivatives of the Poisson structure) provides alternative Hamiltonian descriptions. Proceeding instead as in the previous discussion with* $T = G^2$ *and:*

$$G = \begin{vmatrix} 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \\ -m\Omega_1^2 & 0 & 0 & 0 \\ 0 & -m\Omega_2^2 & 0 & 0 \end{vmatrix} \tag{3.49}$$

and:

$$\Lambda = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \tag{3.50}$$

one finds:

$$G^2 = \begin{vmatrix} -\Omega_1^2 & 0 & 0 & 0 \\ 0 & -\Omega_2^2 & 0 & 0 \\ 0 & 0 & -\Omega_1^2 & 0 \\ 0 & 0 & 0 & -\Omega_2^2 \end{vmatrix} \tag{3.51}$$

and:

$$G^{-2}\Lambda\widetilde{G}^{-2} = \begin{vmatrix} 0 & 0 & 1/\Omega_1^4 & 0 \\ 0 & 0 & 0 & 1/\Omega_2^4 \\ -1/\Omega_1^4 & 0 & 0 & 0 \\ 0 & -1/\Omega_2^4 & 0 & 0 \end{vmatrix} \tag{3.52}$$

So, but for the isotropic case $\Omega_1 = \Omega_2$ in which $G^{-2}\Lambda\widetilde{G}^{-2}$ and $\Lambda H \Lambda H \Lambda$ become proportional, the two approaches appear to be genuinely different.

Example 11 *As a last (almost trivial but explanatory) example let us take the most general linear vector field in $\mathbf{R}^2 = \{(x,y)\}$:*

$$\Gamma = (ax+by)\frac{\partial}{\partial x} + (cx+dy)\frac{\partial}{\partial y} \tag{3.53}$$

corresponding to the matrix

$$G = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \ , \ a,b,c,d \in \mathbf{R}. \tag{3.54}$$

with $TrG^{2k+1} = 0$ if and only if $a = -d$. Given then the constant symplectic structure $\Omega = \alpha dx \wedge dy$ ($\alpha \in \mathbf{R}$):

$$\Omega = \alpha \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}. \tag{3.55}$$

Γ will be Hamiltonian with Hamiltonian: $H = \alpha axy + \alpha\left(by^2 - cx^2\right)/2$, corresponding to: $H = -\Omega G$.

Three situations are possible:

1. *The eigenvalues of G are $\pm\lambda; \lambda \equiv \sqrt{a^2+bc} \in \mathbf{R}$. Then there exist coordinates (x,y) such that the matrix (3.54) is of the form*

$$G = \begin{vmatrix} 0 & \lambda \\ \lambda & 0 \end{vmatrix}. \tag{3.56}$$

If we set:

$$x = A\cosh\Phi \ , \ y = A\sinh\Phi \tag{3.57}$$

then:

$$\begin{aligned} \Omega &= dH \wedge d\Phi \\ H &= \tfrac{\lambda\alpha}{2} A^2. \end{aligned} \tag{3.58}$$

2. *The eigenvalues of G are $\pm i\lambda; \lambda \equiv \sqrt{|a^2+bc|} \in \mathbf{R}$. Then G may be put in the form:*

$$G = \begin{vmatrix} 0 & \lambda \\ -\lambda & 0 \end{vmatrix}. \tag{3.59}$$

   We can now define

$$x = A\cos\Phi \ , \ y = A\sin\Phi \tag{3.60}$$

   that allow to write the symplectic form and the hamiltonian as in (3.58).

3. *Finally we consider the case $a^2 + bc = 0$, when there exist coordinates (x, y) such that G assumes the form*

$$G = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}. \tag{3.61}$$

   Now $H = \alpha y^2/2$.

Returning now to the general case, we have seen that a necessary condition for a linear vector field Γ with representative matrix G to be Hamiltonian is that the traces of odd powers of G vanish. Whether or not this is also sufficient requires a rather long analysis of the decomposition of G into Jordan blocks [22], for whose details we refer to the literature, and whose main result is contained in the following [83]:

Proposition 12 *A linear vector field Γ is Hamiltonian iff the representative matrix G satisfies $TrG^{2k+1} = 0$ and:*

i) no further condition if the eigenvalues are non-degenerate or purely imaginary,

ii) for degenerate real or genuinely complex (i.e not purely imaginary) eigenvalues the Jordan block belonging to a given eigenvalue λ has the same structure as the block belonging to $-\lambda$, this meaning that the Jordan block associated with the eigenvalue λ can be brought to the form:

$$G_{\{\lambda\}} = \left\| \begin{matrix} \mathbb{J} & \mathbf{0} \\ \mathbf{0} & -\widetilde{\mathbb{J}} \end{matrix} \right\| \tag{3.62}$$

iii) zero eigenvalues have even multiplicity.

This solves the problem of under which conditions a linear vector field is Hamiltonian, but does not tell us how many genuinely different Hamiltonians (and symplectic structures) are permissible for a given vector field. A more stringent result has also been proved in Ref.[83] and precisely that:

Proposition 13 *If Γ has non-complex (i.e. either real or purely imaginary) non-degenerate eigenvalues, then it has a minimal family (a "pencil" [81, 107]) of equivalent admissible symplectic forms parametrized by a number of parameters equal to the number of couples $(\lambda, -\lambda)$ of eigenvalues minus one (i.e. a $(n-1)$-parameter family).*

The case in which Γ has (only) purely imaginary eigenvalues is of particular interest for the analysis of (finite-dimensional, for the time being) quantum systems. Indeed, we can remark that:

- If the eigenvalues are purely imaginary, then all the motions of the system will be stable [4, 5]. Considering the decomposition: $G = -\Lambda H$ of Eq.(3.23), if H is positive, it will define an Euclidean metric[39] and, after possibly a rescaling that will be discussed in the next Chapter, Λ will define the Poisson tensor and G will become the complex structure. The system will become what we will call a *quantum system*, and that because the evolution is unitary with respect to the Hermitian structure associated with G and Λ. In this sense, as we will see shortly, the analysis of this Chapter provides also a way to classify the possible, and alternative, Hamiltonian descriptions for quantum systems.

- With reference in particular to Ch.1, if the (quantum) Hamiltonian H has a real spectrum, then (cfr. Eq.(1.19)) (the realified of) $-iH/\hbar$ will turn out to have purely imaginary eigenvalues. Even if H is not Hermitian w.r.t. the given Hermitian structure, one can always find [14, 175, 220] a modified scalar product (see again Ch.1) w.r.t. which H turns out to be Hermitian.

All this material will be expanded and put into use in the next Chapter.

3.3 Inequivalent Descriptions

In this section we discuss some methods to obtain inequivalent descriptions for a given classical system defined by a dynamical vector field Γ, not necessarily a linear one.

3.3.1 Alternative Hamiltonian descriptions

As explained in Appendix A, given any 1-1 tensor T, we can define an antiderivation d_T which acts on functions as

$$d_T f \equiv T(df) \ . \tag{3.63}$$

[39] Or a pseudo-Euclidean one if it is non-degenerate but not necessarily positive.

In the sequel we will use extensively this construction with $T = J$, the complex structure. Suppose now that the function F be a constant of motion and that the tensor T be invariant under the action of Γ so that

$$L_\Gamma F = 0 \ , \ L_\Gamma T = 0 \ . \tag{3.64}$$

Then we can define a closed two-form

$$\omega_F \equiv d(d_T F) \tag{3.65}$$

which is invariant under action of Γ since $L_\Gamma \omega_F = d(L_\Gamma d_T F)$ and $L_\Gamma d_T F = 0$ because of (3.64). Assuming that ω_F be non-degenerate, it will define a new invariant symplectic structure. To obtain the alternative Hamiltonian function H associated to ω_F it is sufficient to notice that:

$$0 = L_\Gamma d_T F = i_\Gamma d(d_T F) + d i_\Gamma (d_T F) = i_\Gamma \omega_F + dF(T(\Gamma)) = i_\Gamma \omega_F + d(L_{T(\Gamma)} F) \ . \tag{3.66}$$

Hence:

$$H = -L_{T(\Gamma)} F = -\left(d_T F\right)(\Gamma) \tag{3.67}$$

Remark 14 *The above construction may turn out to be empty if the function F is in the kernel of dd_T : $dd_T F = 0$. For example, in $\mathbb{R}^2 \approx \mathbb{C}$ with (real) coordinates (q, p), take T to be the complex structure :*

$$J = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p} \tag{3.68}$$

which is invariant under the dynamics of the $1D$ harmonic oscillator. Then, it is immediate to check that:

$$dd_J F = \left(\frac{\partial^2 F}{\partial q^2} + \frac{\partial^2 F}{\partial p^2} \right) dq \wedge dp \tag{3.69}$$

and hence all the harmonic functions in the plane will be in the kernel of dd_J.

Remark 15 *Suppose now that $\Gamma = G^i{}_j x^j \frac{\partial}{\partial x^i}$ be a linear vector field and $T = T^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}$ a constant invariant 1-1 tensor. Then it is not difficult to check that ω_F is constant if and only if F is a quadratic function:*

$$F = \frac{1}{2} F_{ij} x^i x^j \ , \ F_{ij} = F_{ji} \ . \tag{3.70}$$

In this case, using the matrix notation of sect. 3.2, we have:

$$H = \frac{1}{2} H_{ij} x^i x^j \ , \ H_{ij} = H_{ji} = -(FTG)_{ij} - (FTG)_{ji} \ , \tag{3.71}$$

$$\omega_F = \frac{1}{2} \Omega_{ij} dx^i \wedge dx^j \ , \ \Omega_{ij} = -\Omega_{ji} = (FT)_{ij} - (FT)_{ji} \ . \tag{3.72}$$

Using the fact that Eqs. (3.64) are equivalent to the conditions: $(FG)_{ij} = -(FG)_{ji}$ *and* $(GT)^i{}_j = (TG)^i{}_j$*, one can show that, as it should be, the relation* $\Omega G = -H$ *is trivially satisfied.*

As an example, let us consider the two-dimensional isotropic harmonic oscillator whose dynamics is described by the vector field

$$\Gamma = p^a \frac{\partial}{\partial q^a} - q^a \frac{\partial}{\partial p^a} \ , \tag{3.73}$$

where the summed-over index a assumes the values: $a = 1, 2$. We will take for T the complex structure of the phase space $\mathbb{R}^4$, i.e:

$$T = J = dp^a \otimes \frac{\partial}{\partial q^a} - dq^a \otimes \frac{\partial}{\partial p^a} \ . \tag{3.74}$$

Thus the representative matrices will be:

$$G = \left| \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right| \ , \ T = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| \ . \tag{3.75}$$

With $T = J$ we have:

$$-J(\Gamma) = \Delta \equiv q^a \frac{\partial}{\partial q^a} + p^a \frac{\partial}{\partial p^a} \tag{3.76}$$

with Δ the dilation (Liouville) field associated with the standard linear structure on $\mathbb{R}^4$ and:

$$d_J F = \frac{\partial F}{\partial q^a} dp^a - \frac{\partial F}{\partial p^a} dq^a \Rightarrow \omega_F = d d_J F = \left(\frac{\partial^2 F}{\partial q^a \partial q^b} + \frac{\partial^2 F}{\partial p^a \partial p^b} \right) dq^a \wedge dp^b \tag{3.77}$$

as well as:

$$H = -L_{J(\Gamma)} F = L_\Delta F \tag{3.78}$$

for any function $F = F(\mathbf{q}, \mathbf{p})$.

It is well known that a basis of constants of motion is given, for example, by the four independent functions

$$\begin{array}{ll} F_0 = \frac{1}{4} \left[(p^1)^2 + (q^1)^2 + (p^2)^2 + (q^2)^2 \right] \ , & F_1 = \frac{1}{4} \left[(p^1)^2 + (q^1)^2 - (p^2)^2 - (q^2)^2 \right] \ , \\ F_2 = \frac{1}{2} \left[p^1 p^2 + q^1 q^2 \right] , & F_3 = \frac{1}{2} \left[q^1 p^2 - q^2 p^1 \right] \ . \end{array} \tag{3.79}$$

All four functions being quadratic[40], the above construction yields then the following four alternative hamiltonian descriptions:

$$\begin{array}{ll} H_0 = \frac{1}{2} \left[(p^1)^2 + (q^1)^2 + (p^2)^2 + (q^2)^2 \right] \ , & \omega_0 = dq^1 \wedge dp^1 + dq^2 \wedge dp^2 \ ; \\ H_1 = \frac{1}{2} \left[(p^1)^2 + (q^1)^2 - (p^2)^2 - (q^2)^2 \right] \ , & \omega_1 = dq^1 \wedge dp^1 - dq^2 \wedge dp^2 \ ; \\ H_2 = p^1 p^2 + q^1 q^2 \ , & \omega_2 = dq^1 \wedge dp^2 + dq^2 \wedge dp^1 \ ; \\ H_3 = q^1 p^2 - q^2 p^1 \ , & \omega_3 = -dq^1 \wedge dq^2 + dp^1 \wedge dp^2 \ . \end{array} \tag{3.80}$$

[40] $L_\Delta F_i = 2F_i$ for $i = 0, 1, 2, 3$.

Both the Liouville field Δ and the complex structure J are associated with the standard linear structure on $\mathbb{R}^4$. As we will now see, this observation may be exploited to obtain alternative Hamiltonian descriptions by defining inequivalent linear structures on phase space.

3.3.2 Inequivalent Descriptions from Alternative Linear Structures

We recall here [66] some known facts about the possibility of defining alternative (i.e. not linearly related) linear structures on a vector space and/or of using the linear structure of a vector space to endow with a linear structure manifolds that are related to the given vector space.

Let E be a (real or complex) linear vector space with addition $+$ and multiplication by scalars $\cdot$, and a nonlinear diffeomorphism:

$$\phi : E \leftrightarrow E. \tag{3.81}$$

We can define a new linear structure if we define:

- Addition of $u, v \in M$ as:

$$u +_{(\phi)} v =: \phi(\phi^{-1}(u) + \phi^{-1}(v)). \tag{3.82}$$

- Multiplication by a scalar $\lambda \in \mathbb{R}$ or $\mathbb{C}$ of $u \in M$ as:

$$\lambda \cdot_{(\phi)} u =: \phi\left(\lambda \phi^{-1}(u)\right). \tag{3.83}$$

Obviously, the two linear spaces $(E, +, \cdot)$ and $(E, +_{(\phi)}, \cdot_{(\phi)})$ are finite dimensional vector spaces of the same dimension and hence are isomorphic. However, the change of coordinates defined by ϕ that we are using to "deform" the linear structure is a nonlinear diffeomorphism. In other words, we are using two different (diffeomorphic but not linearly related) global charts to describe the same manifold space E

Within the framework of the new linear structure, it makes sense to consider the mapping:

$$\Psi : M \times \mathbb{R} \to M \quad , \quad \Psi(u, t) =: e^t \cdot_{(\phi)} u =: u(t), \tag{3.84}$$

that defines a one-parameter group as it can be easily checked. Its infinitesimal generator, the dilation (Liouville) field, is given by

$$\Delta(u) = \left[\frac{d}{dt} u(t)\right]_{t=0} = \left[\frac{d}{dt} \phi\left(e^t \phi^{-1}(u)\right)\right]_{t=0}. \tag{3.85}$$

As an example[41] consider $T^*\mathbb{R}$ with coordinates (q, p) and linear structure defined by the dilation field:

$$\Delta = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p}, \tag{3.86}$$

[41] More examples may be found in Ref.[66].

which is such that $i_{\Delta}\omega = qdp - pdq$ with respect to the standard symplectic form $\omega = dq \wedge dp$.
As it is well known the dynamics of the $1D$ harmonic oscillator is described, in appropriate units, by the vector field:

$$\Gamma = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}, \tag{3.87}$$

which is ω-Hamiltonian: $i_{\Gamma}\omega = dH$ with Hamiltonian: $H = \left(q^2 + p^2\right)/2$. We can also define the complex structure:

$$J = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p} \tag{3.88}$$

which is such that:

$$J^2 = -\mathbb{I},\ \ J\left(\Delta\right) = \Gamma,\ \ J\left(\Gamma\right) = -\Delta \tag{3.89}$$

The composition of the symplectic and the complex structures gives rise to a compatible [160] metric tensor g:

$$\omega \circ J =: -g, \quad g = dq \otimes dq + dp \otimes dp \tag{3.90}$$

Notice also that the complex structure and the Hamiltonian are connected by:

$$\omega = \frac{1}{2} dd_J H \tag{3.91}$$

Let us consider now the nonlinear change of coordinates on $T^*\mathbb{R}$ [174]: $(q, p) \to (Q, P)$ with:

$$Q = q\left(1 + f\left(H\right)\right) \tag{3.92}$$
$$P = p\left(1 + f\left(H\right)\right). \tag{3.93}$$

Under very mild assumptions on the function $f\left(H\right)$ the mapping (3.93) will be smooth and invertible with a smooth inverse. One might assume, e.g., that $f\left(\cdot\right)$ be nonnegative and monotonically increasing for positive argument.With the dynamics given by Eq.(3.87), it is immediate to check that:

$$L_{\Gamma}Q = P, \quad L_{\Gamma}P = -Q \tag{3.94}$$

Hence, although the two coordinates ystem are *not* linearly related, the vector field Γ will be given, in the new coordinate system, by:

$$\Gamma = P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P} \tag{3.95}$$

which will be again Hamiltonian with respect to the symplectic form $\omega' = dQ \wedge dP$ with $H' = \left(Q^2 + P^2\right)/2 = H\left(1 + f\left(H\right)\right)^2$ as Hamiltonian. Now the new Liouville field Δ', defined via $i_{\Delta'}\omega' = QdP - PdQ$, is given by:

$$\Delta' = Q\frac{\partial}{\partial Q} + P\frac{\partial}{\partial P}\ , \tag{3.96}$$

Notice also that we can define a new 1-1 tensor (the new complex structure):

$$J' = dP \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial P} \ , \tag{3.97}$$

which is again such that $J'(\Gamma) = -\Delta'$. J' and ω' will generate then the new metric tensor: $g' = dQ \otimes dQ + dP \otimes dP$. Thus, following the construction outlined in the previous section, we might have obtained this alternative description of the dynamics of the one-dimensional harmonic oscillator also by setting:

$$T = J' \ , \tag{3.98}$$

$$\omega' = \frac{1}{2} dd_{J'} H' \ . \tag{3.99}$$

One obtains in this way a new linear structure, which is in some sense "adapted" to the chosen Hamiltonian description.

Finally, we observe that the above construction to obtain alternative descriptions may be easily generalized to the n-dimensional harmonic oscillator by defining

$$\omega_F \equiv \alpha_a d\left(\frac{\partial F}{\partial p^a}\right) \wedge d\left(\frac{\partial F}{\partial q^a}\right) \tag{3.100}$$

and

$$H_F \equiv \frac{1}{2}\alpha_a \left[\left(\frac{\partial F}{\partial p^a}\right)^2 + \left(\frac{\partial F}{\partial q^a}\right)^2\right] \ , \tag{3.101}$$

where F is a constant of the motion such that ω_F is non-degenerate.

3.3.3 Alternative Lagrangian Descriptions Coming from "Adapted" Linear Structures

Switching now to the Lagrangian framework, we recall [186] that a regular Lagrangian $\mathcal{L}$ will define the symplectic structure on TQ:

$$\omega_{\mathcal{L}} = d\theta_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \wedge dq^i; \ \theta_{\mathcal{L}} = \left(\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i. \tag{3.102}$$

We look now [155] for Hamiltonian vector fields X_j, Y^j such that:

$$i_{X_j}\omega_{\mathcal{L}} = -d\left(\frac{\partial \mathcal{L}}{\partial u^j}\right), \ i_{Y^j}\omega_{\mathcal{L}} = dq^j \tag{3.103}$$

Explicitly this implies:

$$L_{X_j} q^i = \delta^i_j, \ L_{X_j} \tfrac{\partial \mathcal{L}}{\partial u^i} = 0, \tag{3.104}$$

$$L_{Y^j} q^i = 0, \ L_{Y^j} \tfrac{\partial \mathcal{L}}{\partial u^i} = \delta^j_i. \tag{3.105}$$

Using then the identity $i_{[Z,W]} = L_Z \circ i_W - i_W \circ L_Z$, and the fact that the Lie derivative of the Hamiltonian of every field of the set (3.103) with respect to any other of the fields is either zero or a constant (actually unity), one can show that:

$$i_{[Z,W]}\omega_{\mathcal{L}} = 0 \text{ whenever } [Z,W] = [X_i, X_j], \left[X_i, Y^j\right], \left[Y^i, Y^j\right], \tag{3.106}$$

which proves that:

$$[X_i, X_j] = \left[X_i, Y^j\right] = \left[Y^i, Y^j\right] = 0. \tag{3.107}$$

This defines an infinitesimal action of an Abelian Lie group on TQ. If this integrates to an action of the group $\mathbb{R}^{2n}$ ($\dim Q = n$) that is free and transitive, this will define a new vector space structure on TQ that is "adapted" to the Lagrangian two-form $\omega_{\mathcal{L}}$. More explicitly, defining dual forms $\left(\alpha^i, \beta_i\right)$ via: $\alpha^i(X_j) = \delta^i_j$, $\alpha^i\left(Y^j\right) = 0$; $\beta_i\left(Y^j\right) = \delta^j_i$, $\beta_i(X_j) = 0$, it is immediate to see that:

$$\alpha^i = dq^i \tag{3.108}$$

$$\beta_i = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \tag{3.109}$$

and that the symplectic form can be written as:

$$\omega_{\mathcal{L}} = \beta_i \wedge \alpha^i. \tag{3.110}$$

Basically, what this means is that, to the extent that the definition of vector fields and dual forms is global, we have found in this way a global Darboux chart.

As an example of this construction, we may consider a particle in a (time-independent) magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The corresponding second-order vector field is given by ($e = m = c = 1$):

$$\Gamma = u^i \frac{\partial}{\partial q^i} + \delta^{is}\epsilon_{ijk} u^j B^k \frac{\partial}{\partial u^s}. \tag{3.111}$$

The Lagrangian is given in turn by :

$$\mathcal{L} = \frac{1}{2}\delta_{ij} u^i u^j + u^i A_i. \tag{3.112}$$

while the symplectic form is:

$$\omega_{\mathcal{L}} = \delta_{ij} dq^i \wedge du^j - \frac{1}{2}\varepsilon_{ijk} B^i dq^j \wedge dq^k. \tag{3.113}$$

The field Γ is hamiltonian, the Hamiltonian being given by:

$$H = \frac{1}{2}\delta_{ij} u^i u^j. \tag{3.114}$$

Now it is easy to see that:

$$X_j = \frac{\partial}{\partial q^j} - \delta^{ik}\frac{\partial A_k}{\partial q^j}\frac{\partial}{\partial u^i}, \tag{3.115}$$

$$Y^j = \delta^{jk}\frac{\partial}{\partial u^k}. \tag{3.116}$$

The dual forms $\alpha^i, \beta_i, i = 1, ..., n = \dim Q$ are given by:

$$\alpha^i = dq^i, \tag{3.117}$$

$$\beta_i = \delta_{ij} d(u^j + \delta^{jk} A_k). \tag{3.118}$$

Therefore the mapping

$$Q^i = q^i \tag{3.119}$$

$$U^i = u^i + \delta^{ik} A_k, \tag{3.120}$$

provides us with a symplectomorphism that reduces $\omega_{\mathcal{L}}$ to the canonical form

$$\omega_{\mathcal{L}} = dq^i \wedge d\pi_i, \tag{3.121}$$

where $\pi_i = \delta_{ij} U^j$. We may say that the chart (Q, U) is a Darboux chart "adapted" to the vector potential $\overrightarrow{A}$.

The Liouville field will be[42] then:

$$\Delta = Q^i \frac{\partial}{\partial Q^i} + \left[U^i + \delta^{ik}\left(Q^j \frac{\partial A_k}{\partial Q^j} - A_k\right)\right]\frac{\partial}{\partial U^i}. \tag{3.122}$$

Denoting collectively the old and new coordinates as (q, u) and (Q, U) respectively, Eq. (3.120) defines a mapping:

$$(q, u) \xrightarrow{\phi} (Q, U). \tag{3.123}$$

It is then a straightforward application of the definitions (3.82) and (3.83) to show that the rules of addition and multiplication by a constant become, in this specific case:

$$(Q, U) +_{(\phi)} (Q', U') = (Q + Q', U + U' + [A(Q + Q') - (A(Q) + A(Q'))]) \tag{3.124}$$

and:

$$\lambda \cdot_{(\phi)} (Q, U) = (\lambda Q, \lambda U + [A(\lambda Q) - \lambda A(Q)]). \tag{3.125}$$

[42] We notice that Δ depends on the gauge choice. The symplectic form will be however gauge-independent

3.4 Symmetries and Constants of the Motion for Systems Admitting of Alternative Descriptions

3.4.1 Introduction

In our setting, according to which the primitive (or the more physically relevant [167]) object is the vector field Γ describing the dynamics on some carrier space $\mathcal{M}$, a *symmetry* will be defined as a one-parameter group of diffeomorphisms of the carrier space that maps solutions (i.e. integral curves of Γ) into solutions. At the infinitesimal level, if $X \in \mathfrak{X}(\mathcal{M})$ is the associated infinitesimal generator of the one-parameter group, this means [167] that it must commute with Γ, i.e.:

$$[X, \Gamma] = 0 \tag{3.126}$$

It is a straightforward consequence of the Jacobi identity on the commutator bracket that[43]:

$$[X_1, \Gamma] = 0, [X_2, \Gamma] = 0 \Rightarrow [[X_1, X_2], \Gamma] = 0 \tag{3.127}$$

(but not viceversa, of course). Hence: *All the vector fields satisfying the condition (3.126) for a given dynamical vector field* Γ *close on a Lie algebra, the Lie algebra of (infinitesimal) symmetries of* Γ.

On the other hand, *constants of the motion* are, as is well known, functions $f \in \mathcal{F}(\mathcal{M})$ that are invariant under the flow of Γ, i.e.:

$$L_\Gamma f = 0 \tag{3.128}$$

where L_Γ is the Lie derivative. A considerable effort is usually devoted in textbooks (both in point-particle Mechanics and/or in Field Theory, both elementary and more advanced) to try and define a clear-cut procedure allowing to associate constants of the motion (i.e. conserved quantities) with symmetries (and the other way around). This goes usually through the use of Nöther's Theorem[44], that, for completeness, we will revisit briefly here both in the Lagrangian and Hamiltonian formulations of point-particle Mechanics.

3.4.2 The Nöther Theorem

1. Lagrangian Formalism. In this case $\mathcal{M} = TQ$, with Q a base manifold with (local) coordinates $q^1, ..., q^n$, $n = \dim(Q)$. Before proceeding, we recall how vector fields on the base manifold can be lifted to vector fields on TQ. Given:

$$X = X^i \frac{\partial}{\partial q^i} \in \mathfrak{X}(Q), \quad X^i \in \mathcal{F}(Q) \tag{3.129}$$

[43]This is very much reminiscent of Poisson's theorem of Hamiltonian Mechanics.

[44]See however, e.g., Ref.[186] for the discussion of different approaches.

the *tangent lift* (sometimes called also the *complete lift*) X^c of X is defined as[45]:

$$X^c = X^i \frac{\partial}{\partial q^i} + (L_{\Gamma_0} X^i) \frac{\partial}{\partial u^i} \in \mathfrak{X}(TQ) \tag{3.130}$$

where the u^i's are coordinates along the fibers and Γ_0 is *any* second-order vector field.

If $\mathcal{L}$ is a Lagrangian appropriate for the description, via the Euler-Lagrange equations, of the dynamics associated with a given second-order vector field Γ, a *Nöther symmetry* [186] is, by definition, a tangent lift X^c that is a symmetry for Γ, i.e. such that:

$$[\Gamma, X^c] = 0 \tag{3.131}$$

and such that:

$$L_{X^c}\mathcal{L} = L_\Gamma h \tag{3.132}$$

where[46]: $h = \pi^* g$, $g \in \mathcal{F}(Q)$ and: $\pi : TQ \to Q$ is the canonical projection. The Lagrangian will be said to be *strictly invariant* if $h = 0$ (i.e. $g = 0$)[47], *quasi-invariant* [161] if $g \neq 0$ Nöther's theorem states then that:

$$F_{X^c} =: i_{X^c}\theta_{\mathcal{L}} - h \tag{3.133}$$

is a constant of the motion. Here:

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial u^i} dq^i \tag{3.134}$$

is the Lagrangian one-form associated with $\mathcal{L}$. In local coordinates:

$$F_{X^c} = X^i \frac{\partial \mathcal{L}}{\partial u^i} - h \tag{3.135}$$

2. <u>*Hamiltonian Formalism.*</u> In this case $\mathcal{M} = T^*Q$, the cotangent bundle of the base manifold, with local coordinates (q^i, p_i), $i = 1, ..., n$, equipped with the Cartan form:

$$\theta_0 = p_i dq^i \tag{3.136}$$

and the symplectic structure:

$$\omega_0 = -d\theta_0 = dq^i \wedge dp_i \tag{3.137}$$

Here too there is a standard procedure for lifting vector fields from $\mathfrak{X}(Q)$ to $\mathfrak{X}(T^*Q)$. namely, given a vector field $X \in \mathfrak{X}(Q)$ of the form (3.129),

[45] Here, with abuse of notation, we write X^i for what should be instead $\pi^* X^i$, with: $\pi : TQ \to Q$ the canonical projection.

[46] Of course this is nothing but the familiar statement that, under the action of X^c, the Lagrangian changes by the total time derivative of a function of the q's alone.

[47] Barring the trivial case h (i.e. g)= *const.*, a second-order vector field does not admit of constants of the motion that are functions of the q's alone.

the *cotangent lift* (sometimes called the *natural lift*) X^* of X is given by:

$$X^* = X^i \frac{\partial}{\partial q^i} - \left(\frac{\partial X^j}{\partial q^i} \right) p_j \frac{\partial}{\partial p_i} \in \mathfrak{X} \left(T^* Q \right) \tag{3.138}$$

and it is easy to show that it is the *unique* vector field that projects down to X on the base manifold and that leaves the Cartan form invariant, i.e. such that:

$$L_{X^*} \theta_0 = 0 \tag{3.139}$$

Remark 16 *In a more intrinsic way, both lifts can be defined [167] as the infinitesimal generators of the tangent or, respectively, cotangent lift of the one-parameter group of diffeomorphisms of Q that has X as its infinitesimal generator.*

Remark 17 *Symmetries for the dynamics that are (tangent or cotangent) lifts of vector fields on the base manifold are also called point symmetries.*

A vector field $\Gamma \in \mathfrak{X}(T^*Q)$ is *Hamiltonian* if there exists a (Hamiltonian) function $H \in \mathcal{F}(T^*Q)$ such that:

$$i_\Gamma \omega_0 = dH \tag{3.140}$$

Given then a function $F \in \mathcal{F}(TQ)$, let X_F be the associated Hamiltonian vector field (not necessarily a cotangent lift), i.e.: $i_{X_F} \omega_0 = dF$. Then:

$$L_{X_F} H = i_{X_F} dH = i_{X_F} i_\Gamma \omega_0 = -i_\Gamma i_{X_F} \omega_0 = -i_\Gamma dF = -L_\Gamma F \tag{3.141}$$

Hence:

$$L_\Gamma F = 0 \Leftrightarrow L_{X_F} H = 0 \tag{3.142}$$

Therefore, if X_F is a symmetry for the Hamiltonian (i.e.: $L_{X_F} H = 0$), then F will be a constant of the motion and viceversa. Moreover, using the identity [186]:

$$i_{[X,Y]} = i_X \circ L_Y - L_Y \circ i_X \tag{3.143}$$

valid for any pair of vector fields, it follows that, if X is at least locally Hamiltonian (i.e.: $L_X \omega_0 = 0$), then:

$$dL_X H = L_X i_\Gamma \omega_0 = -i_{[X,\Gamma]} \omega_0 \tag{3.144}$$

Hence, if X is a symmetry for the Hamiltonian, and as ω_0 is non-degenerate:

$$L_X \omega_0 = 0 \;\; \textit{and} \;\; L_X H = 0 \Rightarrow [X, \Gamma] = 0 \tag{3.145}$$

i.e. X is also a symmetry for the dynamics. The converse however is not true [167]: from: $[X, \Gamma] = 0$ one can only infer that: $L_X H = \textit{const.}$, i.e. X need not be a symmetry for the Hamiltonian.

So far for the standard derivation of the Nöther Theorem. As a simple example, considering, e.g., the $3D$ harmonic oscillator with the standard Lagrangian: $\mathcal{L} = (1/2)\sum_{i=1}^{3}\left[\left(u^i\right)^2 - \left(q^i\right)^2\right]$ or the corresponding Hamiltonian leads to the well-known association of (strict) rotational invariance (of the Lagrangian and/or of the Hamiltonian) with the conservation of angular momentum.

The motivation for having gone here to some length through essentially standard material has been to emphasize the crucial rôle that "intermediate" structures such as the Lagrangian or the Hamiltonian, as well as the symplectic structure, play along the way that leads to the association of symmetries with constants of the motion. When these "intermediate" structures are not unique, as it happens when more non-equivalent (Lagrangian (on TQ) or Hamiltonian (on T^*Q)) descriptions are available [36, 47, 130, 154, 155, 156, 159, 164, 186, 200], the connection becomes more ambiguous, and different (non-equivalent) descriptions of the same dynamical system may lead to the association of different constants of the motion with the same group of symmetries, or of the same constants of the motion with different groups of symmetry or to no association at all, as we shall discuss now.

3.4.3 Alternative Descriptions and Symmetries in the Lagrangian Formalism

We will consider here some simple examples:

1. Let $Q = \mathbb{R}^3$, and let Γ be the dynamics of an isotropic harmonic oscillator (with unit mass and frequency for simplicity). Then it is immediate to show all the Lagrangians of the form:

$$\mathcal{L}_B = \frac{1}{2}B_{ij}\left(u^i u^j - q^i q^j\right) \tag{3.146}$$

where: $B = \|B_{ij}\|$ is a real and (necessarily) symmetric matrix are admissible Lagrangians for the isotropic harmonic oscillator, and, moreover, regular ones iff the matrix B is non-singular. By "admissible" we mean obviously that the Euler-Lagrange equations associated with any one of the Lagrangians (3.146) reproduce the dynamics of the isotropic harmonic oscillator. As we can always diagonalize B with the aid of an orthogonal transformation, we can limit ourselves to considering only either the standard Lagrangian:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3;\ \mathcal{L}_i = \frac{1}{2}\left[\left(u^i\right)^2 - \left(q^i\right)^2\right],\ i = 1,2,3 \tag{3.147}$$

or (up to an overall sign and an overall factor):

$$\mathcal{L}' = \mathcal{L}_1 + \mathcal{L}_2 - \mathcal{L}_3 \tag{3.148}$$

Now, it is obvious that the Lagrangian (3.147) is (strictly) invariant under the (lifted) action of $O(3)$, while the Lagrangian (3.148) is (again, strictly) invariant under the (lifted) action of $O(2,1)$, the Lorentz group in $(2+1)$ dimensions. As it can be proved [186] that, in any number n of dimensions, the most general group of point symmetries for the dynamics of the isotropic harmonic oscillator is $GL(n,\mathbb{R})$, the above two groups are groups of Nöther symmetries. While invariance under $O(3)$ associates, via Nöther's theorem, the three components of the angular momentum with the three generators of the group if the Lagrangian (3.147) is chosen as the Lagrangian of the system, in the case in which one chooses $\mathcal{L}'$ as the Lagrangian the situation is different. The three generators of $O(2,1)$ are given by the tangent lifts of the vector fields:

$$X_1 = q^3\frac{\partial}{\partial q^1} + q^1\frac{\partial}{\partial q^3},\ X_2 = q^3\frac{\partial}{\partial q^2} + q^2\frac{\partial}{\partial q^3},\ J = q^1\frac{\partial}{\partial q^2} - q^2\frac{\partial}{\partial q^1} \quad (3.149)$$

While X_1 and X_2 correspond to "boosts" in the q^1 and q^2 directions, J represents ordinary rotations in the $\left(q^1 - q^2\right)$ plane. They close on the Lie algebra $\mathfrak{o}(2,1)$, namely:

$$[X_1, X_2] = J,\ \ [X_1, J] = X_2,\ \ [J, X_2] = X_1 \quad (3.150)$$

and the same will hold true for the tangent lifts X_1^c, X_2^c and J^c.

Applying now Nöther's theorem we find the following constants of the motion:

$$F_1 =: i_{X_1^c}\theta_{\mathcal{L}'} = q^3u^1 - q^1u^3;\ F_2 =: i_{X_2^c}\theta_{\mathcal{L}'} = q^3u^2 - q^2u^3 \quad (3.151)$$

while, as before: $i_{J^c}\theta_{\mathcal{L}'} = q^1u^2 - q^2u^1$. Therefore, we find that *the angular momentum is the (vector) constant of the motion associated not with the rotation group but instead with the Lorentz group* $O(2,1)$.

2. Suppose however that we want to look for infinitesimal (strict) symmetries of the Lagrangian (3.146) without performing changes of coordinates (i.e. without diagonalizing the matrix B). We will consider here only linear vector fields that are generators of point symmetries, i.e. vector fields of the form[48]:

$$X = A^i{}_j\left(q^j\frac{\partial}{\partial q^i} + u^j\frac{\partial}{\partial u^i}\right) \quad (3.152)$$

for some matrix $A = \left\|A^i{}_j\right\| \in End(Q)$. Then:

$$L_X\mathcal{L}_B = (BA)_{jk}\left(u^ju^k - q^jq^k\right) \quad (3.153)$$

and strict invariance requires: $(BA)_{jk} + (BA)_{kj} = 0$, i.e.(as B is symmetric):

$$A^tB + BA = 0 \quad (3.154)$$

[48] This is the case of the symmetries (3.149).

which means that the matrix AB has to be antisymmetric. For example, with the Lagrangian (3.148): $B = diag\,(1,1,-1)$ and, e.g. for the first symmetry X_1 of Eq.(6.22):

$$A = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{3.155}$$

($A = A^t$) and it is easy to check that the condition (3.154) is indeed satisfied.

By assumption, the matrix B in Eq.(3.146) can be diagonalized with the aid of an orthogonal transformation: $B = OB'O^t$ with B' diagonal and: $OO^t = O^tO = Id$. Then it is easy to see that Eq.(3.154) becomes:

$$A'^t B' + B'A' = 0 \tag{3.156}$$

with:

$$A' = O^t A O \tag{3.157}$$

defining the transformed infinitesimal symmetry in the new coordinate system.

3. Consider, as a further example, the (isotropic) harmonic oscillator in $2D$. Apart from the standard Lagrangian ($\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ in the notation of Eq.(3.147)) we may consider the (regular) Lagrangian:

$$\mathcal{L}' = u^1 u^2 - q^1 q^2 \tag{3.158}$$

This Lagrangian is (strictly) invariant under the "squeeze" transformation, i.e. the tangent lift of the one-parameter group:

$$\left(q^1, q^2\right) \mapsto \left(q^1 e^t, q^2 e^{-t}\right);\ t \in \mathbb{R} \tag{3.159}$$

whose infinitesimal generator is:

$$S = q^1 \frac{\partial}{\partial q^1} - q^2 \frac{\partial}{\partial q^2} \tag{3.160}$$

that lifts to:

$$S^c = q^1 \frac{\partial}{\partial q^1} - q^2 \frac{\partial}{\partial q^2} + u^1 \frac{\partial}{\partial u^1} - u^2 \frac{\partial}{\partial u^2} \tag{3.161}$$

Nöther's theorem yields then the constant of the motion:

$$F =: i_{s^c} \theta_{\mathcal{L}'} = q^1 u^2 - q^2 u^1 \tag{3.162}$$

Hence: *with the Lagrangian $\mathcal{L}'$ angular momentum is associated with invariance under squeeze.*

In the notation of the previous example, here: $B = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \sigma_1$ and: $A = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \sigma_3$, and, again, they satisfy the condition (3.154).

4. The Lagrangian (3.158) can be diagonalized via a rotation of $\pi/4$ to new coordinates:
$$Q^1 = \frac{q^1 + q^2}{\sqrt{2}}, \; Q^2 = \frac{q^1 - q^2}{\sqrt{2}} \tag{3.163}$$
(and similarly for the velocities), whereby the Lagrangian becomes (cfr.Eqs.(3.147) and (3.148)):
$$\mathcal{L} \to \mathcal{L}_1 - \mathcal{L}_2 \tag{3.164}$$
Now, the "squeeze" transformation (3.159) becomes:
$$\begin{vmatrix} Q^1 \\ Q^2 \end{vmatrix} \longrightarrow \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} \cdot \begin{vmatrix} Q^1 \\ Q^2 \end{vmatrix} \tag{3.165}$$
whose infinitesimal generator is:
$$X = Q^2 \frac{\partial}{\partial Q^1} + Q^1 \frac{\partial}{\partial Q^2} \tag{3.166}$$
(corresponding to the matrix: $A = \sigma_1$) i.e., as expected, a (the unique) Lorentz boost with the parameter t playing the rôle of the rapidity of the boost.

3.5 The Transition to the Hamiltonian Formalism

3.5.1 Preliminaries and Recollections

Restricting ourselves for simplicity to dynamical systems described by regular Lagrangians, we recall [167],[186] that the Euler-Lagrange equations for the second-order vector field Γ associated with a regular Lagrangian $\mathcal{L}$ can be written, in intrinsic terms, as:
$$L_\Gamma \theta_{\mathcal{L}} - d\mathcal{L} = 0 \tag{3.167}$$
where:
$$\theta_{\mathcal{L}} =: \frac{\partial \mathcal{L}}{\partial u^i} dq^i \tag{3.168}$$
is the Lagrangian one-form or in the equivalent, "Hamiltonian" form:
$$i_\Gamma \Omega_{\mathcal{L}} = dE_{\mathcal{L}} \tag{3.169}$$
where:
$$\Omega_{\mathcal{L}} =: -d\theta_{\mathcal{L}} \tag{3.170}$$
is the "Lagrangian two-form", which is symplectic if $\mathcal{L}$ is regular, and:
$$E_{\mathcal{L}} =: i_\Gamma \theta_{\mathcal{L}} - \mathcal{L} \tag{3.171}$$
is known [167],[186] as the "energy function" associated with the Lagrangian $\mathcal{L}$.

The transition to the Hamiltonian formulation on T^*Q is accomplished, as is well known[167], with the aid of the "fiber derivative" (or "Legendre map"): $F\mathcal{L}: TQ \to T^*Q$ that is defined by:

$$F\mathcal{L} : \left(q^i, u^i\right) \mapsto \left(q^i, p_i = \partial\mathcal{L}/\partial u^i\right) \tag{3.172}$$

If, as assumed here, the Lagrangian is regular, the fiber derivative is invertible and has the following properties (see Ref.[186] for details):

- $(F\mathcal{L})_* \, \theta_{\mathcal{L}} = \theta_0$ and: $(F\mathcal{L})_* \, \Omega_{\mathcal{L}} = \omega_0$

  where $(F\mathcal{L})_*$ denotes the "push-forward" associated with the fiber derivative, i.e.: $(F\mathcal{L})_* = \left((F\mathcal{L})^{-1}\right)^*$;

- Via push-forward, the vector field Γ is mapped onto a vector field $\widetilde{\Gamma} \in \mathfrak{X}\left(T^*Q\right)$ that is Hamiltonian with respect to the canonical symplectic form ω_0 with an Hamiltonian H given by:

$$H =: (F\mathcal{L})_* \, E_{\mathcal{L}} = E_{\mathcal{L}} \circ (F\mathcal{L})^{-1} \tag{3.173}$$

- Explicitly (and locally):

$$\widetilde{\Gamma} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{3.174}$$

All this can be summarized in the following scheme:

$$\left(\Gamma, \Omega_{\mathcal{L}}, dE_{\mathcal{L}}\right) \xrightarrow{F\mathcal{L}} \left(\widetilde{\Gamma}, \omega_0, dH\right) \tag{3.175}$$

3.5.2 Consequences of the existence of alternative descriptions

It is clear that the transition to T^*Q summarized in the scheme (3.175) will be non-ambiguous and unique if and only if, apart from trivial equivalencies, the Lagrangian is unique.

When more than one Lagrangian description is available, the situation can become more involved. To be more specific, let, say, $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ be two alternative Lagrangians for the same dynamical system, Γ, on TQ. Each one defining its own fiber derivative, we can obtain different Hamiltonian descriptions on T^*Q with different vector fields and Hamiltonians but *the same* symplectic structure (i.e. ω_0) using alternatively the two fiber derivatives according to the scheme:

$$\underbrace{\begin{array}{ccc} & & \Omega_{\mathcal{L}^{(1)}}, dE_{\mathcal{L}^{(1)}} \\ & \nearrow & \\ \Gamma & & \\ & \searrow & \\ & & \Omega_{\mathcal{L}^{(2)}}, dE_{\mathcal{L}^{(2)}} \end{array}}_{TQ} \begin{array}{c} \xrightarrow{F\mathcal{L}^{(1)}} \\ \\ \\ \\ \xrightarrow{F\mathcal{L}^{(2)}} \end{array} \underbrace{\begin{array}{ccccc} \widetilde{\Gamma}^{(1)} & & & & dH^{(2)} \\ & \searrow & & \nearrow & \\ & & \omega_0 & & \\ & \nearrow & & \searrow & \\ \widetilde{\Gamma}^{(2)} & & & & dH^{(1)} \end{array}}_{T^*Q} \tag{3.176}$$

Although the vector fields $\widetilde{\Gamma}^{(1)}$ and $\widetilde{\Gamma}^{(2)}$ may look different, it is worth stressing that nonetheless they offer different descriptions of the same dynamical system. Indeed, in both cases their trajectories in T^*Q project down to the *same* set of trajectories in the physical space Q. Stated otherwise, the two sets of first-order differential equations on T^*Q associated with $\widetilde{\Gamma}^{(1)}$ and $\widetilde{\Gamma}^{(2)}$ give rise to the *same* set of second-order differential equations on Q.

Example 18 *Let* Γ *represent, as in Sect.3.4.3, the dynamics of the two-dimensional isotropic harmonic oscillator:*

$$\Gamma = u^1 \frac{\partial}{\partial q^1} + u^2 \frac{\partial}{\partial q^2} - q^1 \frac{\partial}{\partial u^1} - q^2 \frac{\partial}{\partial u^2} \tag{3.177}$$

and let, again in the notation of Sect.3.4.3, the two Lagrangians be: $\mathcal{L}^{(1)} = \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ *(the standard Lagrangian) and:* $\mathcal{L}^{(2)} = \mathcal{L}'$ *(cfr. Eq.(3.158)). Then, omitting unnecessary details,* $H^{(1)}$ *has the standard form:*

$$H^{(1)} = \frac{1}{2}\left[\left(p^1\right)^2 + \left(p^2\right)^2 + \left(q^1\right)^2 + \left(q^2\right)^2\right] \tag{3.178}$$

and:

$$\widetilde{\Gamma}^{(1)} = p^1 \frac{\partial}{\partial q^1} + p^2 \frac{\partial}{\partial q^2} - q^1 \frac{\partial}{\partial p^1} - q^2 \frac{\partial}{\partial p^2} \tag{3.179}$$

As to $\mathcal{L}^{(2)}$*, we find instead:*

$$F\mathcal{L}^{(2)} : \left(q^1, q^2, u^1, u^2\right) \mapsto \left(q^1, q^2, p^2, p^1\right) \tag{3.180}$$

and:

$$\widetilde{\Gamma}^{(2)} = p_2 \frac{\partial}{\partial q^1} + p_1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial p_1} - q^1 \frac{\partial}{\partial p_2} \tag{3.181}$$

with the Hamiltonian:

$$H^{(2)} = p_1 p_2 + q^1 q^2 \tag{3.182}$$

Concerning symmetries, while the Hamiltonian (3.178) is rotationally-invariant and we obtain, via Nöther's theorem, the usual association of the angular momentum with rotations, The Hamiltonian (3.182) is squeeze-invariant, the squeeze transformation being generated by the cotangent lift of the vector field (3.160), i.e.:

$$S^* = S - p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} \tag{3.183}$$

Now:

$$i_{S^*}\omega_0 = dF \tag{3.184}$$

where now the (Hamiltonian) constant of the motion is:

$$F = q^1 p_1 - q^2 p_2 \tag{3.185}$$

which, although it doesn't look such at first sight, is again the (only component of the) angular momentum,as:

$$\left(F\mathcal{L}^{(2)}\right)^{*} F = q^1 u^2 - q^2 u^1 \tag{3.186}$$

The scheme (3.176) outlined above is not the only possible one, though. We might decide instead to perform the Legendre map by using only one of the two fiber derivatives in both cases. If we select, e.g., $F\mathcal{L}^{(1)}$, we obtain the following scheme for the transition from TQ to $T^{*}Q$:

$$\underbrace{\begin{array}{ccc} & & \Omega_{\mathcal{L}^{(1)}}, dE_{\mathcal{L}^{(1)}} \\ & \nearrow & \\ \Gamma & & \\ & \searrow & \\ & & \Omega_{\mathcal{L}^{(2)}}, dE_{\mathcal{L}^{(2)}} \end{array}}_{TQ} \begin{array}{c} \stackrel{F\mathcal{L}^{(1)}}{\longrightarrow} \\ \\ \\ \\ \stackrel{F\mathcal{L}^{(1)}}{\longrightarrow} \end{array} \underbrace{\begin{array}{cccc} \searrow & & & \omega^{(1,2)}, dH^{(1,2)} \\ & & \nearrow & \\ & \widetilde{\Gamma}^{(1)} & & \\ & & \searrow & \\ \nearrow & & & \omega_0, dH^{(1)} \end{array}}_{T^{*}Q} \tag{3.187}$$

where now:

$$\omega^{(1,2)} = \left(F\mathcal{L}^{(1)}\right)_{*} \Omega_{\mathcal{L}^{(2)}} = \left(F\mathcal{L}^{(1)}\right)_{*} \left(F\mathcal{L}^{(2)}\right)^{*} \omega_0 = \left(F\mathcal{L}^{(2)} \circ \left(F\mathcal{L}^{(1)}\right)^{-1}\right)^{*} \omega_0 \tag{3.188}$$

and similarly for $H^{(1,2)}$.

Remark 19 *If we forget about the "TQ part" of the scheme (3.187) and retain only the "$T^{*}Q$ part", we see that this procedure exhibits an example of a given dynamical system ($\widetilde{\Gamma}^{(1)}$) on $T^{*}Q$ that is bihamiltonian.*

Example 20 *For the same system as in Example 18 above, $\widetilde{\Gamma}^{(1)}$ is again given by Eq.(3.179), but we find instead:*

$$\omega^{(1,2)} = dq^1 \wedge dp_2 + dq^2 \wedge dp_1 \tag{3.189}$$

while:

$$H^{(1,2)} = p_1 p_2 + q^1 q^2 \tag{3.190}$$

as in the previous example. However, now:

$$i_{S^{*}} \omega^{(1,2)} = q^1 dp_2 - p_2 dq^1 + p_1 dq^2 - q^2 dp_1 \tag{3.191}$$

and:

$$d\left(i_{S^{*}} \omega^{(1,2)}\right) = L_{S^{*}} \omega^{(1,2)} = 2\left(dq^1 \wedge dp_2 - dq^2 \wedge dp_1\right) \neq 0 \tag{3.192}$$

Therefore, although: $L_{S^{*}} H^{(1,2)} = 0$ and hence S^{*} is a symmetry for the Hamiltonian $H^{(1,2)}$, it is not Hamiltonian with respect to the symplectic form $\omega^{(1,2)}$, and ceases therefore to be the generator of a Nöther symmetry.

To conclude this Section, we would like to "re-visit", in the Hamiltonian formalism, the consequences of the use, for the isotropic harmonic oscillator, of one of the Lagrangians (3.146), parametrized by the family of symmetric and nonsingular matrices: $B = \|B_{ij}\|$.

Let us specialize here too to $n = 3$. The canonical momenta are defined by:

$$p_i = B_{ij}u^j \Leftrightarrow u^i = A^{ij}p_j,\ i = 1, 2, 3 \tag{3.193}$$

where: $A = \left\|A^{ij}\right\|$ is the matrix inverse of $B : A^{ij}B_{jk} = \delta^i_k$. The Hamiltonian is therefore:

$$H = \frac{1}{2}\left(A^{ij}p_ip_j + B_{ij}q^iq^j\right) \tag{3.194}$$

while the three components of the angular momentum: $J_i = \varepsilon_{ijk}q^ju^k$ are given, in the canonical formalism on $T^*\mathbb{R}^3$, by:

$$J_i = \varepsilon_{ijk}A^{kl}q^jp_l \tag{3.195}$$

The J_i's are of course constants of the motion, i.e.:

$$\{J_i, H\} = 0,\ i = 1, 2, 3 \tag{3.196}$$

where $\{.,.\}$ is the canonical Poisson bracket on $T^*\mathbb{R}^3$. Now, some long but straightforward algebra [164] shows that the Poisson brackets among the J_i's are given by:

$$\{J_h, J_k\} = \varepsilon_{hkr}A^{rs}J_s \tag{3.197}$$

Eq.(3.197) defines a Lie algebra whose derived algebra is spanned by the vectors of the form: $J_{hk} =: \varepsilon_{hkr}A^{rs}J_s$. As the Ricci tensor is antisymmetric, there are only three independent such vectors and, as the matrix A is symmetric, they are independent. Therefore, the derived algebra is three-dimensional, and the Lie algebra can be only [113] (apart from a sign) that of $O(3)$ or that of $O(2,1)$[49]. Denoting by X_i and X_{hk} the associated Hamiltonian vector fields, defined by:

$$i_{X_i}\omega_0 = dJ_i;\ i_{X_{hk}}\omega_0 = dJ_{hk} \tag{3.198}$$

($X_{hk} = \varepsilon_{hkr}A^{rs}X_s$) which implies, in particular: $\mathcal{L}_{X_i}\omega_0 = 0$, Eq.(3.196) is equivalent to the statement that: $\mathcal{L}_{X_i}H = 0$. Hence (see Sect.3.4.2), the X_i's are also symmetries for the dynamics. Moreover, using the identity (3.143), one sees at once that:

$$i_{[X_h,X_k]}\omega_0 = -\mathcal{L}_{X_h}i_{X_k}\omega_0 = -\mathcal{L}_{X_h}dJ_k = -d\mathcal{L}_{X_h}J_k \tag{3.199}$$

i.e. that:

$$i_{[X_h,X_k]}\omega_0 = d\{J_h, J_k\} = dJ_{hk} = i_{X_{hk}}\omega_0 \tag{3.200}$$

which implies in turn, as ω_0 is nondegenerate:

$$[X_h, X_k] = \varepsilon_{hkr}A^{rs}X_s \tag{3.201}$$

Hence, the X_i's generate the same algebra of symmetries (that of $O(3)$ or that of $O(2,1)$), and this is in agreement with the results of Sect.3.4.3.

[49] These are called $su(2)$ and $su(1,1)$ in Ref.[113], but the Lie algebras are isomorphic.

4 Geometry of Quantum Mechanics and Alternative Structures

4.1 Introduction

Alternative descriptions for both classical and quantum systems have been discussed already all along the previous Chapters. In particular, in Sect. 1.2 we have discussed how one can obtain alternative descriptions both in the Schrödinger and Heisenberg pictures either by modifying the Hermitian structure using constants of the motion (Sect. 1.2.1) or, in the infinite-dimensional case (Sect. 1.2.3) by changing the symplectic structure (as well as the Hamiltonian) using powers of the original Hamiltonian.

The discussion was carried on systematically within the framework of the description of states as vectors on some (finite- or infinite-dimensional) complex Hilbert space $\mathcal{H}$ (with the associated Hermitian structure $\langle .|.\rangle$) and of observables as self-adjoint linear operators on $\mathcal{H}$.

Hilbert spaces were introduced and used in a systematic way first by Dirac [56] as a consequence of the fact that one needs a superposition rule (and hence a linear structure) in order to accommodate a consistent description of the interference phenomena that are fundamental for Quantum Mechanics. Parenthetically, we should note that a *complex* Hilbert space carries with it in a natural way a "complex structure" (multiplication of vectors by the imaginary unit). The rôle of the latter was discussed in the early Forties by Reichenbach [202]. Later on Stückelberg [217] emphasized the rôle of the complex structure in deducing in a consistent way the uncertainty relations of Quantum Mechanics (see also the discussion in Refs.[69] and [170]).

However, it is well known that a "complete" measurement in Quantum Mechanics (a simultaneous measurement of a complete set of commuting observables[50] [56, 69, 183]) does not provide us with an uniquely defined vector in some Hilbert space, but rather with a "ray", i.e. an equivalence class of vectors differing by multiplication through a nonzero complex number. Even fixing the normalization, an overall phase[51] will remain unobservable. Quotienting w.r.t. both multiplications leads, for a finite-dimensional Hilbert space $\mathcal{H}$ ($\dim_{\mathbb{C}} \mathcal{H} = n$), to the following double fibration:

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathcal{H}_0 = \mathcal{H} - \{\mathbf{0}\} \\ & & \downarrow \\ U(1) & \longrightarrow & \mathbb{S}^{2n-1} \\ & & \downarrow \\ & & P(\mathcal{H}) \end{array} \tag{4.1}$$

[50] We will not worry at this stage about the technical complications that can arise, in the infinite-dimensional case, when the spectrum of some observable has a continuum part.

[51] Not a *relative* phase in a superposition of vectors, of course.

whose final result is the *projective Hilbert space* $P\mathcal{H}$, and it is clear that:

$$\begin{array}{c} P(\mathcal{H}) \simeq \mathbb{C}P^{n-1} = \{[|\psi\rangle] \ : |\psi\rangle, |\psi'\rangle \in [|\psi\rangle] \Leftrightarrow \ |\psi\rangle = \lambda|\psi'\rangle\} \\ |\psi\rangle, |\psi'\rangle \in \mathcal{H}- \{\mathbf{0}\}\,,\ \lambda \in \mathbb{C}_0 = \mathbb{C}- \{\mathbf{0}\}\} \end{array} \tag{4.2}$$

where $[|\psi\rangle]$ denotes the equivalence class to which $|\psi\rangle \in \mathcal{H}$ belongs under multiplication by a non-zero complex number.

Remark 21 *Notice that in this way the Hilbert space $\mathcal{H}$ acquires the structure of a principal fiber bundle [104, 167, 215], with base $P\mathcal{H}$ and typical fiber $\mathbb{C}_0$.*

The self-duality of $\mathcal{H}$ determined by the Hermitian structure allows for the (unique) association of every equivalence class $[|\psi\rangle]$ with the rank-one projector:

$$\rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \tag{4.3}$$

with the known properties:

$$\begin{array}{l} \rho_\psi^\dagger = \rho_\psi \\ Tr\rho_\psi = 1 \\ \rho_\psi^2 = \rho_\psi \end{array} \tag{4.4}$$

It is clear by construction that the association depends on the Hermitian structure we consider.

The space of rank-one projectors is usually denoted [86] as $\mathcal{D}_1^1(\mathcal{H})$. It is then clear that in this way we can identify it with the projective Hilbert space $P\mathcal{H}$. Hence, what the best of measurements will yield will be always (no more and not less than) a *rank-one projector* (also called a *pure state* [95]).

Also, transition probabilities that, together with the expectation values of self-adjoint linear operators that represent dynamical variables, are among the only observable quantities one can think of, will be insensitive to overall phases, i.e. they will depend only on the (rank-one) projectors associated with the states. If $A = A^\dagger$ is any such observable, then the expectation value $\langle A\rangle_\psi$in the state $|\psi\rangle$ will be given by:

$$\langle A\rangle_\psi = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \equiv Tr\,\{\rho_\psi A\} \tag{4.5}$$

Transition probabilities are in turn expressed via a binary product that can be defined on pure states. Again, if $|\psi\rangle$ and $|\phi\rangle$ are any two states, then the (normalized) transition probability from $|\psi\rangle$ to $|\phi\rangle$ will be given by:

$$\frac{|\,\langle\phi|\psi\rangle\,|^2}{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle} = Tr\,\{\rho_\psi\rho_\phi\} \tag{4.6}$$

and the trace on the r.h.s. of Eq.(4.12) will define the binary product among pure states (but more on this shortly below).

It appears therefore that the most natural setting for Quantum Mechanics is not primarily the Hilbert space itself but rather the projective Hilbert space, or, equivalently, the space of rank-one projectors $\mathcal{D}_1^1(\mathcal{H})$, whose convex hull will provide us with the set of all density states. [222, 223, 71].

On the other hand, the superposition rule, which leads to interference phenomena, remains one of the fundamental building blocks of Quantum Mechanics, one that, among other things, lies at the very heart of the modern formulation of Quantum Mechanics in terms of path integrals [29, 74, 75, 85], an approach that goes actually back to earlier suggestions by Dirac [56, 57].

To begin with, if we consider, for simplicity, two orthonormal states:

$$|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}, \langle\psi_i|\psi_j\rangle = \delta_{ij}, i, j = 1, 2 \tag{4.7}$$

with the associated projection operators:

$$\rho_1 = |\psi_1\rangle\langle\psi_1|, \ \rho_2 = |\psi_2\rangle\langle\psi_2| \tag{4.8}$$

a linear superposition with (complex) coefficients c_1 and c_2 with: $|c_1|^2+|c_2|^2 = 1$ will yield the normalized vector:

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{4.9}$$

and the associated projector:

$$\rho_\psi = |\psi\rangle\langle\psi| = |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + (c_1c_2^*\rho_{12} + h.c.) \tag{4.10}$$

where: $\rho_{12} =: |\psi_1\rangle\langle\psi_2|$, which cannot however be expressed directly in terms of the initial projectors.

A procedure to overcome this difficulty by retaining at the same time the information concerning the relative phase of the coefficients can be summarized as follows [44, 144, 145, 146, 148, 170].

Considering a third, fiducial vector $|\psi_0\rangle$ with the only requirement that it be not orthogonal[52] neither to $|\psi_1\rangle$ nor to $|\psi_2\rangle$, it is possible to associate normalized vectors $|\phi_i\rangle$ with the projectors ρ_i $(i = 1, 2)$ by setting:

$$|\phi_i\rangle = \frac{\rho_i|\psi_0\rangle}{\sqrt{Tr(\rho_i\rho_0)}}, \ i = 1, 2 \tag{4.11}$$

Remark 22 *Note that, as all the ρ's involved are rank-one projectors*[53]*:*

- $$Tr(\rho_i\rho_0)\, Tr(\rho_j\rho_0) = Tr(\rho_i\rho_0\rho_j\rho_0) \ \ \forall i, j \tag{4.12}$$

  and that:

[52] In terms of the associated rank-one projections, we require: $Tr(\rho_i\rho_0) \neq 0, i = 1, 2$, with: $\rho_0 = |\psi_0\rangle\langle\psi_0|$.

[53] The proof of Eqs.(4.12) and (4.13)is elementary and will not be given here.

- $$|\phi_i\rangle\langle\phi_i| = \frac{\rho_i\rho_0\rho_i}{\sqrt{Tr\left(\rho_i\rho_0\rho_i\rho_0\right)}} \equiv \rho_i, \;\; i = 1, 2 \tag{4.13}$$

Forming now the linear superposition: $|\phi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$ and the associated projector: $\rho = |\phi\rangle\langle\phi|$, one finds easily, using also Eqs.(4.12) and (4.13), that:

$$\rho = |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + \frac{c_1 c_2^* \rho_1\rho_0\rho_2 + h.c.}{\sqrt{Tr\left(\rho_1\rho_0\rho_2\rho_0\right)}} \tag{4.14}$$

which can be written in a compact form as:

$$\rho = \sum_{i,j=1}^{2} c_i c_j^* \frac{\rho_i\rho_0\rho_j}{\sqrt{Tr\left(\rho_i\rho_0\rho_j\rho_0\right)}} \tag{4.15}$$

The results (4.14) and (4.15) are now written entirely in terms of rank-one projectors. Thus, a superposition of rank-one projectors which yields another rank-one projector is possible, but requires the arbitrary choice of the fiducial projector ρ_0. This procedure is equivalent to the introduction of a connection on the bundle, usually called the Pancharatnam connection [185, 197].

Remark 23 *If the (normalized) probabilities $|c_1|^2$ and $|c_2|^2$ are given, Eq.(4.9) describes a one-parameter family of linear superposition of states, and the same will be true in the case of Eq.(4.14). Both families will be parametrized by the relative phase of the coefficients.*

Remark 24 *Comparison of Eqs.(4.10) and (4.14) shows that, while the first two terms on the r.h.s. of both are identical, the last terms of the two differ by an extra (fixed) phase, namely that:*

$$\frac{\rho_1\rho_0\rho_2}{\sqrt{Tr\left(\rho_1\rho_0\rho_2\rho_0\right)}} = \rho_{12}\exp\left\{i\left[\arg\left(\langle\psi_1|\psi_0\rangle - \arg\left(\langle\psi_2|\psi_0\rangle\right)\right)\right]\right\} \tag{4.16}$$

Remark 25 *The result of Eq.(4.15) can be generalized in an obvious way to the case of an arbitrary number, say n, of orthonormal states none of which is orthogonal to the fiducial state. The corresponding family of rank-one projectors will be parametrized in this case by the $(n-1)$ relative phases.*

If, now, we are given two[54] (rank-one) projectors and only the relative probabilities are given, we are led to conclude that the system is described by the convex combination (a rank-two density matrix): $\rho = |c_1|^2 \rho_1 + |c_2|^2 \rho_2$, which is again Hermitian and of trace one, but now: $\rho - \rho^2 > 0$ (strictly). The procedure leading from this "impure" state to one of the pure states given by, say, Eq.(4.15), i.e. the procedure that associates a pure state with a pair of pure

[54] Or more, with an obvious generalization.

states, is a composition law for pure states that has been termed in the literature [145] as a "purification" of "impure" states.

In the Hilbert space formulation of Quantum Mechanics one needs also to find the spectral family associated with any observable, represented by a self-adjoint operator on the Hilbert space of states. Limiting ourselves for simplicity to observables with a pure point-spectrum, these notions can be made easily to "descend" to the projective Hilbert space $P\mathcal{H}$ by noticing that, if $A = A^{\dagger}$ is an observable, and considering from now on only normalized vectors, the expectation value (4.5) associates with the observable A a (real) functional on $P\mathcal{H}$. The standard variational principle of Quantum Mechanics [69, 183] can be rephrased [31, 44] by saying that the critical points of this functional are the eigenprojectors of A and that the critical values yield the corresponding eigenvalues.

Unitary (and, as a matter of fact, also anti-unitary[55]) operators play also a relevant rôle in Quantum Mechanics [69, 183]. In particular, self-adjoint operators can act as infinitesimal generators of one-parameter groups of unitaries. Both unitary and anti-unitary operators share the property of leaving all transition probabilities invariant. At the level of the projective Hilbert space they represent then *isometries* of the binary product (4.6). The converse is also true. Indeed, it was proved long ago by Wigner [227, 230] that bijective maps on $P\mathcal{H}$ that preserve transition probabilities (i.e., isometries of the projective Hilbert space) are associated with unitary or anti-unitary transformations on the original Hilbert space[56]. For a recent version of this theorem, see Ref.[87].

To summarize the content of this Section, we have argued that all the relevant building blocks of Quantum Mechanics can be re-formulated in terms of parent objects that "live" in the projective Hilbert space $P\mathcal{H}$. The latter, however, is no more a linear vector space. As will be discussed in the following Sections, it carries instead a rich manifold structure. In this context, the very notion of linear transformations looses meaning, and we are led in a natural way to consider a non-linear manifold and (non-linear) diffeomorphisms thereof. This given, only objects that have a tensorial character will be allowed. We will have then, as a preliminary step, to proceed to, so-to-speak, "tensorialize" all the notions that have been established in the context of the linear Hilbert space. We will do that in the second part of this Chapter, where we will discuss the geometry of Quantum Mechanics. In the last part of the Chapter, having achieved this goal, we will re-discuss the problem of alternative structures in the context of Quantum Mechanics.

[55] Think of the operation [69, 183] of time-reversal.

[56] The association being up to a phase, this may lead to the appearance of "ray" (or "projective") representations [11, 69, 95, 132, 133, 183, 207] of unitary groups on the Hilbert space instead of ordinary ones, a problem that we will not discuss here, though.

4.2 The Geometry of Quantum Mechanics

4.2.1 Some Preliminaries

We recall here some basic notions, in order mainly to fix the language and notations to be employed in what follows.

1. Given an n-dimensional vector space $\mathcal{H}$ over the field $\mathbb{C}$ of the complex numbers, the *realified* [5] $\mathcal{H}_{\mathbb{R}}$ of $\mathcal{H}$ is a real vector space that coincides with $\mathcal{H}$ as a group (abelian group under addition) but in which only multiplication by real scalars is allowed. The realified of $\mathcal{H}$ can be constructed as follows. Let $(e_1, ..., e_n)$ be a basis for $\mathcal{H}$. Then, a basis for $\mathcal{H}_{\mathbb{R}}$ will be provided by $(e_1, ..., e_n, ie_1, ..., ie_n)$ and $\mathcal{H}_{\mathbb{R}} \approx \mathbb{R}^{2n}$. Once a basis has been chosen, $\mathcal{H} \approx \mathbb{C}^n$. If: $x = x^k e_k, x^k = u^k + iv^k; u^k, v^k \in \mathbb{R}$ (in short: $x = u + iv; u, v \in \mathbb{R}^n$), then the corresponding vector in $\mathcal{H}_{\mathbb{R}}$ is represented by $\left(u^1, ..., u^n, v^1, ..., v^n\right)$, or (u, v), again for short, and it is immediate to check that the group property is satisfied. Let now: $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on $\mathcal{H}$. The *realified* of A will be the linear operator: $A_{\mathbb{R}} : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ that coincides with A pointwise, i.e., if: $Ax = x'$, $x = u + iv, x' = u' + iv'$, then: $A_{\mathbb{R}}(u, v) = (u', v')$. In any given basis for $\mathcal{H}$, A will be represented by a matrix of the form: $A = \alpha + i\beta$, with α, β real $n \times n$ matrices. Then it is also immediate to check that $A_{\mathbb{R}}$ will be represented by the $2n \times 2n$ real matrix:
$$A_{\mathbb{R}} = \begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix} \tag{4.17}$$
It is also immediate to check that: $(A + B)_{\mathbb{R}} = A_{\mathbb{R}} + B_{\mathbb{R}}$, as well as that: $(AB)_{\mathbb{R}} = A_{\mathbb{R}} B_{\mathbb{R}}$, and hence the set of the linear operators that are realifications of complex operators on $\mathcal{H}$ is both a subspace of the vector space of all linear operators on $\mathcal{H}_{\mathbb{R}}$ as well as a subalgebra of the associative algebra $\mathfrak{gl}(2n, \mathbb{R})$. In particular, multiplication in $\mathcal{H}$ by the imaginary unit will be represented by the linear operator:
$$J = \begin{vmatrix} \mathbf{0}_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & \mathbf{0}_{n\times n} \end{vmatrix} \tag{4.18}$$
(or: $(u, v) \to (-v, u)$) with the property:
$$J^2 = -\mathbb{I}_{2n\times 2n} \tag{4.19}$$

2. A *complex manifold* [40, 210] is a manifold Z that can be locally modeled on $\mathbb{C}^n$ for some n, and for which the chart-compatibility conditions are required to be $\mathbb{C}^\omega$ diffeomorphisms. Then, on the tangent bundle TZ one can define the *complex structure* J_0 via:
$$J_0 : TZ \to TZ; \quad J_0(v) =: iv, \quad v \in TZ. \tag{4.20}$$
Clearly: $J_0^2 = -\mathbb{I}$. Also:

3. An *almost complex* manifold [186] is an even-dimensional real manifold M endowed with a $(1,1)$-type tensor field J , called an *almost complex structure,* satisfying:

$$J^2 = -\mathbb{I} \tag{4.21}$$

It was proved in Ref.[195] that an almost complex manifold becomes a complex one iff the almost complex structure J satisfies the *Nijenhuis condition* $N_J = 0$, where N_J is the Nijenhuis torsion associated with J.

4. Finally, let $\mathcal{K}$ be a real, even-dimensional, manifold with a complex structure and a closed two-form satisfying the compatibility condition:

$$\omega\left(x, Jy\right) + \omega\left(Jx, y\right) = 0;\ \ x, y \in T\mathcal{K} \tag{4.22}$$

Notice that this implies that:

$$g\left(.,.\right) =: \omega\left(., J\left(.\right)\right);\ \ \left(x, y\right) \mapsto g\left(x, y\right) =: \omega\left(x, Jy\right) \tag{4.23}$$

is *symmetric* $\left(g\left(x,y\right) = g\left(y,x\right) \forall x, y\right)$ and nondegenerate iff ω is, hence a metric. When g is positive, then $\mathcal{K}$ is a *Kähler manifold* [40, 210, 224][57]. Also, $J^2 = -\mathbb{I}$ implies:

$$\omega\left(Jx, Jy\right) = \omega\left(x, y\right);\ g\left(Jx, Jy\right) = g\left(x, y\right)\ \ \forall x, y \tag{4.24}$$

Notice that Eq.(4.23) implies the analog of Eq.(4.22) for g, namely:

$$g\left(x, Jy\right) + g\left(Jx, y\right) = 0 \tag{4.25}$$

A tensorial triple (g, J, ω), with g a metric, J a complex structure and ω a symplectic structure satisfying the conditions (4.22),(4.23) and (4.24) will be called an *admissible triple.* Eq.(4.23) and the parent equation, obtained by substituting: $y \to Jy$ in it tell us also that:

$$\omega\left(.,.\right) = -g\left(., J\left(.\right)\right) \tag{4.26}$$

Coming back now to the complex vector space $\mathcal{H}$, let it be endowed also with an Hermitian structure $h(.,.) = \langle .|. \rangle$, i.e. a positive-definite sesquilinear form,nondegenerate, linear in the second factor and antilinear in the first one. Then $\mathcal{H}$ will become a (finite-dimensional: $\dim_{\mathbb{C}} \mathcal{H} = n$) Hilbert space. We will keep denoting vectors in $\mathcal{H}$ with Latin letters (i.e.: x, y etc.) and we will use Dirac's notation ($|x\rangle, |y\rangle$ etc.) only when convenient. Separating real and imaginary parts, we can write:

$$\begin{aligned} h\left(x, y\right) &= g\left(x, y\right) + i\omega\left(x, y\right) \\ g\left(x, y\right) &= \operatorname{Re} h\left(x, y\right) \\ \omega\left(x, y\right) &= \operatorname{Im} h\left(x, y\right) \end{aligned} \tag{4.27}$$

[57] If not, then $\mathcal{K}$ is also called [186] a *pseudo-Kähler* manifold.

g is clearly symmetric, positive and nondegenerate, while ω is antisymmetric and nondegenerate too.

Now we can consider $\mathcal{H}_{\mathbb{R}}$ together with its tangent bundle $T\mathcal{H}_{\mathbb{R}} \approx \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}}$. Points in $\mathcal{H}_{\mathbb{R}}$, i.e. in the first factor, will be again denoted by the same Latin letters[58], and we will use Greek letters for the second factor. Then, e.g., (x, ψ) will denote a point in $\mathcal{H}_{\mathbb{R}}$ and a tangent vector at x: $\psi \in T_x\mathcal{H}_{\mathbb{R}} \approx \mathcal{H}_{\mathbb{R}}$. We can associate with every point $x \in \mathcal{H}_{\mathbb{R}}$ the constant vector field:

$$X_{\psi} =: (x, \psi) \tag{4.28}$$

Then, we can "promote" g and ω to $(0, 2)$ tensor fields by defining:

$$g\,(x)\,(X_{\psi}, X_{\phi}) =: g\,(\psi, \phi) \tag{4.29}$$

and similarly for ω. In this way, g becomes a Riemannian metric and ω a symplectic structure. Proceeding in a similar way, we define:

$$J\,(x)\,(X_{\psi}) = (x, J\psi) \tag{4.30}$$

where: $J\psi = i\psi$ (i.e.: $J\,(u, v) = (-v, u)$) and in this way J too is "promoted" to a $(1, 1)$ tensor field. As all these tensors fields are translationally invariant, and hence the Nijenhuis condition for J is trivially satisfied, and as all the compatibility conditions are also satisfied, $\mathcal{H}_{\mathbb{R}}$ becomes in this way a *linear* Kähler manifold, with J playing the rôle of the complex structure. Explicitly, if $(e_1, ..., e_n)$ is an orthonormal basis for $\mathcal{H}$, and: $x = (u, v)\,, y = (u', v')$, then:

$$\begin{aligned} g\,(x, y) = u \cdot u' + v \cdot v' \\ \omega\,(x, y) = u \cdot v' - v \cdot u' \end{aligned} \tag{4.31}$$

It may be convenient to give explicit expressions by introducing real coordinates $x^1, ..., x^{2n}$ on $\mathcal{H}_{\mathbb{R}} \approx \mathbb{R}^{2n}$. Then, e.g., g and J will be explicitly represented as:

$$g = g_{ij} dx^i \otimes dx^j \tag{4.32}$$

and[59]:

$$J = J^i_j dx^j \otimes \frac{\partial}{\partial x^i} \tag{4.33}$$

Hence:

$$J^2 = -\mathbb{I} \Longleftrightarrow J^i{}_k J^k{}_j = -\delta^i{}_j \tag{4.34}$$

[58] With reference to a basis, $x = u + iv$ will stand (see item 1 above) for the (real) pair (u, v)

[59] Here: $Jx = \left\{(Jx)^i\right\}_1^{2n}; (Jx)^i = J^i{}_j x^j$.

Remark 26 *With the given metric, orthogonal matrices will be those leaving the scalar product invariant, and they will provide a representation of $O(2n)$ which need not be the standard one. Eq.(4.24) tells us that J is what we might call a "g-orthogonal" matrix. In this context, it is worth recalling that the adjoint $\mathbb{A}^{\dagger}$ w.r.t. g of any linear operator $\mathbb{A}$ (a $(1,1)$ tensor)is defined by:*

$$g(x, \mathbb{A}y) = g\left(\mathbb{A}^{\dagger}x, y\right) \tag{4.35}$$

In terms of matrices:

$$\mathbb{A}^{\dagger} = g^{-1}\widetilde{\mathbb{A}}g \tag{4.36}$$

where $\widetilde{\mathbb{A}}$ stands for the transpose matrix and hence, for a generic metric tensor, (real) symmetric matrices need not be self-adjoint. Eq.(4.25) tells us then that J is skew-adjoint w.r.t. g, i.e. that: $J^{\dagger} = -J$, which implies, according to Eq.(1.46):

$$J^{\dagger}J = \mathbb{I} \tag{4.37}$$

Remark 27 *ii) If we consider a one-parameter group $\{\exp(t\mathbb{A})\}_{t\in\mathbb{R}}$ of g-orthogonal matrices, then: $g\left(e^{t\mathbb{A}}x, e^{t\mathbb{A}}y\right) = g(x,y)$ implies, at the infinitesimal level:*

$$g(\mathbb{A}x, y) + g(x, \mathbb{A}y) = 0 \tag{4.38}$$

Hence, J acts at the same time as a generator of finite and infinitesimal orthogonal transformations (rotations).
iii) in terms of the representative matrices, the condition $g(Jx,y)+g(x,Jy) = 0$ can be written as:

$$\widetilde{J} \circ g + g \circ J = 0 \tag{4.39}$$

i.e., as g is symmetric: $\widetilde{(g \circ J)} = -g \circ J$, i.e. $g \circ J$ must be a skew-symmetric matrix.

Using g and J we can construct, as discussed before, the skew-symmetric tensor ω (cfr Eq.(4.26)). ω will be nondegenerate iff g is, hence a symplectic form. In terms of matrices:

$$\omega = -g \circ J \tag{4.40}$$

$(\omega_{ij} = -g_{ik}J^{k}{}_{j})$, Moreover. Eqs.(4.24) and (4.22), i.e.:

$$\omega(Jx, Jy) = \omega(x,y)\,\forall x, y \tag{4.41}$$

and:

$$\omega(Jx, y) + \omega(x, Jy) = 0 \forall x, y \tag{4.42}$$

tell us that J will generate (both finite and infinitesimal) symplectic transformations as well. Notice that, for $y = Jx$:

$$\omega(x, Jx) = g(x,x) \tag{4.43}$$

and hence: $\omega(x, Jx) > 0$ if g is positive-definite.

One could start instead from the datum of a symplectic form and of a complex structure, requiring the admissibility condition $\omega(Jx,y)+\omega(x,Jy)=0$ (which implies $\omega(Jx,Jy)=\omega(x,y)$ and viceversa), and define then:

$$g(x,y)=:\omega(x,Jy) \tag{4.44}$$

($g=\omega\circ J$ in terms of representative matrices), the only difference being that, although g will be still nondegenerate iff ω is, it need not be positive unless $\omega(x,Jx)>0\forall x$.

Finally, one could start from g and ω and require the admissibility condition that: $J=:g^{-1}\circ\omega$ be a complex structure,i.e.: $J^2=-\mathbb{I}$. In conclusion, a third tensor is determined whenever any other admissible two are given.

Remark 28 *We have already encountered examples of admissible triples* (g,ω,J) *in Sect. 3.3. E.g., for the isotropic two-dimensional harmonic oscillator we may consider* (H_0,ω_0,J) *or* (H_3,ω_3,J) *as given in Eqns. (3.76) and (3.80), while for the one-dimensional harmonic oscillator we may choose (see again Sect.3.3)* (H,ω,J) *or* $H',\omega',J')$*, as long as the Hamiltonian is positive definite.*

4.2.2 Geometric Quantum Mechanics

Here and in the following we will exploit the already-discussed connection between the space $P(\mathcal{H})$ of rays and the space $\mathcal{D}_1^1(\mathcal{H})$ of density states of rank one to see how it is possible to use symplectic methods to study quantum systems. This geometric approach is based on some observations that will be developed in the following.

We have just proved that the realification $\mathcal{H}_{\mathbb{R}}$ of the Hilbert space $\mathcal{H}$ (the space of states) is a linear Kälher manifold, equipped with an admissible triple (J,g,ω). Now, taking into account that $P(\mathcal{H})$ is not a linear space, we will have to use a tensorial description of these structures. Via a momentum map on $P(\mathcal{H})$ that we shall define shortly below, the space of Hermitian operators (the observables) will be identified with the dual $u^*(\mathcal{H})$ of the Lie algebra of the unitary group $U(\mathcal{H})$, which can be thought of as the intersection of the Lie algebras of the symplectic and orthogonal groups. By exploiting the fact that the action of the latter is Hamiltonian, we will use the momentum map to define contravariant metric and Poisson tensors on $u^*(\mathcal{H})$. Finally we will study how these structures behave under the $U(\mathcal{H})$-action on $u^*(\mathcal{H})$ and see how $\mathcal{D}_1^1(\mathcal{H})$ itself becomes a Kälher manifold.

4.2.3 Tensors on Hilbert spaces

We have seen how we can construct the tensor fields g,J and ω on $T\mathcal{H}_{\mathbb{R}}$. The $(0,2)$-tensors g and ω define maps from $T\mathcal{H}_{\mathbb{R}}$ to $T^*\mathcal{H}_{\mathbb{R}}$. The two being both non-degenerate, we can also consider their inverses, i.e. the $(2,0)$ contravariant tensors G (a metric tensor) and Λ (a Poisson tensor) mapping $T^*\mathcal{H}_{\mathbb{R}}$ to $T\mathcal{H}_{\mathbb{R}}$ and such that:

$$G \circ g = \Lambda \circ \omega = \mathbb{I}_{T\mathcal{H}_{\mathbb{R}}} \tag{4.45}$$

i.e., in short: $G = g^{-1}, \Lambda = \omega^{-1}$. G and Λ can be used together to define an Hermitian product between any two α, β in the dual $\mathcal{H}^*_{\mathbb{R}}$ equipped with the dual complex structure J^*[60]:

$$\langle \alpha, \beta \rangle_{\mathcal{H}^*_{\mathbb{R}}} = G(\alpha, \beta) + i\Lambda(\alpha, \beta). \tag{4.46}$$

This induces two (non-associative) real brackets on smooth, real-valued functions on $\mathcal{H}_{\mathbb{R}}$:

- the (symmetric) Jordan bracket $\{f, h\}_g =: G(df, dh)$, and:
- the (antisymmetric) Poisson bracket $\{f, h\}_\omega =: \Lambda(df, dh)$.

By extending both these brackets to complex functions via complex linearity we obtain eventually a complex bracket $\{.,.\}_{\mathcal{H}}$ defined as:

$$\{f, h\}_{\mathcal{H}} = \langle df, dh \rangle_{\mathcal{H}^*_{\mathbb{R}}} =: \{f, h\}_g + i\{f, h\}_\omega. \tag{4.47}$$

To make these structures more explicit, we may introduce an orthonormal basis $\{e_k\}_{k=1,\cdots,n}$ in $\mathcal{H}$ and global coordinates (q^k, p^k) for $k = 1, \cdots, n$ on $\mathcal{H}_{\mathbb{R}}$ defined as

$$\langle e_k, x \rangle = (q^k + ip^k)(x), \ \forall x \in \mathcal{H}. \tag{4.48}$$

Then[61]:

$$J = dp^k \otimes \frac{\partial}{\partial q^k} - dq^k \otimes \frac{\partial}{\partial p^k} \tag{4.49}$$

$$g =: dq^k \otimes dq^k + dp^k \otimes dp^k \tag{4.50}$$

$$\omega =: dq^k \otimes dp^k - dp^k \otimes dq^k \tag{4.51}$$

as well as:

$$G = \frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial q^k} + \frac{\partial}{\partial p^k} \otimes \frac{\partial}{\partial p^k} \tag{4.52}$$

$$\Lambda = \frac{\partial}{\partial p^k} \otimes \frac{\partial}{\partial q^k} - \frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial p^k} \tag{4.53}$$

and hence:

$$\{f, h\}_g = \frac{\partial f}{\partial q^k}\frac{\partial h}{\partial q^k} + \frac{\partial f}{\partial p^k}\frac{\partial h}{\partial p^k} \tag{4.54}$$

$$\{f, h\}_\omega = \frac{\partial f}{\partial p^k}\frac{\partial h}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial h}{\partial p^k} \tag{4.55}$$

Introducing complex coordinates: $z^k =: q^k + ip^k$, $\bar{z}^k =: q^k - ip^k$, we can also write

$$G + i \cdot \Lambda = 4\frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial \bar{z}^k}, \tag{4.56}$$

[60]Which will act (see Footnote 59) via the transpose matrix of J.

[61]Summation over repeated indices being understood here and in the rest of the Section.

where
$$\frac{\partial}{\partial z^k} =: \frac{1}{2}\left(\frac{\partial}{\partial q^k} - i\frac{\partial}{\partial p^k}\right), \; \frac{\partial}{\partial \bar{z}^k} =: \frac{1}{2}\left(\frac{\partial}{\partial q^k} + i\frac{\partial}{\partial p^k}\right). \tag{4.57}$$

Complex coordinates are employed here and also elsewhere in this paper only as a convenient shorthand or as a stenographic notation. Their use does not mean at all that vector fields like those in Eq.(4.57) should operate on functions that are holomorphic (or anti-holomorphic) in the z^k's. They must rather be seen as complex-valued vector fields that operate on (smooth) complex-valued functions defined on a real differentiable manifold.

With this in mind, we have :

$$\{f, h\}_{\mathcal{H}} = 4\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k}, \tag{4.58}$$

or, in more detail:

$$\{f, h\}_g = 2\left(\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k} + \frac{\partial h}{\partial z^k}\frac{\partial f}{\partial \bar{z}^k}\right); \quad \{f, h\}_\omega = \frac{2}{i}\left(\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k} - \frac{\partial h}{\partial z^k}\frac{\partial f}{\partial \bar{z}^k}\right) \tag{4.59}$$

Notice also that:
$$J = -i\left(dz^k \otimes \frac{\partial}{\partial z^k} - d\bar{z}^k \otimes \frac{\partial}{\partial \bar{z}^k}\right) \tag{4.60}$$

In particular, for any $A \in gl(\mathcal{H})$ we can define the quadratic function:

$$f_A(x) = \frac{1}{2}\langle x, Ax\rangle = \frac{1}{2}z^\dagger Az \tag{4.61}$$

where z is the column vector $(z_1, ..., z_n)$. It follows immediately from Eq.(4.59) that, for any $A, B \in gl(\mathcal{H})$:

$$\{f_A, f_B\}_g = f_{AB+BA} \tag{4.62}$$

$$\{f_A, f_B\}_\omega = f_{\frac{AB-BA}{i}} \tag{4.63}$$

So, the Jordan bracket of any two quadratic functions f_A and f_B is related to the (commutative) Jordan bracket of A and B, $[A, B]_+$, defined[62] as:

$$[A, B]_+ =: AB + BA \tag{4.64}$$

while their Poisson bracket is related to the commutator product (the Lie bracket) $[A, B]_-$ defined as:

$$[A, B]_- =: \frac{1}{i}(AB - BA) \tag{4.65}$$

[62]This is actually *twice* the Jordan Bracket as it is usually defined in the literature [65], but we find here more convenient to employ this slightly different definition.

In particular, if A and B are Hermitian, their Jordan product (4.64) and their Lie bracket will be Hermitian as well. Hence, the set of Hermitian operators on $\mathcal{H}_{\mathbb{R}}$, equipped with the binary operations (4.64) and (4.65), becomes a *Lie-Jordan algebra* [65, 108, 109], and the binary product [65]:

$$(A, B) = \frac{1}{2}\left([A, B]_{+} + i\,[A, B]_{-}\right) \tag{4.66}$$

is an associative product (Indeed: $(A, B) \equiv AB$). We remark parenthetically that all this extends without modifications [65] to the infinite-dimensional case, if we assume: $A, B \in \mathcal{B}_{sa}(\mathcal{H})$, the set of bounded self-adjoint operators on the Hilbert space $\mathcal{H}$.

Coming back to quadratic functions, it is not hard to check that:

$$\{f_A, f_B\}_{\mathcal{H}} = 2f_{AB}, \tag{4.67}$$

which proves the associativity of the bracket (4.47) on quadratic functions, i.e.:

$$\{\{f_A, f_B\}_{\mathcal{H}}, f_C\}_{\mathcal{H}} = \{f_A, \{f_B, f_C\}_{\mathcal{H}}\}_{\mathcal{H}} = 4f_{ABC}, \quad \forall A, B, C \in gl(\mathcal{H}). \tag{4.68}$$

We look now at real, smooth functions on $\mathcal{H}_{\mathbb{R}}$.

First of all, it is clear that f_A will be a real function iff A is Hermitian. The Jordan and Poisson brackets will define then a Lie-Jordan algebra structure on the set of real, quadratic functions, and, according to Eq.(4.68), the bracket $\{\cdot,\cdot\}_{\mathcal{H}}$ will be an associative bracket.

For any such $f \in \mathcal{F}(\mathcal{H}_{\mathbb{R}})$ we may define two vector fields, the *gradient* ∇f of f and the *Hamiltonian vector field* X_f associated with f, defined by:

$$\begin{array}{lcl} g(\cdot, \nabla f) = df & & G(\cdot, df) = \nabla f, \\ \omega(\cdot, X_f) = df & or & \Lambda(\cdot, df) = X_f \end{array} \quad . \tag{4.69}$$

which allow us also to obtain the Jordan and the Poisson brackets as:

$$\{f, h\}_g = g(\nabla f, \nabla h), \tag{4.70}$$

$$\{f, h\}_{\omega} = \omega(X_f, X_h). \tag{4.71}$$

Explicitly, in coordinates:

$$\nabla f = \frac{\partial f}{\partial q^k}\frac{\partial}{\partial q^k} + \frac{\partial f}{\partial p^k}\frac{\partial}{\partial p^k} = 2\left(\frac{\partial f}{\partial z^k}\frac{\partial}{\partial \bar{z}^k} + \frac{\partial f}{\partial \bar{z}^k}\frac{\partial}{\partial z^k}\right) \tag{4.72}$$

$$X_f = \frac{\partial f}{\partial p^k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p^k} = 2i\left(\frac{\partial f}{\partial z^k}\frac{\partial}{\partial \bar{z}^k} - \frac{\partial f}{\partial \bar{z}^k}\frac{\partial}{\partial z^k}\right) \tag{4.73}$$

which are such that $J(\nabla f) = X_f$.

Turning to linear operators, to any $A : \mathcal{H} \to \mathcal{H}$ we can associate:

1. A quadratic function as in Eq. (4.61), and (cfr. also below, Sect.4.4),

2. A vector field: $X_A : \mathcal{H} \to T\mathcal{H}$ via: $x \longmapsto (x, Ax)$, and:

3. A $(1,1)$ tensor field: $T_A : T_x\mathcal{H} \ni (x,y) \longmapsto (x, Ay) \in T_x\mathcal{H}$. Clearly, as already remarked, f_A is real if and only if A is Hermitian. In this case:

$$\nabla f_A = X_A \tag{4.74}$$

and:

$$X_{f_A} = J(X_A) \tag{4.75}$$

Indeed, denoting with $(\cdot,\cdot)$ the pairing between vectors and covectors, Eq.(4.74) holds because:

$$\begin{aligned} g\left(y, X_A\left(x\right)\right) &= g\left(y, Ax\right) = \frac{1}{2}\left(\langle y, Ax\rangle_{\mathcal{H}} + \langle Ax, y\rangle_{\mathcal{H}}\right) = \\ &= \left(df_A\left(x\right), y\right) \end{aligned} \tag{4.76}$$

while Eq.(4.75) follows from the second expression in Eq.(4.23), i.e. from : $g\left(y, Ax\right) = \omega\left(y, (JX_A)(x)\right) = \omega\left(y, iAx\right)$. ■

Thus, we will write:

$$\nabla f_A = A \;\; \text{and:} \;\; X_{f_A} = iA \tag{4.77}$$

In particular, if we consider the identity operator $\mathbb{I}$, we obtain the dilation (or *Liouville*) field (cfr. also Eq.(4.28)):

$$\Delta : x \longmapsto (x, x) \tag{4.78}$$

or, in real coordinates:

$$\Delta = q^k \frac{\partial}{\partial q^k} + p^k \frac{\partial}{\partial p^k} \tag{4.79}$$

which is such that:

$$X_A = T_A(\Delta). \tag{4.80}$$

Finally we can also define the *phase vector field*:

$$\Gamma = J(\Delta) = p^k \frac{\partial}{\partial q^k} - q^k \frac{\partial}{\partial p^k} \tag{4.81}$$

that will be considered in the next Section.

4.2.4 The complex projective space

We would like now to discuss in some detail the structure of the complex projective Hilbert space $P\mathcal{H}$, which, as we have already mentioned, represents the right context to describe a geometric formulation of Quantum Mechanics. Indeed, given any vector $|x\rangle \in \mathcal{H} - \{\mathbf{0}\}$, the corresponding element in $P\mathcal{H}$ may be represented by the rank-one projector: $\widehat{\rho}_x =: |x\rangle\langle x| / \langle x|x\rangle$ in $D_1^1(\mathcal{H})$ (or simply: $\widehat{\rho}_x =: |x\rangle\langle x|$ if the vector is already normalized), and this will encode all the relevant physical information contained in $|x\rangle$.

In more geometric terms, we can consider the distribution generated by the dilation field Δ and the phase field $\Gamma = J(\Delta)$, which is involutive as $[\Delta, J(\Delta)] = 0$. Going to the quotient with respect to the foliation associated with this distribution (cfr.Eq.(4.1)) will be a way of generating the ray space $P\mathcal{H}$ which is independent on any Hermitian structure. Contravariant tensorial objects on $\mathcal{H}$ will "pass to the quotient" (i.e. will be projectable) if and only if they are left invariant by both Δ and Γ, i.e. if they are homogeneous of degree zero and invariant under multiplication of vectors by a phase. Typical quadratic functions that "pass to the quotient" will be normalized expectation values of the form:

$$\rho_x(A) =: Tr\{\widehat{\rho}_x A\} = \frac{\langle x|A|x\rangle}{\langle x|x\rangle} \tag{4.82}$$

with A any linear operator and for any Hermitian structure on $\mathcal{H}$. We note parenthetically that the subalgebra of functions on $\mathcal{H}$ that are invariant under Γ and Δ will define, via the construction of the Gel'fand-Kolmogoroff theorem [157], a manifold which can again be identified with $P\mathcal{H}$.

Concerning projectability of tensors, the complex structure J, being (cfr., e.g., Eq.(4.60)) homogeneous of degree zero and phase-invariant, will be a projectable tensor, while it is clear that the Jordan and Poisson tensors G and Λ defined respectively in Eq.(4.52) or, for that matter, the complex-valued tensor of Eq.(4.56) will *not* be projectable (as they are phase-invariant but homogeneous of degree -2). To turn them into projectable objects we will have to multiply them [86] by the "conformal factor": $\theta(z) =: z^\dagger z$, thus defining new tensors:

$$\widetilde{\Lambda}(z) =: \theta(z)\Lambda(z) \tag{4.83}$$

and similarly for G.

Let us examine these structures directly on $P\mathcal{H}$ more closely[63]. Recall that, in the finite dimensional case, $P\mathcal{H}$ is homeomorphic to $\mathbb{CP}^n$ and it is therefore made up of the equivalence classes of vectors $\mathbf{Z} = (Z^0, Z^1, \cdots, Z^n) \in \mathbb{C}^{n+1}$ w.r.t. the equivalence relation $Z \approx \lambda Z$; $\lambda \in \mathbb{C} - \{0\}$. The space $\mathbb{CP}^n$ is a Kähler manifold when endowed with the Fubini-Study metric [19, 105], whose pull-back to $\mathbb{C}^{n+1}$ is given by:

$$g_{FS} = \frac{1}{(\mathbf{Z}\cdot\bar{\mathbf{Z}})^2}\left[(\mathbf{Z}\cdot\bar{\mathbf{Z}})d\mathbf{Z}\otimes_S d\bar{\mathbf{Z}} - (d\mathbf{Z}\cdot\bar{\mathbf{Z}})\otimes_S(\mathbf{Z}\cdot d\bar{\mathbf{Z}})\right] \tag{4.84}$$

where $\mathbf{Z}\cdot\bar{\mathbf{Z}} = Z^a\bar{Z}^a$, $d\mathbf{Z}\cdot\bar{\mathbf{Z}} = dZ^a\bar{Z}^a$, $d\mathbf{Z}\otimes_S d\bar{\mathbf{Z}} = dZ^a d\bar{Z}^a + d\bar{Z}^a dZ^a$, and so on (the sum over repeated indices has to be understood), together with the compatible symplectic form:

$$\omega_{FS} = \frac{i}{(\mathbf{Z}\cdot\bar{\mathbf{Z}})^2}\left[(\mathbf{Z}\cdot\bar{\mathbf{Z}})d\mathbf{Z}\wedge d\bar{\mathbf{Z}} - (d\mathbf{Z}\cdot\bar{\mathbf{Z}})\wedge(\mathbf{Z}\cdot d\bar{\mathbf{Z}})\right] = d\theta_{FS} \tag{4.85}$$

where:

$$\theta_{FS} = \frac{1}{2i}\frac{\overline{\mathbf{Z}}d\mathbf{Z} - \mathbf{Z}d\overline{\mathbf{Z}}}{\mathbf{Z}\cdot\overline{\mathbf{Z}}} \tag{4.86}$$

[63] In the following of this Section, we will use the $(0,2)$-tensors g, ω instead of their (inverse) $(2,0)$-tensors G, Λ since calculations result to be more easily performed.

The isometries are just the usual unitary transformations which, in infinitesimal form, are written as:

$$\dot{Z}^a = iA^{ab}Z^b \tag{4.87}$$

where $A = [A^{ab}]$ is a Hermitian matrix. These are the equations for the flow of a generic Killing vector field, which therefore has the form[64]:

$$X_A = \dot{Z}^a \partial_{Z^a} - \dot{\bar{Z}}^a \partial_{\bar{Z}^a} = iA^{ab}(Z^b \partial_{Z^a} - \bar{Z}^a \partial_{\bar{Z}^b}) \tag{4.88}$$

A straightforward calculation shows that:

$$\begin{aligned} \omega_{FS}(\cdot, X_A) &= \frac{1}{\mathbf{Z} \cdot \bar{\mathbf{Z}}}[d\bar{Z}^a A^{ab} Z^b + \bar{Z}^a A^{ab} dZ^b] - \frac{\bar{Z}^a A^{ab} Z^b}{(\mathbf{Z} \cdot \bar{\mathbf{Z}})^2}[dZ^c \bar{Z}^c + Z^c d\bar{Z}^c] = \\ &= d\,(i_{X_A}\theta_{FS}) \end{aligned} \tag{4.89}$$

i.e. that X_A is the Hamiltonian vector field X_{f_A}, $\omega_{FS}(\cdot, X_{f_A}) = df_A$ associated with the (real) quadratic function:

$$f_A = \frac{\bar{\mathbf{Z}} \cdot A\mathbf{Z}^b}{\mathbf{Z} \cdot \bar{\mathbf{Z}}} = \frac{\bar{Z}^a A^{ab} Z^b}{Z^c \bar{Z}^c} = i_{X_A}\theta_{FS} \tag{4.90}$$

for the Hermitian matrix A. Also, some algebra shows that, given any two real quadratic functions f_A, f_B (A, B being Hermitian matrices), their corresponding Hamiltonian vector fields satisfy:

$$\omega_{FS}(X_{f_A}, X_{f_B}) = X_{f_A}(df_B) = f_{\frac{AB-BA}{i}} \tag{4.91}$$

Therefore, the Poisson brackets associated with the symplectic form:

$$\{f, g\}_{\omega_{FS}} := -\omega(X_f, X_g) \tag{4.92}$$

are such that:

$$\{f_A, f_B\}_{\omega_{FS}} = f_{\frac{AB-BA}{i}} \tag{4.93}$$

In a similar way, one can prove that the gradient vector field ∇_{f_A}, $g_{FS}(\cdot, \nabla_{f_A}) = df_A$, of f_A has the form:

$$\nabla_A = A^{ab}(Z^b \partial_{Z^a} + \bar{Z}^a \partial_{\bar{Z}^b}) \tag{4.94}$$

so that

$$g_{FS}(\nabla_{f_A}, \nabla_{f_B}) = \nabla_{f_A}(df_B) = f_{AB+BA} - f_A \cdot f_B \tag{4.95}$$

Given any two real quadratic functions f_A, f_B, we can therefore define a Jordan bracket by setting:

$$\{f_A, f_B\}_g := g_{FS}(\nabla_{f_A}, \nabla_{f_B}) + f_A \cdot f_B = f_{AB+BA} \tag{4.96}$$

[64]Notice that these are exactly the Killing vector fields of S^{2n+1}. In particular, for $A = \mathbb{I}$ we obtain $X_k = \Gamma$ which is a vertical vector field w.r.t. the Hopf projection $\pi_H : S^{2n+1} \to \mathbb{CP}^n$.

One says [42] that a real function on $P\mathcal{H}$ is Kählerian iff its Hamiltonian vector field is also Killing. Such functions represent quantum observables. The above calculations show that the space $\mathcal{F}(P\mathcal{H})$ of real quadratic functions on $P\mathcal{H}$ consists exactly of all Kählerian functions. To extend this concept to the complex case, one says that a complex valued function on $P\mathcal{H}$ is Kählerian iff are so its real and imaginary parts. Clearly, any such f is a quadratic function of the form (4.90) with now $A \in \mathcal{B}(\mathcal{H})$. Also, on the space, $\mathcal{F}^{\mathbb{C}}(P\mathcal{H})$, of Kählerian complex functions one can define both an Hermitian two-form:

$$h(\cdot,\cdot) = g_{FS}(\cdot,\cdot) + i\omega_{FS}(\cdot,\cdot) \tag{4.97}$$

and and associative bilinear product (star-product) via:

$$f \star g := f \cdot g + \frac{1}{2} h(df, dg) = \frac{1}{2}\left[\{f,g\}_g + i\{f,g\}_\omega\right] + f \cdot g \tag{4.98}$$

under which the space $\mathcal{F}^{\mathbb{C}}(P\mathcal{H})$ is closed since $f_A \star f_B = f_{AB}$, thus obtaining a particular realization of the $\mathbb{C}^*$-algebra of bounded operators $\mathcal{B}(\mathcal{H})$.

Let us suppose now that $(\mathcal{M}, \tilde{h})$ be a generic Kähler manifold. Also in this generic case, given any two functions f, g in the space of Kählerian (w.r.t. the metric $\tilde{g} = Re(\tilde{h})$) complex functions $\mathcal{F}^{\mathbb{C}}(\mathcal{M})$ one can define a $\star$-product:

$$f \star g := f \cdot g + \frac{1}{2}\tilde{h}(df, dg) \tag{4.99}$$

but now this product, although inner, will be not in general associative unless the functions are Kählerian.The condition that $\mathcal{F}^{\mathbb{C}}(\mathcal{M})$ be closed puts very restrictive conditions on the Kähler structure of $\mathcal{M}$ which imply [43] that $\mathcal{M}$ be a projective Hilbert space $P\mathcal{H}$. At the end of Sect. (4.3), after the discussion of the so called GNS construction, we will see how realizations of a $\mathbb{C}^*$-algebra as bounded operators on a suitable Hilbert space are in one-to-one correspondence with the action of the unitary group on the Kähler manifold.

4.2.5 The momentum map

We shall consider now the action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$, which is the group of linear transformations that preserve the triple (g, ω, J). In the following, we will denote with $u(\mathcal{H})$ the Lie algebra of $\mathcal{U}(\mathcal{H})$ of anti-Hermitian operators and identify the space of all Hermitian operators with the dual $u^*(\mathcal{H})$ of $u(\mathcal{H})$ via the pairing:

$$\langle A, T\rangle =: \frac{i}{2} Tr(AT), A \in u^*(\mathcal{H}), T \in u(\mathcal{H}) \tag{4.100}$$

On $u^*(\mathcal{H})$ we can define a Lie bracket (cfr.also Sect.4.2.3):

$$[A, B]_- =: \frac{1}{i}(AB - BA), \tag{4.101}$$

with respect to which it becomes a Lie algebra, and also a Jordan bracket:

$$[A,B]_+ =: AB + BA. \tag{4.102}$$

with the two together giving $u^*(\mathcal{H})$ the structure of a Lie-Jordan algebra [65].

In addition, $u^*(\mathcal{H})$ is equipped with the scalar product

$$\langle A,B\rangle_{u^*} = \frac{1}{2}Tr(AB) \tag{4.103}$$

which satisfies:

$$\langle [A,\xi]_-,B\rangle_{u^*} = \frac{1}{2}Tr([A,\xi]_-B) = \frac{1}{2}Tr(A,[\xi,B]_-) = \langle A,[\xi,B]_-\rangle_{u^*} \tag{4.104}$$

$$\langle [A,\xi]_+,B\rangle_{u^*} = \frac{1}{2}Tr([A,\xi]_+B) = \frac{1}{2}Tr(A,[\xi,B]_+) = \langle A,[\xi,B]_+\rangle_{u^*} \tag{4.105}$$

With any $A \in u^*(\mathcal{H})$, we can associate the fundamental vector field X_A on the Hilbert space corresponding to the element $\frac{1}{i}A \in u(\mathcal{H})$ defined by the formula:

$$\frac{d}{dt}e^{-\frac{t}{i}A}(x)|_{t=0} = iA(x),\ \forall x \in \mathcal{H} \tag{4.106}$$

In other words, $X_A = iA$. We already know from Sect. 4.2.3 that iA has f_A as its Hamiltonian function: $\omega(\cdot, X_A) = df_A$. Thus, for any $x \in \mathcal{H}_\mathbb{R}$ we obtain a $\mu(x) \in u^*(\mathcal{H})$ such that:

$$\langle \mu(x), \frac{1}{i}A\rangle = f_A(x) = \frac{1}{2}\langle x, Ax\rangle_\mathcal{H} \tag{4.107}$$

In such a way we obtain a mapping:

$$\mu : \mathcal{H}_\mathbb{R} \to u^*(\mathcal{H}) \tag{4.108}$$

which is called the momentum map [167].

More explicitly, it follows from Eq.(4.100) that:

$$\langle \mu(x), \frac{1}{i}A\rangle = \frac{1}{2}Tr(\mu(x)A) \tag{4.109}$$

which, when compared with Eq.(4.107), yields:

$$\mu(x) = |x\rangle\langle x| \tag{4.110}$$

We may therefore conclude that the unit sphere in $\mathcal{H}$ can be projected onto $u^*(\mathcal{H})$ in an equivariant way with respect to the coadjoint action of $\mathcal{U}(\mathcal{H})$. Also, in finite dimensions, the unit sphere is odd dimensional and the orbit in $u^*(\mathcal{H})$ is symplectic.

With every $A \in u^*(\mathcal{H})$ we can associate, with the by now familiar identification (as with every other linear vector space) of the tangent space at every point of $u^*(\mathcal{H})$ with $u^*(\mathcal{H})$ itself, the linear function (hence a one-form) $\hat{A}$:

$u^*(\mathcal{H}) \to \mathbb{R}$ defined as: $\hat{A} =: \langle A, \cdot \rangle_{u^*}$. Then, we can define two contravariant tensors, a symmetric (Jordan) tensor:

$$R(\hat{A}, \hat{B})\,(\xi) =: \langle \xi, [A, B]_+ \rangle_{u^*} \tag{4.111}$$

and a Poisson (Konstant-Kirillov-Souriau [113, 114, 115, 214]) tensor:

$$I(\hat{A}, \hat{B})\,(\xi) = \langle \xi, [A, B]_- \rangle_{u^*} \tag{4.112}$$

($A, B, \xi \in u^*(\mathcal{H})$). We notice that the quadratic function f_A is the pull-back of $\hat{A}$ via the momentum map since, for all $x \in \mathcal{H}$:

$$\mu^*(\hat{A})(x) = \hat{A} \circ \mu(x) = \langle A, \mu(x) \rangle_{u^*} = \frac{1}{2}\langle x, Ax \rangle_{\mathcal{H}} = f_A(x) \tag{4.113}$$

This means also that, if: $\xi = \mu(x)$:

$$(\mu_* G)(\hat{A}, \hat{B})\,(\xi) = G(df_A, df_B)\,(x) = \{f_A, f_B\}_g(x) = f_{[A,B]_+}(x) = R(\hat{A}, \hat{B})\,(\xi) \tag{4.114}$$

where the last equality follows from Eq.(4.62), i.e.:

$$\mu_* G = R \tag{4.115}$$

Similarly, by using now Eq.(4.63), we find:

$$(\mu_* \Lambda)(\hat{A}, \hat{B})\,(\xi) = \Lambda(df_A, df_B)\,(x) = \{f_A, f_B\}_\omega(x) = f_{[A,B]_-}(x) = I(\hat{A}, \hat{B})\,(\xi) \tag{4.116}$$

i.e.:

$$\mu_* \Lambda = I \tag{4.117}$$

Thus, the momentum map relates the contravariant metric tensor G and the Poisson tensor Λ with the corresponding contravariant tensors R and I. Together they form the complex tensor:

$$(R + iI)(\hat{A}, \hat{B})\,(\xi) = 2\langle \xi, AB \rangle_{u^*} \tag{4.118}$$

which is related to the Hermitian product on $u^*(\mathcal{H})$.

Example 29 *Let $\mathcal{H} = \mathbb{C}^2$ (the Hilbert space appropriate for a two-level system). We can write any $A \in u^*(\mathbb{C}^2)$ as:*

$$A = y^0 \mathbb{I} + \mathbf{y} \cdot \sigma \tag{4.119}$$

where $\mathbb{I}$ is the 2×2 identity, $\mathbf{y} \cdot \sigma = y^1\sigma_1 + y^2\sigma_2 + y^3\sigma_3$ and: $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices:

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{4.120}$$

with the well-known identities [183]:

$$\sigma_h \sigma_k = \delta_{hk} \mathbb{I} + i\varepsilon_{hkl}\sigma_l \tag{4.121}$$

$(h, k, l = 1, 2, 3)$ *and:*

$$\sigma_j \sigma_k \sigma_l = i\varepsilon_{jkl}\mathbb{I} + \sigma_j \delta_{kl} - \sigma_k \delta_{jl} + \sigma_l \delta_{jk} \tag{4.122}$$

Every $A \in u^*(\mathbb{C}^2)$ is then represented by the (real) "four-vector" $\left(y_A^0, \mathbf{y}_A\right)$, and:

$$y_A^0 = \frac{1}{2} Tr\,(A)\,;\ y_A^k = \frac{1}{2} Tr\,(\sigma_k A)\,;\ k = 1, 2, 3 \tag{4.123}$$

or, in short:

$$y_\mu(A) = \langle A|\sigma_\mu\rangle\,,\ \mu = 0, 1, 2, 3, \sigma_0 = \mathbb{I} \tag{4.124}$$

Digression.

Rank-one projectors $\left(A = \rho, \rho^\dagger = \rho,\ Tr\rho = 1,\ \rho^2 = \rho\right)$ can be parametrized as [187]:

$$\rho = \rho\,(\theta, \phi) = \begin{vmatrix} \sin^2\frac{\theta}{2} & \frac{1}{2}e^{i\phi}\sin\theta \\ \frac{1}{2}e^{-i\phi}\sin\theta & \cos^2\frac{\theta}{2} \end{vmatrix};\ 0 \le \theta < \pi, 0 \le \phi < 2\pi \tag{4.125}$$

Then, they correspond to:

$$y^0 = \frac{1}{2},\ y^1 = \frac{1}{2}\sin\theta\cos\phi,\ y^2 = -\frac{1}{2}\sin\theta\sin\phi,\ y^3 = -\frac{1}{2}\cos\theta \tag{4.126}$$

(hence: $\mathbf{y}^2 = 1/4$ for all rank-one projectors). As already discussed elsewhere, we can associate with every $A \equiv \left(y_A^0, \mathbf{y}_A\right)$ the vector field: $y^0(A)\partial_0 + y^1(A)\partial_1 + y^2(A)\partial_2 + y^3(A)\partial_3$ $\left(\partial_0 = \partial/\partial y^0 \text{ and so on}\right)$. Also (see the discussion immediately above Eq.(4.111)), $\hat{A} = \langle A, \cdot\rangle_{u^*}$ will be represented by the one-form:

$$\hat{A} = y^0(A)dy^0 + y^1(A)dy^1 + y^2(A)dy^2 + y^3(A)dy^3 \tag{4.127}$$

Using then Eq.(4.119) one proves easily that:

$$AB = \left(y_A^0 y_B^0 + \mathbf{y}_A \cdot \mathbf{y}_B\right)\mathbb{I} + \left(y_A^0\mathbf{y}_B + y_B^0\mathbf{y}_A + i\mathbf{y}_A \times \mathbf{y}_B\right)\cdot\sigma \tag{4.128}$$

(with " $\times$ " denoting the standard cross-product of three-vectors) and hence[65]:

$$\langle AB\rangle_{u^*} = \frac{1}{2}Tr\,(AB) = y_A^0 y_B^0 + \mathbf{y}_A \cdot \mathbf{y}_B \tag{4.129}$$

Moreover:

[65] In particular: $\langle\rho\,(\theta, \phi)\,\rho\,(\theta', \phi')\rangle_{u^*} =$ $\{1 + \sin\theta\sin\theta'\cos(\phi - \phi') + \cos\theta\cos\theta'\}\,/4$ for rank-one projectors.

$$[A,B]_+ = 2\left\{\left(y_A^0 y_B^0 + \mathbf{y}_A\cdot\mathbf{y}_B\right)\mathbb{I} + \left(y_A^0\mathbf{y}_B + y_B^0\mathbf{y}_A\right)\cdot\sigma\right\} \tag{4.130}$$

while:

$$[A,B]_- = 2\mathbf{y}_A\times\mathbf{y}_B\cdot\sigma \tag{4.131}$$

Then:

$$\begin{gathered} R(\hat{A},\hat{B})\,(\xi) = \langle\xi,[A,B]_+\rangle_{u^*} = \left\langle[\xi,A]_+,B\right\rangle = \\ 2\xi^0\left(y_A^0y_B^0 + \mathbf{y}_A\cdot\mathbf{y}_B\right) + 2\left(y_A^0\mathbf{y}_B + y_B^0\mathbf{y}_A\right)\cdot\xi = \\ 2\left(y_A^0\xi^0 + \mathbf{y}_A\cdot\xi\right)y_B^0 + 2\left(y_A^0\xi + \xi^0\mathbf{y}_A\right)\cdot\mathbf{y}_B \end{gathered} \tag{4.132}$$

and hence, explicitly [86]:

$$\begin{gathered} R\,(\xi) = 2\partial_0\otimes\left(\xi^1\partial_1 + \xi^2\partial_2 + \xi^3\partial_3\right) + 2\left(\xi^1\partial_1 + \xi^2\partial_2 + \xi^3\partial_3\right)\otimes\partial_0 + \\ 2\xi^0\left(\partial_0\otimes\partial_0 + \partial_1\otimes\partial_1 + \partial_2\otimes\partial_2 + \partial_3\otimes\partial_3\right) \end{gathered} \tag{4.133}$$

Quite similarly, one finds:

$$I(\hat{A},\hat{B})\,(\xi) = 2(\xi\times\mathbf{y}_A)\cdot\mathbf{y}_B = 2(\mathbf{y}_A\times\mathbf{y}_B)\cdot\xi \tag{4.134}$$

and:

$$I\,(\xi) = 2\left(\xi^1\partial_2\wedge\partial_3 + \xi^2\partial_3\wedge\partial_1 + \xi^3\partial_1\wedge\partial_2\right) \tag{4.135}$$

We thus find the following tensor:

$$\begin{aligned} R + iI = 2\,[\,&\partial_0\otimes y^k\partial_k + y^k\partial_k\otimes\partial_0 + \\ &y^0(\partial_0\otimes\partial_0 + \partial_k\otimes\partial_k) + i\epsilon_{hkl}y^h\partial_k\otimes\partial_l\big] \end{aligned} \tag{4.136}$$

To conclude this Section, we define also two $(1,1)$ tensors, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{J}}$: $Tu^*(\mathcal{H}) \to Tu^*(\mathcal{H})$ that will be employed below in Sect.4.2.6 via:

$$\widetilde{\mathcal{R}}_\xi\,(A) =: [\xi,A]_+ = R\left(\widehat{A},.\right)(\xi) \tag{4.137}$$

and:

$$\widetilde{\mathcal{J}}_\xi\,(A) =: [\xi,A]_- = I\left(\widehat{A},.\right)(\xi) \tag{4.138}$$

for any $A \in T_\xi u^*(\mathcal{H}) \approx u^*(\mathcal{H})$, the last passage in both equations following from Eqns.(4.104) and (4.105).

In the previous example ($\mathcal{H} \approx \mathbb{C}^2$) we find explicitly, in coordinates:

$$\widetilde{\mathcal{R}}_\xi\,(A) = 2\left(y_A^0\xi^0 + \mathbf{y}_A\cdot\xi\right)\partial_0 + 2\left(y_A^0\xi^i + \xi^0y_A^i\right)\partial_i \tag{4.139}$$

or:

$$\widetilde{\mathcal{R}}_\xi = 2\left(\xi^0dy^0 + \xi\cdot\mathbf{dy}\right)\otimes\partial_0 + 2\left(\xi^idy^0 + \xi^0dy^i\right)\otimes\partial_i \tag{4.140}$$

and:

$$\widetilde{\mathcal{J}}_\xi\,(A) = 2\varepsilon_{ijk}\xi^iy_A^j\partial_k \tag{4.141}$$

or:

$$\widetilde{\mathcal{J}}_\xi = 2\varepsilon_{ijk}\xi^idy^j\otimes\partial_k \tag{4.142}$$

4.2.6 The space of density states

We have seen in Sect. 4.2.4 that it is possible to obtain $\mathcal{P}(\mathcal{H})$ as a quotient of $\mathcal{H} - \{\mathbf{0}\}$ with respect to the involutive distribution associated with Δ and $J(\Delta)$. Eq. (4.110) shows that the image of $\mathcal{H} - \{\mathbf{0}\}$ under the momentum map consists of the set of all non-negative Hermitian operators of rank one, that will be denoted as $\mathcal{P}^1(\mathcal{H})$, i.e.[66]:

$$\mathcal{P}^1(\mathcal{H}) = \{|x\rangle\langle x|;\ x \in \mathcal{H},\ x \neq 0\} \tag{4.143}$$

On the other hand, the coadjoint action of $\mathcal{U}(\mathcal{H})$: $(U, \rho) \mapsto U\rho U^\dagger$ ($\rho \in \mathcal{P}^1(\mathcal{H}), U \in \mathcal{U}(\mathcal{H})$) foliates $\mathcal{P}^1(\mathcal{H})$ into the spaces $\mathcal{D}^1_r(\mathcal{H}) = \{|x\rangle\langle x| : \langle x, x\rangle_{\mathcal{H}} = r\}$. In particular we have already denoted with $\mathcal{D}^1_1(\mathcal{H})$ the space of one-dimensional projection operators, which is the image via the momentum map of the sphere $S_{\mathcal{H}} = \{x \in \mathcal{H}\,;\, \langle x, x\rangle_{\mathcal{H}} = 1\}$ and can be identified with the complex projective space $P(\mathcal{H})$ via the identification:

$$[x] \in P(\mathcal{H}) \leftrightarrow \frac{|x\rangle\langle x|}{\langle x, x\rangle} \in \mathcal{D}^1_1(\mathcal{H}) \tag{4.144}$$

We have also argued that $P(\mathcal{H})$ is a Kähler manifold. In the following we will examine this fact in more detail, by showing explicitly that $\mathcal{D}^1_1(\mathcal{H})$ is a Kähler manifold.

Let $\xi \in u^*(\mathcal{H})$ be the image through the momentum map of a unit vector $x \in S_{\mathcal{H}}$, i.e. $\xi = |x\rangle\langle x|$ with $\langle x|x\rangle = 1$, so that $\xi^2 = \xi$. The tangent space of the coadjoint $\mathcal{U}(\mathcal{H})$-orbit at ξ is generated by vectors of the form $[A, \xi]_-$, for any Hermitian A. From Eq.(4.104), it follows that the Poisson tensor I defined in (4.112) satisfies:

$$I(\hat{A}, \hat{B})\,(\xi) = \langle \xi, [A, B]_-\rangle_{u^*} = \langle [\xi, A]_-, B\rangle_{u^*} \tag{4.145}$$

This defines an invertible map $\tilde{I}$ that associates to any one-form $\hat{A}$ the tangent vector at ξ: $\tilde{I}(\hat{A}) =: I(\hat{A}, \cdot) = [\xi, A]_-$. We will denote with $\tilde{\eta}_\xi$ its inverse: $\tilde{\eta}_\xi([\xi, A]_-) = \hat{A}$. This allows us to define, on $u^*(\mathcal{H})$, a canonical two-form which is given by:

$$\eta_\xi([A, \xi]_-, [B, \xi]_-) =: (\tilde{\eta}_\xi([\xi, A]_-), [B, \xi]_-) = (\hat{A}, [B, \xi]_-) \tag{4.146}$$

for all $[A, \xi]_-, [B, \xi]_- \in T_\xi u^*(\mathcal{H})$.

It is also easy to check that η satisfies the equalities: $\eta_\xi([A, \xi]_-, [B, \xi]_-) = -\left(\hat{A}, [B, \xi]_-\right) = -\langle A, [B, \xi]_-\rangle_{u^*} = -\langle \xi, [A, B]_-\rangle_{u^*} = \langle [A, \xi]_-, B\rangle_{u^*}$, for any $A, B \in u^*(\mathcal{H})$.

We can summarize these results in the following:

[66] Note that here the vectors are not necessarily normalized.

Theorem 30 *The restriction of the two-form (4.146) to the $\mathcal{U}(\mathcal{H})$-orbit $\mathcal{D}_1^1(\mathcal{H})$ defines a canonical symplectic form η characterized by the property*

$$\eta_\xi([A,\xi]_-,[B,\xi]_-) = \big\langle [A,\xi]_- \,, B \big\rangle_{u^*} = -\langle \xi, [A,B]_- \rangle_{u^*} \tag{4.147}$$

In a very similar way, starting from the symmetric Jordan tensor R given in (4.111) , one can construct a $(1,1)$ tensor $\tilde{R}(\hat{A}) =: R(\hat{A},\cdot) = [\xi, A]_+$ and its inverse: $\tilde{\sigma}([\xi,A]_+) = \hat{A}$. Thus we obtain a covariant tensor σ such that:

$$\sigma_\xi([A,\xi]_+,[B,\xi]_+) = \langle [A,\xi]_+, B\rangle_{u^*} = \langle \xi, [A,B]_+\rangle_{u^*}. \tag{4.148}$$

Notice that, at this stage, σ_ξ is only a partial tensor, being defined on vectors of the form $[A,\xi]_+$, which belong to the image of the map $\tilde{R}$. However, on $\mathcal{D}_1^1(\mathcal{H})$, we have $[A,\xi]_- = [A,\xi^2]_- = [[A,\xi],\xi]_+$, so that, after some algebra, one can also prove that:
$\sigma_\xi([A,\xi]_-,[B,\xi]_-) = \sigma_\xi([[A,\xi]_-,\xi]_+,[[B,\xi]_-,\xi]_+) = \langle \xi, [[A,\xi]_-,[B,\xi]_-]_+\rangle_{u^*} =$
$= \frac{1}{2}Tr(\xi[[A,\xi]_-,[B,\xi]_-]_+) = \frac{1}{2}Tr(\xi[A,\xi]_-[B,\xi]_-) = \langle [A,\xi]_-,[B,\xi]_-\rangle_{u^*}$.
Therefore we have also the following:

Theorem 31 *On the $\mathcal{U}(\mathcal{H})$-orbit $\mathcal{D}_1^1(\mathcal{H})$ we can define a symmetric covariant tensor σ such that:*

$$\sigma_\xi([A,\xi]_-,[B,\xi]_-) = \langle [A,\xi]_-,[B,\xi]_-\rangle_{u^*}. \tag{4.149}$$

holds.

Moreover, going back to the the $(1,1)$ tensor $\tilde{I}$ given above, one has the following result [86]:

Theorem 32 *When restricted to $\mathcal{D}_1^1(\mathcal{H})$, the $(1,1)$ tensor $\tilde{I}$, which satisfies:*

$$\tilde{I}^3 = -\tilde{I} \tag{4.150}$$

will become invertible. Hence: $\tilde{I}^2 = -\mathbb{I}$ and therefore it will define a complex structure $\jmath$ such that:

$$\eta_\xi([A,\xi]_-,\jmath_\xi([B,\xi]_-)) = \sigma_\xi([A,\xi]_-,[B,\xi]_-) \tag{4.151}$$

$$\eta_\xi(\jmath_\xi([A,\xi]_-),\jmath_\xi([B,\xi]_-)) = \eta_\xi([A,\xi]_-,[B,\xi]_-) \tag{4.152}$$

Eq. (4.150) follows from a direct calculation by taking into account that $\xi^2 = \xi$. The last two expressions follow by combining Eqs.(4.147) and (4.149). To prove that $\jmath$ is a complex structure one has first to show that it defines an almost complex structure (which follows easily from the fact that $[[[A,\xi]_-,\xi]_-,\xi]_- = -[A,\xi]_-$) and then that its Nijenhuis torsion vanishes. Detailed calculations of this can be found in Ref.[86].

Putting everything together, we can now conclude that, as expected:

Theorem 33 *$(\mathcal{D}_1^1(\mathcal{H}),\jmath,\sigma,\eta)$ is a Kähler manifold.*

At last, we notice that there is an identification of the orthogonal complement of any unit vector $x \in \mathcal{H}$ with the tangent space of the $\mathcal{U}(\mathcal{H})$-orbit in $u^*(\mathcal{H})$ at $\xi = |x\rangle\langle x|$. Indeed, for any y perpendicular to x ($\|x\|^2 = 1$) the operators:

$$P_y^x =: (\mu_*)_x(y) = |y\rangle\langle x| + |x\rangle\langle y| \tag{4.153}$$

can be written as $P_y^x = [A_y, \xi]$, where A_y is a Hermitian operator such that $A_y x = iy$, $A_y y = -i\|y\|^2 x$ and $A_y z = 0$ for any z perpendicular to both x and y, as it can be directly checked by applying both expressions to a generic vector in $\mathcal{H}$ which can be written as $ax + by + cz$ with $a, b, c \in \mathbb{C}$. Then, from Eqs.(4.147) and (4.149), it follows immediately that, for any y, y' orthogonal to x:

$$\eta_\xi(P_y^x, P_{y'}^x) = -\frac{1}{2}Tr(\xi[A_y, A_{y'}]_-) = -\frac{1}{2i}(\langle y, y'\rangle - \langle y', y\rangle) = -\omega(y, y') \tag{4.154}$$

$$\sigma_\xi(P_y^x, P_{y'}^x) = \frac{1}{2}Tr(\xi[A_y, A_{y'}]_-) = -\frac{1}{2}(\langle y, y'\rangle + \langle y', y\rangle) = g(y, y') \tag{4.155}$$

In conclusion, we have the following:

Theorem 34 *For any $y, y' \in \mathcal{H}$, the vectors $(\mu_*)_x(y), (\mu_*)_x(y)$ are tangent to the $\mathcal{U}(\mathcal{H})$-orbit in $u^*(\mathcal{H})$ at $\xi = \mu(x)$ and:*

$$\sigma_\xi((\mu_*)_x(y), (\mu_*)_x(y)) = g(y, y') \tag{4.156}$$

$$\eta_\xi((\mu_*)_x(y), (\mu_*)_x(y)) = -\omega(y, y') \tag{4.157}$$

$$\jmath_\xi(\mu_*)_x(y)) = (\mu_*)_x(Jy) \tag{4.158}$$

where the last formula follows from Eq.(4.151).

More generally, with minor changes, we can reconstruct similar structures for any $\mathcal{D}_r^1(\mathcal{H})$, obtaining Kähler manifolds $(\mathcal{D}_r^1(\mathcal{H}), \jmath^r, \sigma^r, \eta^r)$. The analog of above theorem shows then that the latter can be obtained from a sort of "Kähler reduction" starting from the original linear Kähler manifold $(\mathcal{H}_\mathbb{R}, J, g, \omega)$.

Example 35 *Let us go back to the previous example of rank-one projectors on $\mathcal{H} = \mathbb{C}^2$. According to (4.126), the latter are described by three dimensional vectors $\xi = (y^1, y^2, y^3)$ such that $\xi^2 = 1/4$ ($y_0 = 1/2$ always), which form a 2-dimensional sphere of radius $1/2$. A generic tangent vector X_A and a generic one form $\hat{A}$ at ξ are of the form $X_A = y_A^0\partial_0 + y_A^1\partial_1 + y_A^2\partial_2 + y_A^3\partial_3$ and $\hat{A} = y_A^0 dy^0 + y_A^1 dy^1 + y_A^2 dy^2 + y_A^3 dy^3$ with $y_A^0 = 0$ and $\mathbf{y}_A \cdot \xi = 0$.*

It is clear from (4.134) that the map $\tilde{I}$ that associates to any one-form $\hat{A}$ the tangent vector at ξ: $\tilde{I}(\hat{A}) =: I(\hat{A}, \cdot) = [A, \xi]_-$ is manifestly invariant and given by: $\tilde{I}(\hat{A}) = 2(\xi \times \mathbf{y}_A) \cdot \vec{\partial}$, where we have set $\vec{\partial} = (\partial_1, \partial_2, \partial_3)$. It follows that the two-form η_ξ is such that:

$$\eta_\xi([A, \xi]_-, [B, \xi]_-) = 2\xi \cdot (\mathbf{y}_A \times \mathbf{y}_B) \tag{4.159}$$

so that

$$\eta_\xi = 2\epsilon^{ijk} y^i dy^j \wedge dy^k \tag{4.160}$$

which is proportional by a factor $\left(y_1^2 + y_2^2 + y_3^2\right)^{-\frac{3}{2}}$ *to the symplectic two-form on a 2-dimensional sphere*[67]*, when pulled back to the sphere.*

In a similar way, from (4.132), one can prove that $\tilde{R}(\hat{A}) =: R(\hat{A}, \cdot) = [\xi, A]_+ = 2(y_A^0 y^0 + \mathbf{y}_A \cdot \xi)\partial_0 + 2(y_A^0 \xi + y_0 \mathbf{y}_A) \cdot \vec{\partial}$*. Thus, because of (4.149), we have:*

$$\sigma_\xi([A, \xi]_-, [B, \xi]_-) = 4(\xi \times \mathbf{y}_A) \cdot (\xi \times \mathbf{y}_B) = \mathbf{y}_A \cdot \mathbf{y}_B \tag{4.161}$$

where the last equality follows from the fact that $\xi^2 = 1/4$ *and* ξ *is orthogonal to both* $\mathbf{y}_A$ *and* $\mathbf{y}_B$*.*

Finally, starting for example from Eq. (4.151), it is not difficult to check that

$$\jmath_\xi([B, \xi]_-) = y'_B \cdot \vec{\partial} \;\; \textit{with}: \; \mathbf{y}'_B = \xi \times \mathbf{y}_B \tag{4.162}$$

A direct calculation shows that $\jmath_\xi^3 = -\jmath_\xi$*.*

4.3 The geometry of quantum mechanics and the *GNS* construction

In the previous Sections of this Chapter, we have worked out the geometrical structures that naturally arise in the standard approach to quantum mechanics, which starts from the Hilbert space and identifies the space of physical states with the associated complex projective space. In this framework, algebraic notions, such that of the $\mathbb{C}^*$-algebra that contains observables as real elements, arises only as a derived concept.

In this Section, we would like to see how geometrical structures emerge also in a more algebraic setting, where one starts from the very beginning with an abstract $\mathbb{C}^*$-algebra containing the algebra of quantum observables as real elements to obtain the Hilbert space of states is a derived concept via the so called Gelfand-Naimark-Segal (*GNS*) construction [26]. A detailed discussion can be found in Ref. [42].

4.3.1 The *GNS* construction

The algebraic approach known as the *GNS* construction started with the work of Haag and Kastler [97], and is also at the basis of the mathematical approach to quantum field theory [95].

The starting point of this construction is an abstract $\mathbb{C}^*$-algebra $\mathcal{A}$ [26, 65] with unity, the latter being denoted as $\mathbb{I}$. The elements $a \in \mathcal{A}$ such that: $a =$

[67] This is also the volume element of a 2-dimensional sphere of radius $r = 1/2$, as it should be.

a^* constitute the set $\mathcal{A}_{re}$ (a vector space over the reals) of the *real elements*[68] of the algebra. In particular: $\mathbb{I} \in \mathcal{A}_{re}$. The obvious decomposition: $a = a_1 + ia_2$, with:

$$a_1 = \frac{a + a^*}{2}; \quad a_2 = \frac{a - a^*}{2i} \tag{4.163}$$

means that, as a vector space, $\mathcal{A}$ is the direct sum of $\mathcal{A}_{re}$ and of the set $\mathcal{A}_{im}$ (also a vector space over the reals) of the *imaginary elements*, i.e. of the elements of the form ia, $a \in \mathcal{A}_{re}$. $\mathcal{A}_{re}$ can be given [42] the structure of a Lie-Jordan algebra [65], where, using here the conventions of Sect.4.2.3, the Lie product is defined as:

$$[a, b] =: \frac{1}{2i}(ab - ba) \tag{4.164}$$

while the Jordan product is given by:

$$a \circ b = \frac{1}{2}(ab + ba) \tag{4.165}$$

for all $a, b \in \mathcal{A}_{re}$. The product in the algebra is then recovered as:

$$ab = a \circ b + i[a, b] \tag{4.166}$$

Remark 36 *A typical example of a $\mathbb{C}^*$-algebra is the algebra $\mathcal{B}(\mathcal{H})$ of the bounded operators on a Hilbert space $\mathcal{H}$. In this case [65]: $\mathcal{A}_{re} \equiv \mathcal{B}_{sa}(\mathcal{H})$, the set of the bounded self-adjoint operators on $\mathcal{H}$.*

The space $\mathcal{D}(\mathcal{A})$ of the *states* over the $\mathbb{C}^*$-algebra $\mathcal{A}$ is the space of the linear functionals $\omega : \mathcal{A} \to \mathbb{C}$ that are [95]:

- *real*: $\omega(a^*) = \overline{\omega(a)}\ \forall a \in \mathcal{A}$,
- *positive:* $\omega(a^* a) \geq 0\ \forall a \in \mathcal{A}$ and
- *normalized*: $\omega(\mathbb{I}) = 1$

Each functional ω defines a non-negative pairing $\langle \cdot | \cdot \rangle_\omega$ between any two elements $a, b \in \mathcal{A}$ via:

$$\langle a | b \rangle_\omega := \omega(a^* b) \tag{4.167}$$

Reality and positivity of the state guarantee that the pairing (4.167) satisfies the Schwartz inequality, i.e.:

$$|\langle a | b \rangle_\omega| \leq \sqrt{\langle a | a \rangle_\omega} \sqrt{\langle b | b \rangle_\omega} \tag{4.168}$$

but the pairing might be degenerate. We are thus led to consider the "Gelfand ideal" [65, 95] $\mathcal{I}_\omega$ consisting of all elements $j \in \mathcal{A}$ such that $\omega(j^* j) = 0$ and to define the set $\mathcal{A}/\mathcal{I}_\omega$ of equivalence classes:

$$\Psi_a =: [a + \mathcal{I}_\omega] \tag{4.169}$$

[68] Also called the *observables*.

It is immediate to see that $\mathcal{A}/\mathcal{I}_\omega$ is a pre-Hilbert space with respect to the scalar product[69]:

$$\langle \Psi_a, \Psi_b \rangle = \omega(a^* b) \tag{4.170}$$

Completing this space with respect to the topology defined by the scalar product, one obtains a Hilbert space $\mathcal{H}_\omega$ on which the original $\mathbb{C}^*$-algebra $\mathcal{A}$ acts via the following representation[70]:

$$\pi_\omega(a)\Psi_b = \Psi_{ab} \tag{4.171}$$

Clearly the equivalence class of the unit element in $\mathcal{A}$, i.e. $\Omega = \Psi_{\mathbb{I}}$, satisfies: $\|\Psi_{\mathbb{I}}\| := \sqrt{\langle \Psi_{\mathbb{I}} | \Psi_{\mathbb{I}} \rangle} = 1$ and provides a cyclic vector[71] for the representation π_ω. Moreover:

$$\langle \Omega | \pi_\omega(a) | \Omega \rangle = \omega(a) \tag{4.172}$$

This tells us that, if we consider that $\mathcal{A}$ acts by duality on $\mathcal{D}(\mathcal{A})$, the Hilbert space corresponding to a given state ω is the orbit of $\mathcal{A}$ through ω itself. Notice that any other element $b \in \mathcal{A}$ such that the vector $\Psi = \pi_\omega(b)\Omega$ is of unit norm, defines a new state ω_Ψ by:

$$\omega_\Psi(a) = \langle \Psi | \pi_\omega(a) | \Psi \rangle = \omega(b^* a b) \tag{4.173}$$

These states are called vector states of the representation π_ω, and are particular examples of more general states of the form:

$$\omega_\rho(a) = Tr[\rho \pi_\omega(a)] \tag{4.174}$$

where $\rho \in \mathcal{B}(\mathcal{H}_\omega)$ is a density operator [65, 95]. States of the form (4.174) are called a "folium" of the representation π_ω. Also, one says that a state ω is pure iff it cannot be written as a convex combination of other states in $\mathcal{D}(\mathcal{A})$, so that the set of pure states $\mathcal{D}^1(\mathcal{A})$ defines a set of extremal points in $\mathcal{D}(\mathcal{A})$.

The universality and uniqueness of the GNS construction is guaranteed [26] by the following:

Theorem 37

1. *If π_ω is a cyclic representation of A on H, any vector representation ω_Ψ for a normalized Ψ, see Eq.(4.174), ie equivalent to π_ω.*
2. *A GNS representation π_ω of $\mathcal{A}$ is irreducible iff ω is a pure state.*

[69] The Schwartz inequality (4.168) implies: $\langle i|a\rangle_\omega = \langle a|i\rangle_\omega = 0 \; \forall a \in \mathcal{A}, i \in \mathcal{I}_\omega$, and hence that the scalar product (4.170) does indeed depend only on the equivalence classes of a and b and not on the specific representatives chosen.

[70] Notice that if such a representation is faithful, i.e. the map $\pi_\omega : a \mapsto \pi_\omega(a)$ is an isomorphism, the operator norm of $\pi_\omega(a)$ equals the $\mathbb{C}^*$-norm of a [26].

[71] We recall [95] that a vector $\Omega \in \mathcal{H}_\omega$ is called *cyclic* if $\pi_\omega(\mathcal{A})$ is dense in $\mathcal{H}_\omega$.

Example 38 *The GNS construction can be very simple for finite- dimensional C^*-algebras. Consider, e.g., the algebra $A = B(C^n)$ of linear operators on C^n, i.e. of the $n \times n$ matrices with complex entries. Any non-negative operator $\omega \in B(C^n)$ defines a state by:*

$$\omega(A) = Tr[\omega A] \ , \ \forall A \in \mathcal{A} \tag{4.175}$$

while we can define the scalar product in H_ω as:

$$\langle A|B\rangle = \omega(A^* B) = Tr[B\omega A^*] \tag{4.176}$$

If ω is a rank-1 projector and $\{e_k\}$ is an orthonormal basis for which $\omega = |e_1\rangle\langle e_1|$, writing A_{km} for the matrix elements of A in such a basis, the scalar product assumes the form:

$$\langle A|B\rangle = \sum_{k=1}^{n} \bar{A}_{k1} B_{k1} \tag{4.177}$$

while the Gelfand ideal I_ω is given by:

$$\mathcal{I}_\omega = \{X \in \mathcal{A} \ : \ X_{k1} = 0\, , \, k = 1, \cdots, n\} \tag{4.178}$$

Thus $H_\omega = A/I_\omega$ is nothing but C^n itself and π_ω is the defining representation. If ω is a rank-m density operator: $\omega = p_1|e_1\rangle\langle e_1| + \cdots + p_m|e_m\rangle\langle e_m|$ with $p_1, \cdots p_m > 0$ and $p_1 + \cdots p_m = 1$, the scalar product is given by:

$$\langle A|B\rangle = \sum_{k=1}^{n}\sum_{j=1}^{m} p_m \bar{A}_{kj} B_{kj} \tag{4.179}$$

and the Gelfand ideal is given by:

$$\mathcal{I}_\omega = \{X \in \mathcal{A} \ : \ X_{kj} = 0\, , \, k = 1, \cdots, n; \ j = 1, \cdots, m\} \tag{4.180}$$

showing that H_ω is the direct sum of m copies of C^n. Now the representation π_ω is no longer irreducible, decomposing into the direct sum of m copies of the defining representation:

$$\pi_\omega(A) = \mathbb{I}_m \otimes A \tag{4.181}$$

where $\mathbb{I}_m$ is the $m \times m$ identity matrix.

4.3.2 Geometric structures over a $\mathbb{C}^*$-algebra

Let V be a vector space and V^* its dual. To any element $v \in V$, there is a corresponding element in the bi-dual $\hat{v} \in (V^*)^*$ given by:

$$\hat{v}(\alpha) = \alpha(v) \ , \ \forall \alpha \in V^* \tag{4.182}$$

Thus any multilinear function on V^*, $f : V^* \times \cdots V^* \to \mathbb{R}$ defines, by restricting it to the diagonal, a polynomial function $\tilde{f} \in \mathcal{F}(V^*)$, $\tilde{f}(\alpha) = f(\alpha, ..., \alpha)$, which can be obtained from the "monomials of degree one", $\hat{v} \in (V^*)^*$, on which one has defined the (commutative) product:

$$(\hat{v}_1 \cdot \hat{v}_2)(\alpha) := \hat{v}_1(\alpha)\, \hat{v}_2(\alpha) \tag{4.183}$$

Suppose now that on V there is defined an additional bilinear operation:

$$B : V \times V \to V \tag{4.184}$$

which induces a (in general noncommutative) product $\times_B$ on $V \subset \mathcal{F}(V^*)$ by:

$$\hat{v}_1 \times_B \hat{v}_2 = \widehat{B(v_1, v_2)} \tag{4.185}$$

Then we can define a 2-tensor τ_B in $\mathcal{F}(V^*)$, at the point α, by the relation:

$$\tau_B(d\hat{v}_1, d\hat{v}_2)(\alpha) := \alpha(B(v_1, v_2)) \tag{4.186}$$

which satisfies the Leibniz rule:

$$\tau_B(d\hat{v}, d(\hat{v}_1 \cdot \hat{v}_2)) = \tau_B(d\hat{v}, \hat{v}_1 \cdot d\hat{v}_2 + d\hat{v}_1 \cdot \hat{v}_2) = \hat{v}_1 \cdot \tau_B(d\hat{v}, \hat{v}_2) + \tau_B(d\hat{v}, \hat{v}_1) \cdot \hat{v}_2 \tag{4.187}$$

Thus, $\tau_B(d\hat{v}, \cdot)$ defines a derivation on $V \subset \mathcal{F}(V^*)$ with respect to the commutative product (4.183).

In particular, suppose that B is a skew-symmetric bilinear operation which satisfies the Jacobi identity, so that $g = (V, B)$ is a Lie algebra. The corresponding 2-tensor $\Lambda := \tau_B$:

$$\Lambda(d\hat{v}_1, d\hat{v}_2) = \widehat{B(v_1, v_2)} \tag{4.188}$$

is a Poisson tensor in $\mathcal{F}(V^*)$ and $\Lambda(d\hat{v}, \cdot)$ is a derivation with respect to the commutative product (4.183). Moreover, $\Lambda(d\hat{v}, \cdot)$ is a derivation also with respect to the product (4.185). Indeed, by using the fact that B is antisymmetric and satisfies the Jacobi identity, one has:

$$\begin{aligned} \Lambda(d\hat{v}, d(\hat{v}_1 \cdot \hat{v}_2)) &= \widehat{B(v, B(v_1, v_2))} = \\ &= \widehat{B(v_1, B(v, v_2))} + \widehat{B(B(v, v_1), v_2)} = \\ &= \hat{v}_1 \cdot \Lambda(d\hat{v}, d\hat{v}_2) + \Lambda(d\hat{v}, d\hat{v}_1) \cdot \hat{v}_2 \end{aligned} \tag{4.189}$$

Similarly, if on V one has a Jordan product B', the corresponding 2-tensor $G := \tau_{B'}$ is a metric tensor and $G(d\hat{v}, \cdot)$ is a derivation with respect to the commutative product (4.183), but not with respect to the product (4.185).

If now $V = \mathcal{A}$ is a $\mathbb{C}^*$-algebra, where we have defined both a Lie product and a Jordan product as:

$$B(a_1, a_2) := [a_1, a_2] = \frac{1}{2i}(a_1 a_2 - a_2 a_1)\ ,\ \forall a_1, a_2 \in \mathcal{A} \tag{4.190}$$

and a Jordan product

$$B'(a_1, a_2) := a_1 \circ a_2 = \frac{1}{2}(a_1 a_2 + a_2 a_1)\ ,\ \forall a_1, a_2 \in \mathcal{A} \tag{4.191}$$

in $\mathcal{F}(\mathcal{A}^*)$ we have defined both a Poisson tensor Λ and a metric tensor G such that $\Lambda(d\hat{a},\cdot)$ and $G(d\hat{a},\cdot)$ are both derivations with respect to the pointwise commutative product, with the former being also a derivation with respect to the Lie product. It is also not difficult to check that the subalgebra $\mathcal{B} \subset \mathcal{A}$ composed of all real elements, when embedded in $\mathcal{F}(\mathcal{A}^*)$, comes equipped with an antisymmetric and a symmetric product, denoted by $[\cdot,\cdot]$ and $\circ$ respectively, such that:

1. The Leibniz rule is satisfied: $[a, b\circ c] = [a,b]\circ c + b\circ[a,c]$,
2. The Jacobi identity is satisfied: $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$, and
3. The identity: $(a\circ b)\circ c - a\circ(b\circ c) = [[a,c],b]$ holds.

meaning that $(\mathcal{B},[\cdot,\cdot],\circ)$ is a Lie-Jordan algebra [65]Finally, we notice that the Hamiltonian vector fields:

$$X_{\hat{a}} := \Lambda(\cdot, d\hat{a}) = -[\hat{a},\cdot] \tag{4.192}$$

are derivations with respect to the Jordan product, since, by using the properties above:

$$\begin{aligned} X_{\hat{a}}(d(\hat{a}_1\circ\hat{a}_2)) &= -[\hat{a},\hat{a}_1\circ\hat{a}_2] = -[\hat{a},\hat{a}_1]\circ\hat{a}_2 + -\hat{a}_1\circ[\hat{a},\hat{a}_2] \\ &= X_{\hat{a}}(d\hat{a}_1)\circ\hat{a}_2 + \hat{a}_1\circ X_{\hat{a}}(d\hat{a}_2) \end{aligned} \tag{4.193}$$

Let us go back now to the GNS construction and consider first a pure state ω over $\mathcal{A}$, which gives rise to the irreducible representation π_ω in the Hilbert space $\mathcal{H}_\omega$. We have already seen (see Sect. 4.2.5) that self-adjoint operators, that correspond to the real elements of $\mathcal{A}$, may be identified with the dual $u^*(\mathcal{H}_\omega)$ of the Lie algebra $u(\mathcal{H}_\omega)$ of the unitary group $U(\mathcal{H}_\omega)$ and how the momentum map

$$\mu_\omega : \mathcal{H}_\omega \to u^*(\mathcal{H}_\omega)\ ,\ \mu_\omega(\psi) = |\psi\rangle\langle\psi| \tag{4.194}$$

relates the Poisson tensors on $u^*(\mathcal{H}_\omega)$ with those on $\mathcal{H}_\omega$, via the pull-back. We will say that a Poisson map $\Phi : S \to M$, with (S,Ω) a Poisson manifold, is a symplectic realization of a Poisson manifold (M,Λ). When S is a vector space we call Φ a *classical Jordan-Schwinger map* [149]; when S is a Hilbert space, as in the case we are considering, we say it is a Hermitian realization.

We have also seen that the unit sphere in $\mathcal{H}_\omega - \{\mathbf{0}\}$ can be projected onto $u^*(\mathcal{H}_\omega)$ in an equivariant way, in such a way that the Poisson and the Riemann tensor in $\mathcal{P}(\mathcal{H}_\omega)$ are both related to the same tensors defined on $u^*(\mathcal{H}_\omega)$ by using the Lie and the Jordan product that are defined on it. Thus the momentum map provides a symplectic realization, which we call a Kählerian realization where S is the complex projective space.

4.4 Recovering a Hilbert Space out of $\mathbb{R}^{2n}$

Given now $\mathbb{A} \in \mathfrak{gl}(2n,\mathbb{R}) \equiv End(\mathbb{R}^{2n})$, $\mathbb{A} = \left\| A^i{}_j \right\|$ we can make two distinct associations, namely:

i) $\mathfrak{gl}(2n,\mathbb{R}) \to (1,1)$ tensor fields, via:

$$\mathbb{A} \to T_{\mathbb{A}} = A^i{}_j dx^j \otimes \frac{\partial}{\partial x^i} \tag{4.195}$$

The correspondence is an isomorphism of associative algebras, i.e.:

$$T_{\mathbb{A}} \circ T_{\mathbb{B}} = T_{\mathbb{A}\mathbb{B}} \tag{4.196}$$

and $T_{\mathbb{A}}$ is homogeneous of degree zero, i.e.:

$$\mathcal{L}_{\Delta} T_{\mathbb{A}} = 0 \tag{4.197}$$

where Δ is the dilation (Liouville) vector field associated with the linear structure of $\mathbb{R}^{2n}$:

$$\Delta = x^i \frac{\partial}{\partial x^i} \tag{4.198}$$

ii) $\mathfrak{gl}(2n,\mathbb{R}) \to \{\text{linear vector fields}\}$, via:

$$\mathbb{A} \to X_{\mathbb{A}} = A^i{}_j x^j \frac{\partial}{\partial x^i} \tag{4.199}$$

The latter is only a Lie algebra (anti)isomorphism, i.e.:

$$[X_{\mathbb{A}}, X_{\mathbb{B}}] = -X_{[\mathbb{A},\mathbb{B}]} \tag{4.200}$$

$X_{\mathbb{A}}$ is also homogeneous of degree zero:

$$[\Delta, X_{\mathbb{A}}] = 0 \ \forall \mathbb{A} \tag{4.201}$$

i) and *ii*) are connected by:

$$T_{\mathbb{A}}(\Delta) = X_{\mathbb{A}} \tag{4.202}$$

Moreover, for any $\mathbb{A}, \mathbb{B} \in \mathfrak{gl}(2n,\mathbb{R})$:

$$\mathcal{L}_{X_{\mathbb{A}}} T_{\mathbb{B}} = -T_{[\mathbb{A},\mathbb{B}]} \tag{4.203}$$

Remark 39 *Going back to the compatibility condition between, say, g and J, and defining the linear vector field: $X_J = J^i{}_j x^j \left(\partial/\partial x^i\right)$, one checks easily that the compatibility condition $\widetilde{J} \circ g + g \circ J = 0$ is identical to requiring:*

$$\mathcal{L}_{X_J} g = 0 \tag{4.204}$$

This clarifies also why J can be associated with infinitesimal g-orthogonal transformations.

Given now a triple, a *Hermitian structure* on $\mathbb{R}^{2n}$ will be a map:

$$h : \mathbb{R}^{2n} \to \mathbb{R}^2; \quad h(x,y) = (g(x,y), \omega(x,y)) \equiv (g(x,y), g(x,Jy)) \tag{4.205}$$

$\mathbb{R}^{2n}$ can be given a complex vector space structure by defining, for $z = \alpha + i\beta \in \mathbb{C}$:

$$(\alpha + i\beta) \cdot x =: \alpha x + \beta J x \tag{4.206}$$

Remark 40 *Notice that, e.g.,* $g(x, Jx) = 0 \ \forall x$*, i.e.* $x, Jx \in \mathbb{R}^{2n}$ *are orthogonal and hence* $\mathbb{R}$*-linearly independent*[72]*, but they are not linearly independent when linear combinations with complex coefficients are allowed, as:* $Jx =: ix$*. This means that the complex dimension is reduced from* $2n$ *to* n*, and* $\mathbb{R}^{2n} \approx \mathbb{C}^n$ *as a complex vector space. One possible (non-canonical i.e. not unique) way of "mapping"* $\mathbb{R}^{2n}$ *onto* $\mathbb{C}^n$ *is to choose a basis in* $\mathbb{R}^{2n}$*, to pick up* n *vectors* $\left(e^1, ..., e^n\right)$ *of the basis and to construct* $\mathbb{C}^n$ *by taking complex linear combinations thereof with the rule given above (i.e.:* $ze^i =: \alpha e^i + \beta Je^i$*).*

Then, we can write:

$$h(x, y) = g(x, y) + i\omega(x, y) \equiv g(x, y) + ig(x, Jy) \tag{4.207}$$

or:

$$h(x, y) = \omega(Jx, y) + i\omega(x, y) \tag{4.208}$$

and in this way h will be a Hermitian scalar product linear in the first factor and antilinear in the *second* factor[73].

For the alternative descriptions obtained in the previous chapter, we get a new Hermitian scalar product by replacing ω in (4.207) with ω_F.

Let now an admissible triple (g, J, ω) be given on $\mathbb{R}^{2n}$. First of all we can construct the quadratic function:

$$\mathbf{g} =: \frac{1}{2} g(\Delta, \Delta) \tag{4.209}$$

and the associated Hamiltonian vector field Γ via:

$$i_\Gamma \omega = -d\mathbf{g} \tag{4.210}$$

Explicit calculation shows that, with ω and g (admissible and) constant, Γ s forced to be a linear vector field:

$$\Gamma = \Gamma^i{}_j x^j \frac{\partial}{\partial x^i} \tag{4.211}$$

and that:

$$\Gamma^i{}_j = J^i{}_j \tag{4.212}$$

i.e.[74]: $\Gamma = J$, for short. This can be written in coordinate-free language as:

$$\Gamma = J(\Delta) \quad \text{and: } \Delta = -J(\Gamma) \tag{4.213}$$

Notice that Γ is symplectic:

$$\mathcal{L}_\Gamma \omega = 0 \tag{4.214}$$

[72]Indeed, if $x \neq 0$ and $\alpha x + \beta Jx = 0$ with $\alpha, \beta \in \mathbb{R}$, then: $0 = g(\alpha x + \beta Jx, \alpha x + \beta Jx) = (\alpha^2 + \beta^2) g(x, x)$, implying $\alpha = \beta = 0$.

[73]Had we been using: $\omega(x, y) = g(Jx, y)$ instead of $\omega(x, y) = g(x, Jy)$ we would have obtained the opposite, which is the most common convention [56, 183] among physicists.

[74]In terms of representative matrices.

with Hamiltonian function $\mathbf{g}$. Therefore:

$$0 = \mathcal{L}_{\Gamma}\mathbf{g} = \frac{1}{2}\left(\mathcal{L}_{\Gamma}g\right)(\Delta, \Delta) + g\left(\Delta, [\Gamma, \Delta]\right) \tag{4.215}$$

But $[\Gamma, \Delta] = 0$, so Γ is also a Killing vector field:

$$\mathcal{L}_{\Gamma}g = 0 \tag{4.216}$$

Thus Γ will preserve both the metric, the symplectic structure and (of course) the complex structure, i.e. all the tensors of the admissible triple. So, there will be two linear vector fields "canonically" associated with every admissible triple, one of them defining the linear structure.

Of course:

$$\mathcal{L}_{\Gamma}h = 0 \tag{4.217}$$

which is a complex condition equivalent to the two real ones: $\mathcal{L}_{\Gamma}g = 0$ and: $\mathcal{L}_{\Gamma}J = 0$. As the linear transformations that leave the Hermitian scalar product unchanged are those of the unitary group on $\mathbb{C}^n$, Γ will be an infinitesimal transformation of this group, and the representative matrix (i.e. J) will belong to its Lie algebra. All the vector fields with this property will be called *quantum systems.* A quantum system will be therefore any linear vector field:

$$X_{\mathbb{A}} = A^{i}{}_{j}x^{j}\frac{\partial}{\partial x^{i}} \tag{4.218}$$

such that:

$$\mathcal{L}_{X_{\mathbb{A}}}h = 0 \tag{4.219}$$

In terms of the defining matrices. The matrix $\mathbb{A}$ belongs then both to the Lie algebra of the orthogonal (g-orthogonal) group and to the Lie algebra of the symplectic group, i.e. Eq.(4.219) splits into the two real conditions:

$$\mathcal{L}_{X_{\mathbb{A}}}g = 0 \text{ and: } \mathcal{L}_{X_{\mathbb{A}}}\omega = 0 \tag{4.220}$$

The intersection of these algebras is the Lie algebra of the unitary group. At the finite level (i.e. by exponentiation) the one-parameter group $\exp\{t\mathbb{A}\}$ will belong to a *real* realization of the unitary group $U(n)$ in $\mathbb{R}^{2n}$. Notice also that the first of Eqs.(4.220) implies, together with Eq.(4.201), that:

$$\mathcal{L}_{X_{\mathbb{A}}}g\left(\Delta, \Delta\right) = 0 \tag{4.221}$$

Example 41 *Consider, e.g., $SU(2)$ in the defining representation, i.e.:*

$$SU(2) \ni U = \begin{vmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{vmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{4.222}$$

(i.e. we are viewing U as a $(1,1)$ tensor). Writing: $U = a + ib$, with a and b real 2×2 matrices, the unitarity condition $U^{\dagger}U = \mathbb{I}$ becomes:

$$\widetilde{a}a + \widetilde{b}b = \mathbb{I}\ ; \quad a\widetilde{b} - b\widetilde{a} = 0 \tag{4.223}$$

(i.e. $a\widetilde{b}$ must be a symmetric matrix). We can realify[75] $\mathbb{C}^2$ *onto* $\mathbb{R}^4$ *as* ($z = x + iy$ *etc.*):

$$z = \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \to x = \begin{vmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{vmatrix} \tag{4.224}$$

and U *as the* 4×4 *real matrix:*

$$G = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} \tag{4.225}$$

Assume for simplicity the metric to be the standard Euclidean metric. Then it can be checked at once that the unitarity condition leads both to:

$$\widetilde{G}G = \mathbb{I} \tag{4.226}$$

and to:

$$\widetilde{G}\mathbb{J}G = \mathbb{J} \tag{4.227}$$

where:

$$\mathbb{J} = \begin{vmatrix} \mathbf{0} & -\mathbb{I} \\ \mathbb{I} & \mathbf{0} \end{vmatrix} \tag{4.228}$$

with $\mathbb{I}$ *the* 2×2 *identity matrix, i.e.* $\mathbb{J}$ *is the realification of the multiplication by the imaginary unit* i *in* $\mathbb{C}^2$. *In this case, as matrices:* $\omega = \mathbb{J}$ *(we stress however that* ω *is a* $(0,2)$ *tensor, while* J *is a* $(1,1)$ *tensor), and one checks easily that:* $h(x,x') = g(x,x') + i\omega(x,x') \Leftrightarrow z\overline{z'}$ *which is the Hermitian scalar product in* $\mathbb{C}^2$ *antilinear in the second factor.* G *provides then also a realization of both* $SO(4)$ *and of* $Sp(4)$, *and hence of:* $SU(2) = SO(4) \cap Sp(4)$. *Explicitly, the vector field associated with* $\mathbb{J}$ *will be:*

$$\Gamma = x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \tag{4.229}$$

This is the dynamical vector field for the $2D$ *harmonic oscillator. In* $\mathbb{C}^2$ *it corresponds of course to:* $\dot{z}_j = iz_j$, $j = 1, 2$.

4.5 Compatible Hermitian structures and Bihamiltonian vector fields

Consider two different Hermitian structures, h_1 and h_2, on $\mathbb{R}^{2n}$, with associated quadratic functions $\mathbf{g}_a(\Delta, \Delta)$ and Hamiltonian vector fields Γ_a ($\Gamma_a = X_{J_a}$), $a = 1, 2$. The two structures will be called *compatible* iff:

$$\mathcal{L}_{\Gamma_1} h_2 = \mathcal{L}_{\Gamma_2} h_1 = 0 \tag{4.230}$$

[75] See, e.g., Ref.[5] Sect.18.

which implies, of course, that the Γ's will be *bi*Hamiltonian. In more detail, this implies: $\mathcal{L}_{\Gamma_1}\omega_2 = \mathcal{L}_{\Gamma_1}g_2 = 0$ as well as: $\mathcal{L}_{\Gamma_1}\mathbf{g}_2 = 0$ (and similarly by interchanging indices).

As already recalled, given a symplectic form ω and/or a metric tensor g and a linear vector field $X_{\mathbb{A}}$, the following statements are equivalent:

$$\mathcal{L}_{X_{\mathbb{A}}}\omega = 0;\ \omega(\mathbb{A}x, y) + \omega(x, \mathbb{A}y) = 0;\ \ \omega\mathbb{A} = \widetilde{(\omega\mathbb{A})} \tag{4.231}$$

as well as:

$$\mathcal{L}_{X_{\mathbb{A}}}g = 0;\ g(\mathbb{A}x, y) + g(x, \mathbb{A}y) = 0;\ \ g\mathbb{A} = -\widetilde{(g\mathbb{A})} \tag{4.232}$$

(remember that ω is skew-symmetric: $\widetilde{\omega} = -\omega$, while g is symmetric: $\widetilde{g} = g$). So, $X_{\mathbb{A}}$ will leave ω invariant iff $\omega\mathbb{A}$ is symmetric[76], and it will leave g invariant iff $g\mathbb{A}$ is skew-symmetric.

Now, as $\mathcal{L}_{\Gamma_1}\omega_2 = 0 = \mathcal{L}_{\Gamma_1}g_2$ and: $i_{\Gamma_2}\omega_2 = -d\mathbf{g}_2$:

$$0 = \mathcal{L}_{\Gamma_1}(i_{\Gamma_2}\omega_2) = \mathcal{L}_{\Gamma_1}\omega_2(\Gamma_2, .) = \omega_2([\Gamma_1, \Gamma_2], .) \tag{4.233}$$

and, as the symplectic forms are non-degenerate:

$$[\Gamma_1, \Gamma_2] = 0 \tag{4.234}$$

which, in view of the fact that: $\Gamma_a = X_{J_a}, a = 1, 2$ implies (and is implied by):

$$[J_1, J_2] = 0 \tag{4.235}$$

Given a symplectic form ω, the Poisson bracket of any two functions f and g is given by:

$$\{f, g\} = \omega(X_g, X_f) \tag{4.236}$$

where X_f and X_g are the Hamiltonian vector fields associated with f and g respectively. Hence, denoting with $\{.,.\}_a$ the Poisson bracket associated with ω_a $(a = 1, 2)$ we have, e.g.:

$$\{\mathbf{g}_1, \mathbf{g}_2\}_2 = \omega_2(\Gamma_2, \Gamma_1) = -d\mathbf{g}_2(\Gamma_1) = -\mathcal{L}_{\Gamma_1}\mathbf{g}_2 = 0 \tag{4.237}$$

and similarly with the other Poisson bracket. All in all:

$$\{\mathbf{g}_1, \mathbf{g}_2\}_1 = \{\mathbf{g}_1, \mathbf{g}_2\}_2 = 0 \tag{4.238}$$

Out of the metric tensors and symplectic structures one can form the $(1, 1)$ tensors:

$$G = g_1^{-1} \circ g_2 \tag{4.239}$$

(not to be confused with the $(2, 0)$ tensor G introduced in Sect.4.2.3) and:

$$T = \omega_1^{-1} \circ \omega_2 \tag{4.240}$$

[76] Compare Ch.3.

In intrinsic terms: $G(X) = g_1^{-1}(g_2(X))$, i.e.:

$$G = G^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}; \quad G^i{}_j = (g_1)^{ik}(g_2)_{kj} \tag{4.241}$$

and similarly for T. The two are not independent, though. Indeed, using: $J_a = (g_a)^{-1} \circ \omega_a$ $(a = 1,2)$ and: $J_a^{-1} = -J_a$:

$$G = -J_1 \circ T \circ J_2 \Leftrightarrow T = -J_1 \circ G \circ J_2 \tag{4.242}$$

Having been built out of invariant tensors, it is clear that: $\mathcal{L}_{\Gamma_a} G = \mathcal{L}_{\Gamma_a} T = 0$. In terms of the defining matrices, this implies (see the previous Section):

$$[G, J_a] = [T, J_a] = 0, \quad a = 1,2 \tag{4.243}$$

Hence: $GT = -J_1 \circ T \circ J_2 \circ T = -T^2 \circ J_1 \circ J_2 = TG$, i.e.:

$$[G, T] = 0 \tag{4.244}$$

By direct calculation, using the representative matrices and the symmetry of the metric tensors, one proves immediately that: $g_1(Gx, y) = g_2(x, y) = g_1(x, Gy)$. Also, by direct calculation: $g_2(Gx, y) = (g_1)^{-1}(g_2(x, .), g_2(y, .)) = g_2(x, Gy)$. Hence, G is *self-adjoint* w.r.t. both metrics:

$$g_a(Gx, y) = g_a(x, Gy), \quad a = 1,2 \tag{4.245}$$

Furthermore, the compatibility condition implies: $\mathcal{L}_{\Gamma_1} \omega_2 = 0$. In terms of the representative matrices, this implies (see above): $\omega_2 J_1 = \widetilde{(\omega_2 J_1)}$. As: $\widetilde{\omega} = -\omega$ and $\widetilde{J_1} = -\omega_1 \circ g_1^{-1}$, we obtain: $\omega_2 \circ g_1^{-1} \circ \omega_1 = \omega_1 \circ g_1^{-1} \circ \omega_2$. This implies: $\left(\omega_1^{-1} \circ \omega_2\right) \circ g_1^{-1} \circ \omega_1 = g_1^{-1} \circ \left(\omega_2 \circ \omega_1^{-1}\right) \circ \omega_1$ or (multiplying on the right by ω_1^{-1} and remembering that: $T = \omega_1^{-1} \circ \omega_2$): $T \circ g_1^{-1} = g_1^{-1} \circ \widetilde{T}$. Remembering the definition of the adjoint of a $(1,1)$ tensor we have then:

$$T = g_1^{-1} \circ \widetilde{T} \circ g_1 \equiv \left(T^\dagger\right)_1 \tag{4.246}$$

i.e., T is self-adjoint w.r.t. the metric g_1. Interchanging indices, one proves that: $\left(T^\dagger\right)_2 = T$ as well. Finally, each J_a $(a = 1,2)$ is *skew*-adjoint w.r.t. the respective metric tensor: $J_a = -\left(J_a^\dagger\right)_a = -g_a^{-1} \circ \widetilde{J_a} \circ g_a$. On top of that we have also, e.g.: $\left(J_1^\dagger\right)_2 = g_2^{-1} \circ \widetilde{J_1} \circ g_2 = -g_2^{-1} \circ g_1 \circ J_1 \circ g_1^{-1} \circ g_2 = -G^{-1} \circ J_1 \circ G = -J_1$, as G and the J's commute. Interchanging indices, one proves a similar result for J_2. All in all:

$$\left(J_a^\dagger\right)_b = -J_a, \quad a, b = 1,2 \tag{4.247}$$

In summary, G, T, J_1 *and* J_2 *are a set of mutually commuting operators.* G *and* T *are self-adjoint, while* J_1 *and* J_2 *are skew-adjoint w.r.t. both metric tensors.*

G being self-adjoint, one can proceed to diagonalize it, and $\mathbb{V} = \mathbb{R}^{2n}$ will split into an orthogonal sum[77] of eigenspaces: $\mathbb{V} = \bigoplus_{k=1,\ldots,r} \mathbb{V}_k$ where: $G|_{\mathbb{V}_k} = \lambda_k \mathbb{I}_k$

[77] The sum will be orthogonal w.r.t. *both* metrics.

and the λ_k's ($k = 1, ..., r \leq 2n$) are the distinct eigenvalues of G, and $\lambda_k > 0$. Notice that, as: $G = g_1^{-1} \circ g_2$, this implies:

$$g_2|_{\mathbb{V}_k} = \lambda_k g_1|_{\mathbb{V}_k} \tag{4.248}$$

T commutes with G and is self-adjoint as well. Then $\mathbb{V}_k$ will decompose further into the (bi)orthogonal sum:

$$\mathbb{V}_k = \bigoplus_{\alpha} \mathbb{W}_{k,\alpha} \tag{4.249}$$

where, denoting as $\mu_{k,\alpha}$ the distinct eigenvalues of T in $\mathbb{V}_k$ (labeled by the index α), $\mathbb{W}_{k,\alpha}$ will be the eigenspace of the eigenvalue $\mu_{k,\alpha}$. Once again: $T|_{\mathbb{W}_{k,\alpha}} = \mu_{k,\alpha}\mathbb{I}_{k,\alpha}$, and hence:

$$\omega_2|_{\mathbb{W}_{k,\alpha}} = \mu_{k,\alpha}\omega_1|_{\mathbb{W}_{k,\alpha}} \tag{4.250}$$

Notice that, neither symplectic form being degenerate by assumption, each $\mathbb{W}_{k,\alpha}$ will be necessarily even-dimensional. The dimension of each $\mathbb{W}_{k,\alpha}$ will be then at least two.

The complex structures J_1 and J_2 commute with both G and T. So, they will leave the subspaces $\mathbb{W}_{k,\alpha}$ invariant. Reconstructing them from the g's and ω's we find:

$$J_2|_{\mathbb{W}_{k,\alpha}} = \frac{\mu_{k,\alpha}}{\lambda_k} J_1|_{\mathbb{W}_{k,\alpha}} \tag{4.251}$$

and, as: $J_1^2 = J_2^2 = -\mathbb{I}$: $(\mu_{k,\alpha}/\lambda_k)^2 = 1$, i.e.: $\mu_{k,\alpha} = \pm\lambda_k$, implying:

$$J_2|_{\mathbb{W}_{k,\alpha}} = \pm J_1|_{\mathbb{W}_{k,\alpha}} \tag{4.252}$$

Therefore, the index α can assume only *at most* two values, corresponding to $\pm\lambda_k$, i.e.: $\mathbb{V}_k = \bigoplus_{\alpha=\pm} \mathbb{W}_{k,\alpha}$ *at most*, with $\mathbb{W}_{k,\pm}$ corresponding to the eigenvalues $\pm\lambda_k$ respectively. The dimension of each eigenspace $\mathbb{V}_k$ will be then at least two if only one of the possible eigenvalues $\pm\lambda_k$ of T is present, at least four if both are present. Hence, the maximum number of distinct eigenvalues of G will be $r \leq n$.

In general, a $(0, 2)$ and a $(2, 0)$ tensors (such as, say, g_2 and g_1^{-1}) can be composed to yield a $(1, 1)$ tensor. They will be said to be *"in a generic position"* iff the resulting $(1, 1)$ tensor has eigenvalues of minimum degeneracy. In the present context, we will say that h_1 and h_2 are in a generic position iff the eigenvalues of both G and T have minimum degeneracy, which means *double* degeneracy. Then: $r = n$ and we will have the (bi)orthogonal decomposition:

$$\mathbb{V} = \bigoplus_{k=1,\ldots,n} \mathbb{E}_k \tag{4.253}$$

where: $\dim \mathbb{E}_k = 2$ and either $\mathbb{E}_k = \mathbb{W}_{k,+}$ or $\mathbb{E}_k = \mathbb{W}_{k,-}$ (only one can be present but *not* both, otherwise λ_k would be fourfold degenerate). One can choose in E_k a g_1-orthogonal basis (e_1, e_2) in such a way that:

$$g_1|_{E_k} = e_1^* \otimes e_1^* + e_2^* \otimes e_2^* \tag{4.254}$$

the e^*'s being the dual basis: $e_i^*(e_j) = \delta_{ij}$. Then the condition: $g_1(x, J_1 y) + g_1(J_1 x, y) = 0$ will imply:

$$J_1|_{E_k} = e_2 \otimes e_1^* - e_1 \otimes e_2^* \tag{4.255}$$

or the opposite (i.e.: $J_1 e_1 = e_2, J_1 e_2 = -e_1$), and hence that:

$$\omega_1|_{E_k} = e_1^* \wedge e_2^* \tag{4.256}$$

Correspondingly, we will have:

$$g_2|_{E_k} = \lambda_k g_1|_{E_k}; \quad J_2|_{E_k} = \pm J_1|_{E_k}; \quad \omega_2|_{E_k} = \pm\lambda_k \omega_1|_{E_k} \tag{4.257}$$

Coming now to the general problem of bihamiltonian fields, every linear vector field Γ preserving both h_1 and h_2 will have a representative matrix commuting with those of G and T. Therefore, it will be block-diagonal in the common eigenspaces of both tensors. In the generic (linear) case, the analysis can be restricted to the two-dimensional eigenspaces E_k. On each one of these Γ will preserve both a symplectic structure and a positive-definite metric. Therefore it will be in $sp(2) \cap so(2) = u(1)$ and it will represent a harmonic oscillator, with a frequency possibly depending on E_k.

Using, say, Γ_1 and T, one can construct the n vectors: $\Gamma_{k+1} = T^k \Gamma_1$, $k = 0, 1, ..., n-1$. First of all one sees immediately, by looking at the representative matrices, that, as that of Γ_1 is J_1, which commutes with T, the Γ_k's will commute pairwise, i.e.:

$$[\Gamma_r, \Gamma_s] = 0 \ \forall r, s = 1, 2, ..., n \tag{4.258}$$

Moreover, we have shown that T can be brought into the diagonal form:

$$T = \bigoplus_{k=1,\dots,n} \rho_k \mathbb{I}_k \tag{4.259}$$

with $\rho_k = \pm\lambda_k$ and $\rho_k \neq \rho_r$ for $k \neq r$. If the Γ's were linearly dependent, there would exist a linear combination such that:

$$\sum_{r=0}^{n-1} \alpha_r T^r = 0 \tag{4.260}$$

But on each E_k this would reduce to:

$$\sum_{r=0}^{n-1} \alpha_r (\rho_k)^r = 0, \quad k = 1, ...n \tag{4.261}$$

The determinant of the coefficients of this system of linear equations being the Vandermonde determinant of the ρ's, it will be nonzero, and hence the α's must all vanish, which proves that the Γ's are linearly independent, and hence a basis. As T is a constant tensor, its Nijenhuis torsion vanishes identically. Therefore, as discussed in Sect.B, T is a *strong recursion operator*.$\blacksquare$

What has been proved up to now is the following. Given two admissible triples: (g_1, ω_1, J_1) and (g_2, ω_2, J_2), on $V \approx \mathbb{R}^{2n}$, each triple defines a $2n$-dimensional real representation $U_r(2n, g_a, \omega_a)$, $a = 1, 2$, of the group that leaves simultaneously invariant both g_a and ω_a (and hence J_a), i.e. of the unitary group. The intersection:

$$W_r = U_r(2n, g_1, \omega_1) \cap U_r(2n, g_2, \omega_2) \tag{4.262}$$

will be the common invariance group of both triples. As shown in a $2D$ example in Ref.[160] and as emerges from the previous analysis, the compatibility condition implies that W_r does not reduce to the identity alone. Any "quantum" bihamiltonian (linear) vector field Γ, i.e. a field such that: $\mathcal{L}_\Gamma \omega_a = 0$ *and* $\mathcal{L}_\Gamma g_a = 0$ will be in the Lie algebra of W_r. In the generic case:

$$W_r = \underbrace{SO(2) \times SO(2) \times ... \times SO(2)}_{n \quad times} \tag{4.263}$$

otherwise:

$$W_r = U_r(2r_1; g, \omega) \times U_r(2r_2; g, \omega) \times ... \times U_r(2r_k; g, \omega) \tag{4.264}$$

where (g, ω) is any one of the pairs (g_a, ω_a) and: $r_1 + ... + r_k = n$. Quite a similar analysis can be done by complexifying V in two different ways using the two complex structures and reasoning in terms of the two Hermitian structures. In the generic case, then:

$$W_r = \underbrace{U(1) \times U(1) \times ... \times U(1)}_{n \quad times} \tag{4.265}$$

For further details, see Ref.[160].

To end this Section, we will like to rephrase the previous results in a way more suitable to be generalized to the infinite dimensional case.

We first notice that, going back to the original complex n-dimensional Hilbert space $\mathbb{H}$, there exist two positive constants α and β, such that:

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \ , \ \forall x \in \mathbb{H} \tag{4.266}$$

This implies, by Riesz's theorem [112, 192, 204], that there exists a bounded[78] positive and self-adjoint operator F such that:

$$h_2(x, y) = h_1(Fx, y) \ , \ \forall x, y \in \mathbb{H} \tag{4.267}$$

[78]With respect to both Hermitian structures.

Formally ($h_a = g_a + i\omega_a$, $a = 1, 2$):

$$F = h_1^{-1} \circ h_2 \tag{4.268}$$

and F replaces the previous G and T.

Then [169, 171] a necessary and sufficient condition for h_1 and h_2 to be in generic position is that F be a cyclic operator, i.e. that there exists a vector x_0 such that the vectors $x_0, Fx_0, \cdots, F^{n-1}x_0$ span the whole Hilbert space. Indeed, when h_1 and h_2 are in generic position, F has n distinct eigenvalues, λ_k. If we now denote with $\{f_k\}$ its eigenvector basis and with $\{\mu^{(k)}\}$ a set of n nonzero complex numbers, we can construct the vectors

$$F^m x_0 = \sum_k \mu^{(k)} \lambda_k^m f_k \ , \ m = 0, 1, \cdots, n-1. \tag{4.269}$$

They are linear independent because the determinant of their components is given by $(\prod_k \mu^{(k)}) V(\lambda_1, \cdots, \lambda_n)$, where the Vandermonde determinant V is nonzero, the eigenvalues λ_k's being distinct. Clearly, the converse is also true.

Also, it has been argued in Ref.[160], that "bi-unitary" operators, i.e. operators that are unitary w.r.t. both Hermitian structures[79], must commute with F (the proof is simple and we refer to the above reference for it), i.e. bi-unitary operators are in the commutant F' of F[80].

The results of this discussion can be summarized in the following:

Proposition 42 *Two Hermitian forms are in a generic position iff the bicommutant of F coincides with the commutant: $F'' = F'$.*

It should be clear from our presentation that many results will carry over to the infinite-dimensional case, although new problems may arise because the algebraic properties do not "control" properties such as continuity and differentiability in infinite dimensions.

4.6 The infinite-dimensional case

In the (genuinely) infinite-dimensional case of a Hilbert space $\mathbb{H}$ there arise two difficulties, namely:
i) Given two Hermitian structures, $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ on $\mathbb{H}$ defining two complex scalar products (both linear in, say, the second factor and antilinear in the first, but this is not a crucial point), they might define two non-equivalent topologies on $\mathbb{H}$, and:

[79] Of course, any linear vector field that leaves both h's invariant will generate a one-parameter group of bi-unitary transformations.

[80] The commutant F' of F is the set of all operators that commute with F. It is of course closed under commutation because of the Jacobi identity, i.e. it is a Lie algebra. The *bi-commutant* F'' is the set of all operators that commute with all those in the commutant. In particular, they will commute with F itself, and hence: $F'' \subset F'$. Moreover, any two operators in F'' must commute among themselves. F'' is therefore a (maximal) Abelian subalgebra of F', i.e. F'' is the *center* of F'

ii) The spectra of self-adjoint operators may have both a point part and a continuum part.

Point i) is taken care of in an almost standard way, assuming that there exist two positive constants α and β, such that formula (4.266) holds. It follows that we can define the operator F as in (4.267). But now, due to point *ii*), we have to better specify what we mean, for example, by requiring F to have nondegenerate eigenvalues. On the other side, the definitions of the commutant and the bicommutant of F are of purely algebraic character and can therefore be generalized to the infinite dimensional situation. Then, following Refs. [169] and [171], we will adopt the following definition:

Definition. *Two Hermitian structures h_1 and h_2 are said to be in generic position iff $F'' = F'$, F being their connecting operator.*

To proceed further in understanding the situation in which F has also a continuous spectrum, one needs suitable mathematical tools such as the spectral theory and the theory of rings of operators in Hilbert spaces [192]. We first observe that F' and $F'' \subset F'$ are both (weakly closed) rings of bounded operators on $\mathbb{H}$. Now, given any set $S \in \mathcal{B}(\mathbb{H})$, it can be proved [192] that the minimal weakly closed ring $R(S)$ containing S contains only those elements $A \in S''$ such that

$$E_0 A = A E_0 = A \tag{4.270}$$

where E_0 is the so called principal identity of the set S, i.e. the projection operator on $(ker S \cap ker S^\dagger)^\perp$. If $S = \{F\}$, F being self-adjoint and positive, we have that $\mathbb{I} \in R(F)$ and $R(F) = F''$, which is therefore commutative.

If we decompose now F in terms of its spectral family $\{P(\lambda)\}$:

$$F = \int_\Delta \lambda \, dP(\lambda) \tag{4.271}$$

where $\Delta = [a, b]$ is a closed interval containing the spectrum of F, it is possible to show that:
a) The weakly closed commutative ring $R(F)$ corresponds to a decomposition of the Hilbert space $\mathbb{H}$ into the direct integral

$$\mathbb{H} = \int_\Delta H_\lambda \, d\sigma(\lambda) \tag{4.272}$$

where the measure $\sigma(\lambda)$ is obtained from the spectral family $\{P(\lambda)\}$ of F.
b) Any operator $A \in F'$ can be represented as

$$A = \int_\Delta A(\lambda) \, d\sigma(\lambda) \tag{4.273}$$

where $A(\lambda)$ is a bounded operator on H_λ, for almost all λ.
c) Every $B \in F'' = R(F)$ is a multiplication by a number $b(\lambda)$ on H_λ, for almost all λ.
Moreover, since $R(F)$ is a maximal commutative ring by itself, the family $F'(\lambda)$

of all operators $A(\lambda)$ corresponding to F', for a fixed λ, is irreducible so that we can rewrite $a, b)$ above as:

a') The spectrum Δ of F is the union of a countable number of measurable sets Δ_k such that, for $\lambda \in \Delta_k$, the spaces H_λ have the same dimension n_k (finite or infinite) and:

$$\mathbb{H} = \bigoplus_k \int_{\Delta_k} H_\lambda \, d\sigma(\lambda) \tag{4.274}$$

b') Any $A \in F'$ can be written as

$$A = \bigoplus_k \int_{\Delta_k} A(\lambda) \, d\sigma(\lambda) \tag{4.275}$$

Now, going back to the two Hermitian structures h_1 and h_2 on $\mathbb{H}$, since the connecting operator F acts on each H_λ as a multiplication by the number λ, we can easily derive the following result generalizing the finite-dimensional situation.

Proposition 43 *There exists a decomposition of $\mathbb{H}$ as direct integral of Hilbert spaces H_λ, of dimension n_k such that in each H_λ: $h_2 = \lambda h_1$.*

It follows that the elements of the unitary group that leave simultaneously invariant h_1 and h_2 have the form (see Eq.(4.5.3)):

$$U = \bigoplus_k \int_{\Delta_k} U_k(\lambda) \, d\sigma(\lambda) \tag{4.276}$$

where $U_k(\lambda)$ is an element of the unitary group $U(n_k)$, for each $\lambda \in \Delta_k$.

Also, it is now immediate to prove that definition (1) is equivalent to:

Definition. *Two Hermitian structures h_1 and h_2 are said to be in generic position iff the spaces H_λ are one-dimensional.*

Indeed, if h_1 and h_2 are in generic position, then $R(F) = F'' = F'$, so that the latter is commutative and $A(\lambda)$, for almost all $\lambda \in \Delta$, acts on a one-dimensional Hilbert space H_λ. Conversely, if $R(F) = F'' \neq F'$, F' is non-commutative and hence there is a subset $\Delta_0 \subset \Delta$ such that H_λ has dimension greater than one for $\lambda \in \Delta_0$. ■

Notice also that, in the generic case, the operators $U_k(\lambda)$ in (4.276) are one-dimensional and reduces to a multiplication by a phase factor $\exp[i\theta(\lambda)]$.

Finally, we may prove the following equivalence between the genericity condition and the cyclicity of the operator F:

Definition. *F is cyclic iff $F'' = F'$.*

This follows from the fact that, if $F'' = F'$, the latter is commutative and each space $H(\lambda)$, where F acts as a multiplication by λ, is one-dimensional. So the vector $x_0 = 1/\lambda$ is a cyclic vector. Viceversa, if we suppose now that F is cyclic, each $H(\lambda)$ is one-dimensional and any $A \in F'$ acts as a multiplication by a number. Hence $F' = F'' = R(F)$. ■

Example 44 ***A particle in a box.*** *We consider the operator $F = 1 + X^2$ where X is the position operator which acts as multiplication by x on the Hilbert space $L^2([-\alpha, \alpha], dx)$. From the spectrum $\Delta_X = [-\alpha, \alpha]$ and the spectral family $\{P_X(\lambda) = \chi_{[-\alpha,\lambda]}\}$ of X ($\chi_{[-\alpha,\lambda]}$ being the characteristic function on $[-\alpha, \lambda]$), one easily sees that the spectrum of F is $\Delta_F = [1, 1+\alpha^2]$ while its spectral family $\{P_F(\lambda)\}$ is given by*

$$P_F(\lambda) = P(\sqrt{\lambda - 1}) - P(-\sqrt{\lambda - 1}) \tag{4.277}$$

In fact, t is easy to check that:

$$P_F^2 = P_F; \;\; P_F\,(1) = 0; \;\; P_F\left(1 + \alpha^2\right) = \mathbb{I} \tag{4.278}$$

We can write F as:

$$F = \int_{[-\alpha,\alpha]} (1 + \lambda^2)\, dP(\lambda) \tag{4.279}$$

If we now divide the interval as $[-\alpha, \alpha] = [-\alpha, 0] \cup [0, \alpha]$ and change variable by setting $\lambda = -\sqrt{\mu - 1}$ or $\lambda = \sqrt{\mu - 1}$ in the negative or positive parts of the interval respectively, we get:

$$F = \int_{[1,1+\alpha^2]} \lambda\, dP_F(\lambda) \tag{4.280}$$

Now F has no cyclic vector on the whole $L^2([-\alpha, \alpha])$ since G', which contains both X and the parity operator is not commutative. On the contrary, $\chi_{[-\alpha,0]}$ is cyclic on $L^2([-\alpha, 0])$ and, similarly, $\chi_{[0,\alpha]}$ is so on $L^2([0, \alpha])$. Thus the Hilbert space splits in two F-cyclic spaces: $L^2([-\alpha, \alpha]) = L^2([-\alpha, 0]) \oplus L^2([0, \alpha])$ and we obtain the decomposition

$$\mathbb{H} = \int_{[1,1+\alpha^2]} H_\lambda\, d\sigma(\lambda) \tag{4.281}$$

where the measure is obtained from:

$$\sigma(\lambda) = P_F(\lambda)\chi_{[-\alpha,0]} = P_F(\lambda)\chi_{[0,\alpha]} = \sqrt{\lambda - 1} \tag{4.282}$$

Notice that the spaces H_λ are one-dimensional if we work in the interval $[0, \alpha]$ or bidimensional if we consider $[-\alpha, \alpha]$. Also, the bi-unitary transformations read, respectively, as:

$$U = \int_{[1,1+\alpha^2]} e^{i\phi(\lambda)}\, d\sigma(\lambda) \tag{4.283}$$

$$U = \int_{[1,1+\alpha^2]} U_2(\lambda)\, d\sigma(\lambda) \tag{4.284}$$

5 From Finite to Infinite Dimensions. Weyl Systems

5.1 An Abstract Setting for Weyl Systems

A known theorem by A.Wintner [232] states that if, say, $\widehat{q}$ and $\widehat{p}$ are quantum-mechanical operators on an infinite-dimensional Hilbert space satisfying a commutation relation of the form: $[\widehat{q},\widehat{p}] = c\widehat{\mathbb{I}}$ (or, better: $[\widehat{q},\widehat{p}] \subseteq c\widehat{\mathbb{I}}$), with c a constant and $\widehat{\mathbb{I}}$ the identity operator, then at least one of them must be unbounded.

Motivated then by the need of formulating Quantum Mechanics without having to do with unbounded operators, it was apparently H.Weyl [225] (see also [218]) who proposed first a different scheme of quantization that goes as follows:

Let $\mathcal{S}$ be a (real) linear vector space endowed with a constant[81] symplectic structure[82] ω. Weyl's approach consists in the following:

- It is a map W from $\mathcal{S}$ to the set of *unitary operators* on a (so far unspecified[83]) Hilbert space $\mathcal{H}$:

$$W : \mathcal{S} \to \mathcal{U}(\mathcal{H}) \tag{5.1}$$

  via:

$$\mathcal{S} \ni z \to \widehat{W}(z) \in \mathcal{U}(\mathcal{H}), \quad \widehat{W}(z)\widehat{W}^{\dagger}(z) = \widehat{W}^{\dagger}(z)\widehat{W}(z) = \widehat{\mathbb{I}} \tag{5.2}$$

  with the following specifications:

- W is a strongly continuous map, and

- For any $z, z' \in \mathcal{S}$:

$$\widehat{W}(z+z') = \widehat{W}(z)\widehat{W}(z')\exp\{-i\omega(z,z')/2\hbar\} \tag{5.3}$$

  with $\hbar$ the reduced Planck constant. It follows then that:

$$\widehat{W}(z)\widehat{W}(z') = \widehat{W}(z')\widehat{W}(z)\exp\{i\omega(z,z')/\hbar\}, \quad \forall z, z' \tag{5.4}$$

Moreover, setting $z' = 0$ in (5.3) we obtain: $\widehat{W}^{-1}(z)\widehat{W}(z) = \widehat{W}(0)$, and hence: $\widehat{W}(0) = \widehat{\mathbb{I}}$, while setting $z' = -z$ we obtain: $\widehat{W}^{-1}(z) = \widehat{W}(-z)$, and hence:

$$\widehat{W}^{\dagger}(z) = \widehat{W}(-z) \tag{5.5}$$

[81] I.e. translationally-invariant.
[82] Hence, necessarily: $\dim(\mathcal{S})$ will be *even*, and: $\mathcal{S} \approx \mathbb{R}^{2n}$ for some n.
[83] That's why the setting we are describing here has been defined as "abstract".

Then, *a Weyl system is a projective unitary representation of the linear vector space $\mathcal{S}$ (thought of as the group manifold of the translation group) in the Hilbert space $\mathcal{H}$.*

As a running example we shall consider $\mathcal{S} = \mathbb{R}^2$ with coordinates (q, p) and the standard symplectic form: $\omega = dq \wedge dp$, which is represented by the matrix:

$$\omega = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{5.6}$$

Hence:

$$\omega\left((q,p),(q',p')\right) = \begin{vmatrix} q & p \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} q' \\ p' \end{vmatrix} = qp' - q'p \tag{5.7}$$

and therefore:

$$\widehat{W}\left((q,p)+(q',p')\right) = \widehat{W}(q,p)\,\widehat{W}(q',p')\exp\left\{-\frac{i}{2\hbar}(qp'-q'p)\right\} \tag{5.8}$$

In the general case, we can decompose $\mathcal{S}$ into the direct sum of two Lagrangian subspaces: $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, and hence any vector z as: $z = (z_1, 0) + (0, z_2)$, $z_1 \in \mathcal{S}_1$, $z_2 \in \mathcal{S}_2$. We can consider then the restrictions of W to the Lagrangian subspaces, i.e.:

$$U = W|_{\mathcal{S}_1} : \mathcal{S}_1 \to \mathcal{H} \tag{5.9}$$

and:

$$V = W|_{\mathcal{S}_2} : \mathcal{S}_2 \to \mathcal{H} \tag{5.10}$$

As: $\omega|_{\mathcal{S}_1} = \omega|_{\mathcal{S}_2} = 0$, U and V are *faithful* representations of the corresponding Lagrangian subspaces:

$$\widehat{U}(z_1 + z_1') = \widehat{U}(z_1)\,\widehat{U}(z_1');\; z_1, z_1' \in \mathcal{S}_1 \tag{5.11}$$

and similarly for V. Moreover:

$$\widehat{U}(z_1)\,\widehat{V}(z_2) = \widehat{V}(z_2)\,\widehat{U}(z_1)\exp\left\{i\omega\left((z_1,0),(0,z_2)\right)/\hbar\right\} \tag{5.12}$$

Viceversa, we have the following:

Proposition: *Given two faithful representations U and V of two transversal Lagrangian subspaces of a symplectic vector space $\mathcal{S}$ satisfying* (5.12), *the map:*

$$z \longrightarrow \widehat{W}(z) = \widehat{U}(z_1)\,\widehat{V}(z_2)\exp\left\{-i\omega\left((z_1,0),(0,z_2)\right)/2\hbar\right\} \tag{5.13}$$

is a Weyl system.

The proof that (5.13) does indeed satisfy the defining property (5.3) can be done by direct calculation, and will be omitted here.■

Consider now a one-dimensional subspace of $\mathcal{H}$ spanned by a fixed vector z. From (5.3) we have, with α, β real numbers:

$$\widehat{W}(\alpha z)\,\widehat{W}(\beta z) = \widehat{W}\left((\alpha+\beta) z\right) \tag{5.14}$$

Therefore, $\left\{\widehat{W}(\alpha z)\right\}_{\alpha\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitaries and, by Stone's theorem [201]:

$$\widehat{W}(\alpha z) = \exp\left\{i\alpha\widehat{G}(z)/\hbar\right\} \tag{5.15}$$

with an infinitesimal generator $\widehat{G}(z)$ which is (essentially) self-adjoint. Furthermore, $\left\{\widehat{W}(\alpha\beta z)\right\}_{\beta\in\mathbb{R}}$ is also a strongly continuous one-parameter group, and therefore:

$$\widehat{W}(\alpha\beta z) = \exp\left\{i\beta\widehat{G}(\alpha z)/\hbar\right\} \tag{5.16}$$

and, setting $\beta = 1$, we find:

$$\widehat{G}(\alpha z) = \alpha\widehat{G}(z) \tag{5.17}$$

In terms of infinitesimal generators and setting: $z \to \alpha z, z' \to \beta z'$, Eq. (5.4) reads:

$$e^{i\alpha\widehat{G}(z)/\hbar}e^{i\beta\widehat{G}(z')/\hbar} = e^{i\alpha\beta\omega(z,z')/\hbar}e^{i\alpha\widehat{G}(z)/\hbar}e^{i\beta\widehat{G}(z')/\hbar} \tag{5.18}$$

and, for α and β infinitesimal, this yields, to the lowest nontrivial order:

$$\left[\widehat{G}(z), \widehat{G}(z')\right] = -i\hbar\omega(z,z') \tag{5.19}$$

5.2 Von Neumann's Representation Theorem

What is lacking in the "abstract" presentation of the previous Section is a concrete realization of the Hilbert space $\mathcal{H}$ on which the mapping W should operate.

Before discussing von Neumann's theorem, let us resume our running example on $\mathbb{R}^2 \approx T^*\mathbb{R}$. Writing (q,p) as: $(q,p) = (q,0)+(0,p)$, whence: $\omega((q,0),(0,p)) = qp$, our Weyl system becomes ($z = (q,p), z_1 = (q,0), z_2 = (0,p)$) (see Eq.(5.8)):

$$\widehat{W}(q,p) = \widehat{W}((q,0)+(0,p)) = \widehat{W}(q,0)\widehat{W}(0,p)\exp\{-iqp/2\hbar\} \tag{5.20}$$

while:

$$\widehat{W}(q+q',0) = \widehat{W}(q,0)\widehat{W}(q',0) \tag{5.21}$$

and similarly for $\widehat{W}(0,p)$. Define then:

$$\widehat{W}(q,0) = \exp\left\{iq\widehat{P}/\hbar\right\};\ \widehat{W}(0,p) = \exp\left\{ip\widehat{Q}/\hbar\right\} \tag{5.22}$$

In other words, as: $(q,0) = q(1,0), (0,p) = p(0,1)$, we are defining:

$$\widehat{G}(0,1) = \widehat{Q}, \quad \widehat{G}(1,0) = \widehat{P} \tag{5.23}$$

with (cfr. Eq. (5.19)):

$$\left[\widehat{Q}, \widehat{P}\right] = i\hbar\mathbb{I} \tag{5.24}$$

Moreover, using the truncated Baker-Campbell-Hausdorff [201] formula[84] one finds easily:

$$\widehat{W}(q,p) = \exp\left\{ i\left(q\widehat{P} + p\widehat{Q}\right)/\hbar \right\} \tag{5.25}$$

Consider now $L_2(\mathbb{R}, dx)$ with the Lebesgue measure, and define the families of operators $\left\{\widehat{U}(q)\right\}_{q\in\mathbb{R}}$ and $\left\{\widehat{V}(p)\right\}_{p\in\mathbb{R}}$ via:

$$\left(\widehat{U}(q)\psi\right)(x) = \psi(x+q) \tag{5.26}$$

and:

$$\left(\widehat{V}(p)\psi\right)(x) = \exp\{ipx/\hbar\}\psi(x) \tag{5.27}$$

for $\psi \in L_2(\mathbb{R}, dx)$. It is easy to show that both families are actually one-parameter, strongly continuous groups of unitaries, and that:

$$\left(\widehat{U}(q)\widehat{V}(p)\psi\right)(x) = \exp\{iqp/\hbar\}\left(\widehat{V}(p)\widehat{U}(q)\psi\right)(x) \tag{5.28}$$

Then:

$$\widehat{W}(q,p) = \widehat{U}(q)\widehat{V}(p)\exp\{-iqp/\hbar\} \tag{5.29}$$

is a *concrete* realization of a Weyl system. Defining again: $\widehat{U}(q) = \exp\left\{iq\widehat{P}/\hbar\right\}$ and: $\widehat{V}(p) = \exp\left\{ip\widehat{Q}/\hbar\right\}$, we find both Eq.(5.25) and, at the infinitesimal level[85]:

$$\left(\widehat{Q}\psi\right)(x) = x\psi(x), \quad \left(\widehat{P}\psi\right)(x) = -i\hbar\frac{d\psi}{dx} \tag{5.30}$$

Moreover:

$$\left(\widehat{W}(q,p)\psi\right)(x) = \exp\{ip[x+q/2]/\hbar\}\psi(x+q) \tag{5.31}$$

A generic matrix element of $\widehat{W}(q,p)$ will be given then by:

$$\left\langle \phi, \widehat{W}(q,p)\psi \right\rangle = \exp\{iqp/2\hbar\}\int_{-\infty}^{+\infty} dx\overline{\phi(x)}\exp\{ipx/\hbar\}\psi(x+q) \tag{5.32}$$

Remark 45 *Viewed as a function on T^*Q, $\left\langle \phi, \widehat{W}(q,p)\psi \right\rangle$ is square-integrable for all $\phi, \psi \in L^2(\mathbb{R})$. Indeed, defining the Lebesgue measure on R^2 as $dqdp/2\pi\hbar$, a direct calculation shows that:*

$$\left\| \left\langle \phi, \widehat{W}(q,p)\psi \right\rangle \right\|^2 =: \iint \frac{dqdp}{2\pi\hbar}\left| \left\langle \phi, \widehat{W}(q,p)\psi \right\rangle \right|^2 = \|\phi\|^2\|\psi\|^2 \tag{5.33}$$

[84] $e^{a+b} = e^a e^b e^{-[a,b]/2}$ whenever: $[a,[a,b]] = [b,[a,b]] = 0$.

[85] And in the appropriate domains.

Instead, for plane-wave states:

$$\phi(x) = (1/\sqrt{2\pi})\exp(ik'x), \psi(x) = (1/\sqrt{2\pi})\exp(ikx) \tag{5.34}$$

and denoting as $\left\langle k'|\widehat{W}(q,p)|k\right\rangle$ the matrix elements of $\widehat{W}(q,p)$ between these states, we obtain:

$$\left\langle k'|\widehat{W}(q,p)|k\right\rangle = \delta(k - k' + p/\hbar)\exp(iq(k+k')/2) \tag{5.35}$$

and, in particular:

$$\left\langle k|\widehat{W}(q,p)|k\right\rangle = \hbar\delta(p)\exp\{ikq\} \tag{5.36}$$

Integrating Eq.(5.36) over k, we obtain for the *trace* of W[86]:

$$Tr\left\{\widehat{W}(q,p)\right\} = 2\pi\hbar\delta(q)\delta(p) \tag{5.37}$$

Coming now to the general case, let's assume that we are given a symplectic vector space $(\mathcal{S},\omega)$ and a decomposition of $\mathcal{S}$ as the direct sum:

$$\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2 \tag{5.38}$$

with $\mathcal{S}_1$ and $\mathcal{S}_2$ Lagrangian subspaces. Every vector $z \in \mathcal{S}$ can then be decomposed in a unique way as: $z = (z_1,0) + (0,z_2), z_i \in \mathcal{S}_i, i = 1,2$. Let us remark first of all that the symplectic structure allows each one of the two subspaces to be identified with the dual of the other. Indeed, we can define a pairing:

$$\langle .,. \rangle : \mathcal{S}_2 \times \mathcal{S}_1 \to \mathbb{R} \tag{5.39}$$

via:

$$\langle z_2, z_1\rangle : \omega((z_1,0),(0,z_2)) \tag{5.40}$$

The details of the proof that in this way $\mathcal{S}_2 \approx \mathcal{S}_1^*$ (and viceversa, of course) can be found in Ref. [135].

Assume now $\mathcal{H}$ to be a separable Hilbert space and let:

$$\begin{array}{c} U : \mathcal{S}_1 \to \mathcal{H} \\ V : \mathcal{S}_2 \to \mathcal{H} \end{array} \tag{5.41}$$

be *unitary, irreducible and strongly continuous* representations of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively on $\mathcal{H}$, satisfying the additional condition that defines the *"Weyl form"* of the commutation relations:

$$\widehat{U}(z_1)\widehat{V}(z_2) = \widehat{V}(z_2)\widehat{U}(z_1)\exp\{i\omega((z_1,0),(0,z_2))/\hbar\} \tag{5.42}$$

[86] Actually, we can define the trace only if we admit distribution-valued traces. Strictly speaking [59], and as Eq.(5.33) shows, $\widehat{W}$ is bounded but *not* trace-class.

Then we can define:

$$\widehat{W}(z) = \widehat{U}(z_1)\,\widehat{V}(z_2)\exp\{-i\omega((z_1,0),(0,z_2))/2\hbar\} \tag{5.43}$$

which is a Weyl system. Let us denote z_1 and z_2 as $(q,0)$ and $(0,p)$ respectively, with q and p n-dimensional vectors ($n = \dim S_1 = \dim S_2$). Correspondingly, we will denote $\widehat{U}(z_1)$ and $\widehat{V}(z_2)$ as $\widehat{U}(q)$ and $\widehat{V}(p)$ respectively.

Von Neumann's theorem [223] states then that there exists a unitary map:

$$T : \mathcal{H} \to \mathcal{L}_2(\mathbb{R}^n, d\mu) \tag{5.44}$$

such that:

$$\left(T\widehat{U}(q)\,T^{-1}\psi\right)(x) = \psi(x+q) \tag{5.45}$$

and (cfr.Eqn.(5.40)):

$$\left(T\widehat{V}(p)\,T^{-1}\psi\right)(x) = e^{i\langle x,p\rangle}\psi(x) \tag{5.46}$$

This theorem proves that all the representations of the Weyl commutation relations are unitarily equivalent to the Schrödinger representation, and hence are unitarily equivalent among themselves (but see below, Sect.7.3.1).

Example 46 *In the case of $L_2(\mathbb{R})$, setting $\hbar = 1$ and using the Fourier transform:*

$$\psi(x) = \int\limits_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}}\widetilde{\psi}(p)\exp\{ipx\} \tag{5.47}$$

one finds easily that:

$$\left(\widetilde{\exp\left(ix\widehat{P}\right)}\,\psi\right)(p) = e^{ixp}\widetilde{\psi}(p) \tag{5.48}$$

(i.e.: $\left(\widehat{P}\widetilde{\psi}\right)(p) = p\widetilde{\psi}(p)$, and:

$$\left(\widetilde{\exp\left(i\pi\widehat{Q}\right)}\,\psi\right)(p) = \widetilde{\psi}(p-\pi) \tag{5.49}$$

($\left(\widehat{Q}\widetilde{\psi}\right)(p) = id\widetilde{\psi}(p)/dp$). Denoting by:

$$\mathcal{F} : \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R}) \tag{5.50}$$

the unitary operator defined by the Fourier transform, we can conclude that:

$$\mathcal{F}^{\dagger}\widehat{Q}\mathcal{F} = -\widehat{P} \tag{5.51}$$

and:

$$\mathcal{F}^{\dagger}\widehat{P}\mathcal{F} = \widehat{Q} \tag{5.52}$$

5.3 Weyl Systems and Linear Transformations

Let's begin by considering linear transformations that preserve the symplectic structure, i.e. linear maps: $T : S \to S$ such that:

$$\omega\left(Tz, Tz'\right) = \omega\left(z, z'\right) \forall z, z' \in S \tag{5.53}$$

In terms of matrices this means:

$$\widetilde{T}\omega T = \omega \tag{5.54}$$

(where $\widetilde{T}$ stands for the transpose of the matrix T), and this defines a realization of the symplectic group $Sp\left(2n, \mathbb{R}\right)$ associated with the symplectic structure ω.

Then, we can define:

$$\widehat{W_T} : \mathcal{S} \to \mathcal{H} \tag{5.55}$$

via:

$$\widehat{W_T}\left(z\right) =: \widehat{W}\left(Tz\right) \tag{5.56}$$

and, as:

$$\begin{aligned} \widehat{W}\left(T\left(z+z'\right)\right) &= \widehat{W}\left(Tz\right)\widehat{W}\left(Tz'\right)\exp\left\{-i\omega\left(Tz, Tz'\right)/2\hbar\right\} = \\ &= \widehat{W}\left(Tz\right)\widehat{W}\left(Tz'\right)\exp\left\{-i\omega\left(z, z'\right)/2\hbar\right\} \end{aligned} \tag{5.57}$$

we find: :

$$\widehat{W_T}\left(z+z'\right) = \widehat{W_T}\left(z\right)\widehat{W_T}\left(z'\right)\exp\left\{-i\omega\left(z, z'\right)/2\hbar\right\} \tag{5.58}$$

i.e. $\widehat{W_T}$ is also a Weyl system, and hence, by von Neumann's theorem, it is unitarily equivalent to $\widehat{W}$.

As a simple example, consider, in $\mathbb{R}^2$, the map:

$$\left(q, p\right) \to \left(-p, q\right) \tag{5.59}$$

which is realized via the transformation[87]:

$$T = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \tag{5.60}$$

Then it is clear that:

$$\widehat{U}\left(q\right) = \widehat{W}\left(\left(q, 0\right)\right) \to \widehat{W}\left(\left(0, -p\right)\right) = \widehat{V}\left(-p\right) \tag{5.61}$$

and:

$$\widehat{V}\left(p\right) = \widehat{W}\left(\left(0, p\right)\right) \to \widehat{W}\left(\left(q, 0\right)\right) = \widehat{U}\left(q\right) \tag{5.62}$$

which is precisely (see the end of the previous Section) what the Fourier transform does.

[87] The *matrix* representing T is simply minus that of the complex structure. However, the two have different transformation properties (see Chapt.1).

As $\widehat{W}_T$ is unitarily equivalent to $\widehat{W}$, to the map T there is associated an automorphism of the group $\mathcal{U}(\mathcal{H})$ of the unitary operators. As every automorphism of $\mathcal{U}(\mathcal{H})$ is inner, there is a unitary operator $\widehat{U}_T$ such that:

$$\widehat{W}_T(z) = \widehat{U}_T^\dagger \left(\widehat{W}(z) \right) \widehat{U}_T \tag{5.63}$$

More generally, we can consider a one-parameter group $\{T_\lambda\}_{\lambda \in \mathbb{R}}$ of linear symplectic transformations. Calling Γ the linear vector field that is the infinitesimal generator of the group, the condition:

$$\omega(T_\lambda z, T_\lambda z') = \omega(z, z') \, \forall z, z' \in S, \forall \lambda \in \mathbb{R} \tag{5.64}$$

becomes:

$$L_\Gamma \omega = 0 \tag{5.65}$$

with L_Γ the Lie derivative. There exists then (globally on a vector space) a function g such that:

$$i_\Gamma \omega = dg \tag{5.66}$$

and, for linear transformations, g will be a quadratic function of the coordinates.

According to what has been said above, the family $\{T_\lambda\}$ defines a (strongly continuous) one-parameter group $\{U_\lambda\}_{\lambda \in \mathbb{R}}$ of unitary operators such that:

$$\widehat{W}(z(\lambda)) = \widehat{U}_\lambda^\dagger \widehat{W}(z) \widehat{U}_\lambda \tag{5.67}$$

where: $z(\lambda) = T_\lambda(z)$. By Stone's theorem, then:

$$\widehat{U}_\lambda = \exp\left\{ -i\lambda \widehat{G}/\hbar \right\} \tag{5.68}$$

with $\widehat{G}$ self-adjoint. *The self-adjoint operator $\widehat{G}$ is the quantum counterpart of the quadratic function g.* In this way we have achieved a way to quantize all the quadratic functions: given G, we can define via Eq.(5.66) the associated Hamiltonian vector field. This in turns defines a one-parameter group of (linear) symplectic transformations, and the corresponding Weyl system allows us to find the (self-adjoint) quantum operator to be associated with g.

Let's consider now a general linear transformation $T \in GL(2n, \mathbb{R})$, not necessarily a symplectic one. We will denote for clarity as ω_0 a reference (comparison) symplectic structure, written in a Darboux chart as:

$$\omega_0 = \begin{vmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{vmatrix} \tag{5.69}$$

We define then a new symplectic structure ω_T via:

$$\omega_T(z, z') =: \omega_0(Tz, Tz') \tag{5.70}$$

That ω_T is a symplectic structure is obvious. It is represented by the matrix:

$$\omega_T = \widetilde{T} \omega_0 T \tag{5.71}$$

Now, if we define again:

$$\widehat{W}_T(z) =: \widehat{W}(Tz) \tag{5.72}$$

it is easy to prove that:

$$\widehat{W}_T(z+z') = \widehat{W}_T(z)\,\widehat{W}_T(z')\exp\{-i\omega_T(z,z')/2\hbar\} \tag{5.73}$$

Therefore, $\widehat{W}_T$ defines a Weyl system, but for $(\mathcal{S},\omega_T)$ an not for $(\mathcal{S},\omega_D)$. Mimicking the analysis that has been done previously, we conclude that:

$$\widehat{W}_T(\lambda z) = \exp\{i\lambda\widehat{G}(z)\} \tag{5.74}$$

and that:

$$\left[\widehat{G}(z), \widehat{G}(z')\right] = -i\hbar\omega_T(z,z') \tag{5.75}$$

Now we are in a position to consider Weyl systems for a vector space with an arbitrary and translationally invariant symplectic structure ω. By Darboux' theorem [1, 4], there exists always an invertible linear transformation T such that:

$$\omega = \widetilde{T}\omega_0 T \tag{5.76}$$

Then, the sequence of transformations:

$$(\mathcal{S},\omega) \xrightarrow{T} (\mathcal{S},\omega_0) \xrightarrow{W} \mathcal{U}(\mathcal{H}) \tag{5.77}$$

defines a Weyl system $W \circ T = W_T$ for $(\mathcal{S},\omega)$ such that:

$$\widehat{W}_T(z) =: \widehat{W}(Tz) \tag{5.78}$$

Remark 47 *The matrix T in Eq.(5.76) is clearly ambiguous by left multiplication by any matrix T' such that $\widetilde{T'}\omega_0 T = \omega_0$. However, as:*

$$\omega_0(T'Tz, T'Tz') = \omega_0(Tz, Tz') = \omega(z,z') \tag{5.79}$$

the Weyl systems associated with T and $T'T$ are unitarily equivalent.

5.4 Some Examples

As is well known [4], a conspicuous example of a one-parameter group of symplectic transformations is provided by the time evolution of a Hamiltonian system. So, let's study some simple examples.

5.4.1 The free particle

In this case, the one-parameter group is given by: $(q,p) \to (q+tp/m,p)$ and is represented by the matrix:

$$\left| \begin{array}{c} q(t) \\ p(t) \end{array} \right| = F(t) \left| \begin{array}{c} q \\ p \end{array} \right|; \quad F(t) = \left| \begin{array}{cc} 1 & t/m \\ 0 & 1 \end{array} \right|; \quad F(t)\,F(t') = F(t+t') \tag{5.80}$$

Then:

$$\begin{aligned} \widehat{W}_t(q,p) &= \widehat{W}(q(t),p(t)) = \exp\left\{(i/\hbar)\left[q(t)\,\widehat{P} + p(t)\,\widehat{Q}\right]\right\} \\ &=: \exp\left\{(i/\hbar)\left[q\widehat{P}_t + p\widehat{Q}_t\right]\right\} \end{aligned} \tag{5.81}$$

where:

$$\widehat{P}_t = \widehat{P}, \quad \widehat{Q}_t = \widehat{Q} + t\widehat{P}/m \tag{5.82}$$

There exists therefore a one-parameter family $\left\{\widehat{F}(t)\right\}_{t\in\mathbb{R}}$ of unitary operators such that:

$$\exp\left\{ip\widehat{Q}_t/\hbar\right\} = \widehat{F}^{\dagger}(t)\exp\left\{ip\widehat{Q}/\hbar\right\}\widehat{F}(t) \tag{5.83}$$

and:

$$\exp\left\{iq\widehat{P}_t/\hbar\right\} = \widehat{F}^{\dagger}(t)\exp\left\{iq\widehat{P}/\hbar\right\}\widehat{F}(t) \tag{5.84}$$

Setting then:

$$\widehat{F}(t) = \exp\left\{-i\widehat{H}t/\hbar\right\} \tag{5.85}$$

using Eq.(5.82) and expanding for small q,p and t, one finds the commutation relations:

$$\left[\widehat{P},\widehat{H}\right] = 0, \quad \left[\widehat{Q},\widehat{H}\right] = \frac{i\hbar}{m}\widehat{P} \tag{5.86}$$

Remark 48 *Note that the previous equation does not specify what are the basic commutation relations between $\widehat{Q}$ and $\widehat{P}$. Stated otherwise, we are not yet specifying what should be the symplectic structure that appears on the r.h.s. of Eq.(5.19), and this is very much in the spirit [229] of Wigner's approach. In what follows, however, and as we are dealing with this and the following Examples only to exhibit simple instances of Weyl systems, we shall assume that q and p are Darboux coordinates, and hence that the basic commutation relations are of the standard form of Eq.(5.24). The only unknown quantity in Eq.(5.86) will be then the Hamiltonian $\widehat{H}$.*

As the generators of linear and homogeneous canonical transformations are quadratic functions, it is natural to look for a quantum operator $\widehat{H}$ that is also a quadratic function:

$$\widehat{H} = a\widehat{P}^2 + b\widehat{Q}^2 + c\left(\widehat{P}\widehat{Q} + \widehat{Q}\widehat{P}\right) \tag{5.87}$$

Then the solution of the previous commutation relations is precisely:

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \lambda\widehat{\mathbb{I}} \tag{5.88}$$

where $\widehat{\mathbb{I}}$ is the identity operator and λ and arbitrary (real) constant. Apart from this, the quantum operator associated with the time evolution is the standard quantum Hamiltonian.

5.4.2 The Harmonic Oscillator

The classical Hamiltonian is:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{5.89}$$

and the solution of the equations of motion is:

$$\begin{aligned} q(t) &= q\cos\omega t + p\frac{\sin\omega t}{m\omega} \\ p(t) &= p\cos\omega t - qm\omega\sin\omega t \end{aligned} \tag{5.90}$$

The matrix $F(t)$ is then:

$$F(t) = \begin{vmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -m\omega\sin\omega t & \cos\omega t \end{vmatrix} \tag{5.91}$$

Proceeding just as in the previous case we find again:

$$\widehat{W_t}(q,p) = \exp\left\{(i/\hbar)\left[q\widehat{P}_t + p\widehat{Q}_t\right]\right\} \tag{5.92}$$

with, now:

$$\widehat{Q}_t = \widehat{Q}\cos\omega t + \widehat{P}\frac{\sin\omega t}{m\omega} \tag{5.93}$$

and:

$$\widehat{P}_t = \widehat{P}\cos\omega t - \widehat{Q}m\omega\sin\omega t \tag{5.94}$$

Defining again: $\widehat{F}(t) = \exp\left\{-i\widehat{H}t/\hbar\right\}$ and working out the commutation relations of $\widehat{H}$ with $\widehat{Q}$ and $\widehat{P}$, that read now:

$$\left[\widehat{Q},\widehat{H}\right] = \frac{i\hbar}{m}\widehat{P} \tag{5.95}$$

just as before, and:

$$\left[\widehat{P},\widehat{H}\right] = -i\hbar m\omega^2\widehat{Q} \tag{5.96}$$

one finds :

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \frac{1}{2}m\omega^2\widehat{Q}^2 + \lambda\widehat{\mathbb{I}} \tag{5.97}$$

i.e., again "modulo" an additive multiple of the identity, the standard quantum Hamiltonian.

5.4.3 A Charged Particle in a Constant Magnetic Field

The equations of motion for a particle of mass m and charge q in a constant magnetic field $\mathbf{B}$ are [7, 184][88](in units $c = 1$):

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{5.98}$$

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{5.99}$$

The vector field is therefore:

$$\Gamma = \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}\mathbf{v} \times \mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}} \tag{5.100}$$

The equations of motion can be derived either from the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 + q\mathbf{v} \cdot \mathbf{A} \tag{5.101}$$

where $\mathbf{A}$ is the vector potential: $\nabla \times \mathbf{A} = \mathbf{B}$, or from the Hamiltonian:

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} \tag{5.102}$$

with the symplectic form:

$$\omega_B = -\frac{1}{2}q\varepsilon_{ijk}B^k dx^i \wedge dx^j + dx^i \wedge dp_i \tag{5.103}$$

where:

$$\mathbf{p} = \pi - q\mathbf{A} \tag{5.104}$$

π is the canonical momentum:

$$\pi = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \tag{5.105}$$

and $\mathbf{p} = m\mathbf{v}$ is the *kinetic* momentum.

We will consider here a field: $B = (0, 0, B)$. As the motion along x^3 is trivial and decouples, we will ignore it and concentrate on the dynamics in the $\left(x^1, x^2\right)$ plane. Among the various gauges that one can employ the most popular are the *Landau gauges*:

$$\mathbf{A}_1 = B\left(x^2, 0, 0\right); \quad \mathbf{A}_2 = B\left(0, -x^1, 0\right) \tag{5.106}$$

or the *symmetric gauge*:

$$\mathbf{A}_s = \frac{B}{2}\left(x^2, -x^1, 0\right) = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{\mathbf{A}_1 + \mathbf{A}_2}{2} \tag{5.107}$$

[88]For an analysis at the quantum level, see [28, 62, 49, 102, 126]

Introducing collective coordinates: $z = \left(z^1, ..., z^4\right)$ with: $\left(z^1, z^2\right) = \left(x^1, x^2\right)$, $\left(z^3, z^4\right) = (p_1, p_2)$ and setting $q = m = 1$, the symplectic form can be written as:

$$\omega_B = \frac{1}{2}\Omega_{ij} dz^i \wedge dz^j \tag{5.108}$$

where Ω is the matrix:

$$\Omega = \begin{vmatrix} 0 & -B & 1 & 0 \\ B & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \tag{5.109}$$

Explicitly:

$$\omega_B = -B dx^1 \wedge dx^2 + dx^1 \wedge dp_1 + dx^2 \wedge dp_2 \tag{5.110}$$

The inverse of Ω :

$$\Lambda = -\Omega^{-1} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{vmatrix} \tag{5.111}$$

defines the Poisson tensor:

$$\Lambda = \frac{1}{2}\Lambda_{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} \tag{5.112}$$

or, explicitly:

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p^2} + B\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} \tag{5.113}$$

A transformation that reduces ω_B to the standard Darboux form, defined by the matrix:

$$\Omega_0 = \begin{vmatrix} \mathbf{0}_{2\times 2} & \mathbb{I}_{2\times 2} \\ -\mathbb{I}_{2\times 2} & \mathbf{0}_{2\times 2} \end{vmatrix} \tag{5.114}$$

is: $z \to \widetilde{z} = Tz$ with:

$$T = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -B & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{5.115}$$

i.e., explicitly:

$$p_1 \to \widetilde{p}_1 = p_1 - Bx^2 \tag{5.116}$$

with all the other coordinates unchanged and:

$$\omega_B\left(z, z'\right) = \omega_D\left(Tz, Tz'\right) = \omega_0\left(\widetilde{z}, \widetilde{z}'\right) \tag{5.117}$$

which implies, as can also be checked by direct calculation on the representative matrices:

$$\widetilde{T}\omega_0 T = \Omega \tag{5.118}$$

Notice that this amounts to the transformation:

$$\mathbf{r} \to \mathbf{r}, \quad \mathbf{p} \to \mathbf{p} - \mathbf{A}_1 \tag{5.119}$$

One could have used instead, e.g., the transformation:

$$\mathbf{r} \to \mathbf{r}, \quad \mathbf{p} \to \mathbf{p} - \mathbf{A}_s \tag{5.120}$$

that is defined by the matrix:

$$T' = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & B/2 & 1 & 0 \\ -B/2 & 0 & 0 & 1 \end{vmatrix} \tag{5.121}$$

and here too:

$$\widetilde{T}'\omega_0 T' = \Omega \tag{5.122}$$

Notice that, defining: $V =: T'T^{-1}$, Eqns. (5.118) and (5.122) imply:

$$\widetilde{V}\omega_0 V = \omega_0 \tag{5.123}$$

i.e.: $V \in Sp\left(\mathbb{R}^6\right)$, and this too can be checked by direct calculation.

Concentrating now on the transformation defined by Eq.(5.115) and following the procedure outlined in Sect.4.3, we define the Weyl system:

$$\begin{aligned} \widehat{W}_T(z) &= \widehat{W}(Tz) = \\ &= \exp\left\{i\left[\widetilde{x}^1\widehat{P}_1 + \ \widetilde{x}^2\widehat{P}_2 + \widetilde{p}_1\widehat{Q}^1 + \widetilde{p}_2\widehat{Q}^2\right]\right\} \end{aligned} \tag{5.124}$$

or, explicitly:

$$\widehat{W}_T(z) = \exp\left\{i\sum_{i=1,2}\left[x_i\widehat{P}_i^{(T)} + p_i\widehat{Q}_i^{(T)}\right]\right\} \tag{5.125}$$

where:

$$\widehat{Q}_i^{(T)} = \widehat{Q}_i,\ i = 1,2, \quad \widehat{P}_1^{(T)} = \widehat{P}_1,\ \widehat{P}_2^{(T)} = \widehat{P}_2 + B\widehat{Q}^1 \tag{5.126}$$

Notice that:

$$\left[\widehat{Q}_i^{(T)}, \widehat{Q}_j^{(T)}\right] = 0; \quad \left[\widehat{Q}_i^{(T)}, \widehat{P}_j^{(T)}\right] = i\delta_{ij} \tag{5.127}$$

while:

$$\left[\widehat{P}_1^{(T)}, \widehat{P}_2^{(T)}\right] = -iB \tag{5.128}$$

Time evolution is given by:

$$\begin{vmatrix} x^1(t) \\ x^2(t) \\ p_1(t) \\ p_2(t) \end{vmatrix} = F(t)\begin{vmatrix} x^1 \\ x^2 \\ p_1 \\ p_2 \end{vmatrix} \tag{5.129}$$

where:

$$F(t) = \begin{vmatrix} 1 & 0 & \frac{\sin(Bt)}{B} & \frac{1-\cos(Bt)}{B} \\ 0 & 1 & -\frac{1-\cos(Bt)}{B} & \frac{\sin(Bt)}{B} \\ 0 & 0 & \cos(Bt) & \sin(Bt) \\ 0 & 0 & -\sin(Bt) & \cos(Bt) \end{vmatrix} \tag{5.130}$$

is a linear symplectic map. Explicitly:

$$\begin{aligned} x^1(t) &= x^1 + p_1 \frac{\sin(Bt)}{B} + p_2 \frac{1-\cos(Bt)}{B} \\ x^2(t) &= x^2 - p_1 \frac{1-\cos(Bt)}{B} + p_2 \frac{\sin(Bt)}{B} \end{aligned} \tag{5.131}$$

and:

$$\begin{aligned} p_1(t) &= p_1 \cos(Bt) + p_2 \sin(Bt) \\ p_2(t) &= -p_1 \sin(Bt) + p_2 \cos(Bt) \end{aligned} \tag{5.132}$$

Following the procedure outlined in the previous examples, we define then the Weyl system:

$$\widehat{W}_T^{(t)}(z) = \widehat{W}(Tz(t)) = \exp\left\{ i \sum_{i=1,2} \left[x_i(t)\, \widehat{P}_i^{(T)} + p_i(t)\, \widehat{Q}_i^{(T)} \right] \right\} \tag{5.133}$$

or:

$$\widehat{W}_T^{(t)}(z) = \exp\left\{ i \sum_{i=1,2} \left[x_i \widehat{P}_i^{(T)}(t) + p_i \widehat{Q}_i^{(T)}(t) \right] \right\} \tag{5.134}$$

where the $\widehat{P}_i^{(T)}(t)$'s and $\widehat{Q}_i^{(T)}(t)$'s are defined by:

$$\sum_{i=1,2} \left[x_i \widehat{P}_i^{(T)}(t) + p_i \widehat{Q}_i^{(T)}(t) \right] =: \sum_{i=1,2} \left[x_i(t)\, \widehat{P}_i^{(T)} + p_i(t)\, \widehat{Q}_i^{(T)} \right] \tag{5.135}$$

and Eqns.(5.131) and (5.132) have to be used on the r.h.s. Here too we conclude that there exists a unitary operator $\widehat{F}(t) = \exp\{-it\mathcal{H}/\hbar\}$ such that:

$$\begin{aligned} \exp\left\{ ip\widehat{Q}_i^{(T)}(t)/\hbar \right\} &= \widehat{F}^\dagger(t) \exp\left\{ ip\widehat{Q}_i^{(T)}/\hbar \right\} \widehat{F}(t) \\ \exp\left\{ iq\widehat{P}_i^{(T)}(t)/\hbar \right\} &= \widehat{F}^\dagger(t) \exp\left\{ iq\widehat{P}_i^{(T)}/\hbar \right\} \widehat{F}(t)\, ;\ i = 1, 2 \end{aligned} \tag{5.136}$$

Expanding again for small q, p, t and using Eq.(5.126) we find the commutation relations:

$$\left[\widehat{P}_1, \mathcal{H} \right] = 0, \quad \left[\widehat{P}_2, \mathcal{H} \right] = i\hbar B \left(\widehat{P}_1 - B\widehat{Q}_2 \right) \tag{5.137}$$

and:

$$\left[\widehat{Q}_1, \mathcal{H} \right] = i\hbar \left(\widehat{P}_1 - B\widehat{Q}_2 \right), \ \left[\widehat{Q}_2, \mathcal{H} \right] = i\hbar \widehat{P}_2 \tag{5.138}$$

and it is easy to conclude that the Hamiltonian operator is now:

$$\mathcal{H} = \frac{1}{2}\left[\left(\widehat{P}_1 - B\widehat{Q}_2\right)^2 + \widehat{P}_2^2\right] \tag{5.139}$$

which corresponds to the "minimal coupling" prescription:

$$\widehat{\mathbf{P}} \to \widehat{\mathbf{P}} - \mathbf{A} \tag{5.140}$$

with: $\mathbf{A} = \mathbf{A}_1$.

5.4.4 Magnetic Translation Groups and Weyl Systems

We will exhibit in this final Subsection another example [184] of a Weyl system, which is provided by the implementation at the quantum level of the group of translations in a two-dimensional electron gas in a constant (perpendicular) magnetic field that has been studied in the previous Subsection.

Reinstating the constants (c, m, q) in the proper places, the Hamiltonian is given by (cfr.Eq.(5.102)):

$$H = \frac{1}{2m}[\pi - \frac{q}{c}\mathbf{A}]^2 \tag{5.141}$$

Introducing complex coordinates: $\zeta = x + iy$, the equations of motion become:

$$\frac{d}{dt}\{\dot{\zeta} + i\Omega\zeta\} = 0; \quad \Omega = \frac{qB}{mc} \tag{5.142}$$

and they have the solution:

$$\zeta(t) = X + A\exp\{-i\Omega t\} \tag{5.143}$$

where:

$$X = \zeta - \frac{i}{\Omega}\dot{\zeta} = const. \tag{5.144}$$

The associated total energy is:

$$E = \frac{\mathbf{p}^2}{2m} \equiv \frac{1}{2}m|\dot{\zeta}|^2 \equiv \frac{1}{2}m\Omega^2|A|^2 \tag{5.145}$$

and the orbits are circles of radius $|A|$ and center:

$$X = x_0 + iy_0; \quad x_0 = x + \frac{p_y}{m\Omega}, \quad y_0 = y - \frac{p_x}{m\Omega} \tag{5.146}$$

The Poisson brackets for the components of the kinetic momentum are:

$$\{p_i, p_j\} = \frac{q}{c}(\partial_i A_j - \partial_j A_i) \equiv \frac{q}{c}\varepsilon_{ijk}B_k \tag{5.147}$$

i.e.:

$$\{p_x, p_y\} = m\Omega \tag{5.148}$$

and hence:

$$\{x_0, y_0| = -\frac{1}{m\Omega} = -\frac{c}{qB} \tag{5.149}$$

The Cartan form:

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot d\mathbf{q} \equiv \{p_i + \frac{q}{c} A_i\} dx_i \tag{5.150}$$

leads to:

$$\omega_{\mathcal{L}} =: -d\theta_{\mathcal{L}} = dx_i \wedge dp_i + \frac{q}{2c}(\partial_i A_j - \partial_j A_i) dx_i \wedge dx_j \tag{5.151}$$

i.e.:

$$\omega_{\mathcal{L}} = dx_i \wedge dp_i + \frac{q}{2c} \varepsilon_{ijk} B_i dx_j \wedge dx_k \tag{5.152}$$

The dynamical vector field is given by (cfr. Eq.(5.100)):

$$\Gamma = \frac{p_i}{m} \frac{\partial}{\partial q_i} + \frac{q}{mc} \varepsilon_{ijk} p_j B_k \frac{\partial}{\partial v_i} \tag{5.153}$$

Nöther's theorem [186] states that, if X^c is a tangent lift[89], and:

$$L_{X^c} \mathcal{L} = L_\Gamma h, \quad h = \pi^* \widehat{G}, \; i.e.: h = h(\mathbf{r}) \tag{5.154}$$

then the associated constant of the motion is:

$$\chi = i_{X^c} \theta_{\mathcal{L}} - h \tag{5.155}$$

For translations in the plane:

$$X_i \equiv X_i^c = \frac{\partial}{\partial x_i} \tag{5.156}$$

Hence:

$$L_{X_i} \mathcal{L} = \frac{q}{mc} (\partial_i A_j) p_j \equiv \frac{q}{c} (\partial_i A_j - \partial_j A_i) + \frac{q}{c} \partial_j A_i \tag{5.157}$$

i.e.:

$$L_{X_i} \mathcal{L} = \frac{q}{c} \frac{d}{dt} (\mathbf{A} + \mathbf{r} \times \mathbf{B})_i \tag{5.158}$$

($\mathbf{h} = \frac{q}{c}(\mathbf{A} + \mathbf{r} \times \mathbf{B})$), and the associated Nöther's constants of the motion are:

$$\chi_i = i_{X_i} \theta_{\mathcal{L}} - h_i = (\mathbf{p} + \frac{q}{c} \mathbf{B} \times \mathbf{r})_i \tag{5.159}$$

[89]We recall [186] that,if: $X = X^i \partial/\partial q^i$, $X^i = X^i(q)$ is a vector field on some manifold M, its *tangent lift* X^c is the vector field on TM defined by: $X^c = X^i \partial/\partial q^i + L_\Gamma(X^i) \partial/\partial v^i$, with Γ *any* second-order vector field.

Notice that the coordinates of the center of the Larmor orbit are:

$$x_0 = \frac{c}{qB}(p_y + \frac{qB}{c}x) \equiv \frac{c}{qB}(\mathbf{p} + \frac{q}{c}\mathbf{B} \times \mathbf{r})_y \tag{5.160}$$
$$y_0 = -\frac{c}{qB}(mv_x - \frac{qB}{c}y) \equiv -(\mathbf{p} + \frac{q}{c}\mathbf{B} \times \mathbf{r})_x$$

Hence:

$$\chi_x = -\frac{qB}{c}y_0, \quad \chi_y = \frac{qB}{c}x_0 \tag{5.161}$$

It follows then that the *P.B.*'s among the Nöther's constants of the motion are:

$$\{\chi_i, \chi_j\} = -\frac{q}{c}\varepsilon_{ijk}B_k \tag{5.162}$$

Following then the standard rules for the implementation of symmetries at the quantum level, we associate with a (finite) translation by a vector $\mathbf{a}$ in the plane the *magnetic translation operator* $\widehat{T}(\mathbf{a})$ [76, 184, 236, 237] defined by:

$$\widehat{T}(\mathbf{a}) = \exp\left\{\frac{i}{\hbar}\widehat{\chi}_{op} \cdot \mathbf{a}\right\} \tag{5.163}$$

where ($\mathbf{p} = \pi - q\mathbf{A}/c$):

$$\widehat{\chi}_{op} = \widehat{\pi} - \frac{q}{c}(\mathbf{B} \times \mathbf{r} - \mathbf{A}); \quad \widehat{\pi} = \frac{\hbar}{i}\nabla \tag{5.164}$$

with the commutation relations:

$$[\widehat{\chi}_i, \widehat{\chi}_j] = i\hbar\frac{q}{c}\varepsilon_{ijk}B_k \tag{5.165}$$

Of course, $\widehat{\chi}_{op}$ commutes with the Hamiltonian, and so does $\widehat{T}(\mathbf{a})$.

Using then the identities:

$$\exp\{A + B\} = \exp\{A\}\exp\{B\}\exp\left(-\frac{1}{2}[A, B]\right) \tag{5.166}$$
$$\exp\{A\}\exp\{B\} = \exp\{B\}\exp\{A\}\exp([A, B]) \tag{5.167}$$

valid whenever: $[A, [A, B]] = [B, [A, B]] = 0$, and noting that:

$$\left[\pi_i, (\mathbf{B} \times \mathbf{r} - \mathbf{A})_j\right] = i\hbar\frac{\partial A_j}{\partial x_i} \tag{5.168}$$

one finds for the action of $\widehat{T}(\mathbf{a})$ on wavefunctions:

$$\left(\widehat{T}(\mathbf{a})\psi\right)(\mathbf{r}) = \exp\left\{-i\frac{q}{\hbar c}\mathbf{a} \cdot \mathbf{A}\right\}\psi(\mathbf{r} + \mathbf{a}) \tag{5.169}$$

in the symmetric gauge, and, e.g.:

$$\left(\widehat{T}(\mathbf{a})\psi\right)(\mathbf{r}) = \exp\left\{-i\frac{q}{\hbar c}Ba_1(y - a_2/2)\right\}\psi(\mathbf{r} + \mathbf{a}) \tag{5.170}$$

in the Landau gauge: $\mathbf{A} = B\,(0, x, 0)$. Then it is easy to prove that:

$$\widehat{T}\,(\mathbf{a})\,\widehat{T}\,(\mathbf{b}) = \exp\left\{\frac{iq}{2\hbar c}\mathbf{B}\cdot\mathbf{a}\times\mathbf{b}\right\}\widehat{T}\,(\mathbf{a}+\mathbf{b}) \tag{5.171}$$

and:

$$\widehat{T}\,(\mathbf{a})\,\widehat{T}\,(\mathbf{b}) = \widehat{T}\,(\mathbf{b})\,\widehat{T}\,(\mathbf{a})\exp\left\{\frac{iq}{\hbar c}\mathbf{B}\cdot\mathbf{a}\times\mathbf{b}\right\} \tag{5.172}$$

But:

$$\frac{q}{c}\mathbf{B}\cdot\mathbf{a}\times\mathbf{b} = \omega_{\mathcal{L}}\,(\mathbf{a},\mathbf{b}) \tag{5.173}$$

and hence:

$$\widehat{T}\,(\mathbf{a})\,\widehat{T}\,(\mathbf{b}) = \exp\left\{\frac{i}{2\hbar}\omega_{\mathcal{L}}\,(\mathbf{a},\mathbf{b})\right\}\widehat{T}\,(\mathbf{a}+\mathbf{b}) \tag{5.174}$$

The magnetic translation operators are therefore an instance [238] of a Weyl system on the configuration space.

6 Quantum Mechanics in Phase Space

6.1 The Weyl and Wigner Maps

We will work in $S \approx \mathbb{R}^2$ for simplicity. Generalizations to higher dimensions are easy to work out.

As a preliminary remark, let's observe that we have the identity ($f \in \mathcal{L}_2\left(\mathbb{R}^2\right)$):

$$\iiiint \frac{d\xi d\eta dq' dp'}{(2\pi\hbar)^2} f\left(q', p'\right) e^{-i\omega_0\left(\left(q',p'\right),(\xi,\eta)\right)/\hbar} e^{i(\xi p+\eta q)/\hbar} \equiv f(q,-p) \tag{6.1}$$

This can also be rewritten as:

$$\iint \frac{d\xi d\eta}{2\pi\hbar}\left[\frac{1}{\hbar}\mathcal{F}_s(f)\left(\frac{\eta}{\hbar},\frac{\xi}{\hbar}\right)\right] e^{i(\xi p+\eta q)/\hbar} = f(q,-p) \tag{6.2}$$

where $\mathcal{F}_s(f)$ is the symplectic Fourier transform[90] [77, 240]:

$$\mathcal{F}_s(f)(\eta,\xi) = \iint \frac{dqdp}{2\pi} f(q,p)\, e^{-i\omega_0((q,p),(\xi,\eta))} \tag{6.3}$$

and, as usual: $\omega_0((q,p),(\xi,\eta)) = q\eta - p\xi$.

Digression.
Allowing also for distribution-valued transforms, we have, in particular:

$$\mathcal{F}_s(q)(\eta,\xi) = 2\pi i\delta'(\eta)\,\delta(\xi) \tag{6.4}$$

and:

$$\mathcal{F}_s(p)(\eta,\xi) = -2\pi i\delta(\eta)\,\delta'(\xi) \tag{6.5}$$

The *Weyl map* [225] amounts to the replacement, in Eq.(6.2):

$$\exp\left\{i(\xi p+\eta q)/\hbar\right\} \to \exp\left\{i\left(\xi\widehat{P}+\eta\widehat{Q}\right)/\hbar\right\} \equiv \widehat{W}(\xi,\eta) \tag{6.6}$$

whereby one obtains the map:

$$\Omega : \mathcal{F}\left(\mathbb{R}^2\right) \to \mathcal{O}p(\mathcal{H}) \tag{6.7}$$

defined by:

$$\begin{aligned}\Omega(f) &= \iint \frac{d\xi d\eta}{2\pi\hbar}\left[\frac{1}{\hbar}\mathcal{F}_s(f)\left(\frac{\eta}{\hbar},\frac{\xi}{\hbar}\right)\right]\widehat{W}(\xi,\eta) = \\ &\equiv \iint \frac{d\xi d\eta}{2\pi}\mathcal{F}_s(f)(\eta,\xi)\,\widehat{W}(\hbar\xi,\hbar\eta)\end{aligned} \tag{6.8}$$

[90] See Appendix C. The fact that we get a change in sign in the second variable is precisely a byproduct of the use of the symplectic Fourier transform. Had we used instead the ordinary Fourier transform we would have obtained of course $f(q,p)$ instead of $f(q,-p)$ on the r.h.s. of (6.2).

It is simple to show that, if f is real, then:

$$\overline{\mathcal{F}_s(f)(\eta,\xi)} = \mathcal{F}_s(f)(-\eta,-\xi) \tag{6.9}$$

and this proves that $\Omega(f)$ is (at least) a *symmetric* [201] operator (more on this later on). Using then:

$$\left(\widehat{W}(\xi,\eta)\,\psi\right)(x) = \exp\{i\eta[x+\xi/2]/\hbar\}\,\psi(x+\xi) \tag{6.10}$$

we obtain:

$$(\Omega(f)\,\psi)(x) = \iint \frac{d\xi d\eta}{2\pi}\mathcal{F}_s(f)(\eta,\xi)\exp[i\eta(x+\hbar\xi/2)]\,\psi(x+\hbar\xi) \tag{6.11}$$

In particular, using (6.4) and (6.5):

$$(\Omega(q)\,\psi)(x) = x\psi(x)\,,\qquad (\Omega(p)\,\psi)(x) = i\hbar\frac{d\psi}{dx} \tag{6.12}$$

In other words:

$$\Omega(q) = \widehat{Q} \tag{6.13}$$

while (cfr. the discussion in the previous footnote):

$$\Omega(p) = -\widehat{P} \tag{6.14}$$

More generally, for arbitrary integers n and m:

$$\mathcal{F}_s(q^n p^m)(\eta,\xi) = 2\pi(-)^m\, i^{n+m}\delta^{(n)}(\eta)\,\delta^{(m)}(\xi) \tag{6.15}$$

which implies:

$$(\Omega(q^n p^m)\,\psi)(x) = \left(i\frac{d}{d\xi}\right)^m [(x+\hbar\xi/2)^n\,\psi(x+\hbar\xi)]\,|_{\xi=0} \tag{6.16}$$

which can be rearranged [240] in the form:

$$(\Omega(q^n p^m)\,\psi)(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x^k\left(i\hbar\frac{d}{dx}\right)^m\left[x^{n-k}\psi(x)\right] \tag{6.17}$$

Hence:

$$\Omega(q^n p^m) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}[\Omega(q)]^k\cdot[\Omega(p)]^m\cdot[\Omega(q)]^{n-k} \tag{6.18}$$

In particular, for $n=m=1$:

$$\Omega(qp) = \frac{1}{2}(\Omega(q)\cdot\Omega(p)+\Omega(p)\cdot\Omega(q)) \tag{6.19}$$

Notice that:

$$\Omega(qp) = \Omega(pq) \tag{6.20}$$

but:

$$\Omega(qp) \neq \Omega(q) \cdot \Omega(p) \tag{6.21}$$

Also, as can be shown on examples, in general:

$$\Omega(fg) \neq \frac{1}{2}\left(\Omega(f) \cdot \Omega(g) + \Omega(g) \cdot \Omega(f)\right) \tag{6.22}$$

as can be seen already from Eq.(6.18) when m and/or $n \neq 1$, i.e. the "Weyl symmetrization procedure" (6.22) [225] holds only in very special cases.

Using Eq.(6.11) we obtain, for the matrix elements of the Weyl operator $\Omega(f)$:

$$\langle\phi|\Omega(f)|\psi\rangle = \int \frac{dx d\xi d\eta}{2\pi} \mathcal{F}_s(f)(\eta,\xi)\, e^{i\eta(x+\hbar\xi/2)}\overline{\phi(x)}\psi(x+\hbar\xi) \tag{6.23}$$

In particular, in a plane-wave basis ($\psi(x) = \left(1/\sqrt{2\pi}\right)\exp\{ikx\}$ etc.):

$$\langle k'|\Omega(f)|k\rangle = \int \frac{d\xi}{2\pi}\mathcal{F}_s(f)(k'-k,\xi)\exp\{i\hbar\xi(k+k')/2\} \tag{6.24}$$

or:

$$\langle K+k/2|\Omega(f)|K-k/2\rangle = \int \frac{d\xi}{2\pi}\mathcal{F}_s(f)(k,\xi)\exp\{i\hbar\xi K/2\} \tag{6.25}$$

Inserting then the explicit form of the symplectic Fourier transform we find eventually:

$$\langle K+k/2|\Omega(f)|K-k/2\rangle = \int \frac{dq}{2\pi} f(q,-\hbar K)\exp\{-ikq\} \tag{6.26}$$

For example, $f(q,p) = p$ yields:

$$\langle K+k/2|\Omega(p)|K-k/2\rangle = -\hbar K\delta(k) \tag{6.27}$$

which (cfr. Eq.(6.14)) is the correct result.

The Weyl map can be inverted, i.e there exists a map, called the *Wigner map*:

$$\Omega^{-1} : \mathcal{O}p(\mathcal{H}) \to \mathcal{F}\left(\mathbb{R}^2\right) \tag{6.28}$$

such that:

$$\Omega^{-1}(\Omega(f)) = f \tag{6.29}$$

In general, given any operator $\widehat{O}$ such that $Tr\left[\widehat{O}\widehat{W}(x,k)\right]$ exists[91], the Wigner map is defined as:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p) =: \iint \frac{dxdk}{2\pi\hbar}\exp\{-i\omega_0((x,k),(q,p))/\hbar\}\, Tr\left[\widehat{O}\widehat{W}^{\dagger}(x,k)\right] \tag{6.30}$$

[91] As W is a bounded operator, this will be granted, e.g., if A is trace-class.

In order to prove Eq.(6.29), we need the trace:

$$Tr[\widehat{W}(x,k)\,\widehat{W}^{\dagger}(\xi,\eta)] = \int dhdh' \left\langle h \left| \widehat{W}(x,k) \right| h' \right\rangle \left\langle h' \left| \widehat{W}^{\dagger}(\xi,\eta) \right| h \right\rangle \tag{6.31}$$

Using Eq. (5.35) we obtain:

$$Tr\left[\widehat{W}(x,k)\,\widehat{W}^{\dagger}(\xi,\eta)\right] = 2\pi\hbar\delta(x-\xi)\,\delta(k-\eta) \tag{6.32}$$

Inserting then (6.32) into (6.30) and using this result, we obtain:

$$\Omega^{-1}(\Omega(f))(q,p) = \int \frac{d\xi d\eta}{2\pi}\mathcal{F}_s(\eta,\xi)\exp\{-i\omega((\xi,\eta),(q,p))\} = f(q,p) \tag{6.33}$$

■

6.2 A Digression on: Phase-Point Operators

Going back to Eq.(6.8), which reads:

$$\Omega(f) = \iint \frac{d\xi d\eta}{(2\pi\hbar)^2}\widehat{W}(\xi,\eta)\iint dqdpe^{-i(q\eta-p\xi)/\hbar}f(q,p) \tag{6.34}$$

and, to the extent that it is legitimate to apply Fubini's theorem, we obtain:

$$\Omega(f) = \iint \frac{dqdp}{2\pi\hbar}f(q,p)\,\widehat{A}(q,p) \tag{6.35}$$

where the symplectic Fourier transform of $\widehat{W}(\xi,\eta)$, *i.e.:*

$$\widehat{A}(q,p) =: \iint \frac{d\xi d\eta}{2\pi\hbar}e^{-i(q\eta-p\xi)/\hbar}\widehat{W}(\xi,\eta) \tag{6.36}$$

defines the so-called "phase-point operators" [38, 39, 68, 125, 233, 234]. It is not hard to prove, using: $\widehat{W}^{\dagger}(\xi,\eta) = \widehat{W}(-\xi,-\eta)$ and Eqns.(5.37) and (6.32) that:

- The phase-point operators are of unit trace:

$$Tr\widehat{A}(q,p) = 1 \tag{6.37}$$

- They are Hermitian:

$$\widehat{A}^{\dagger}(q,p) = \widehat{A}(q,p) \tag{6.38}$$

  and:

- They are trace-orthogonal, i.e.:

$$Tr\left(\widehat{A}(q,p)\,\widehat{A}(q',p')\right) = 2\pi\hbar\delta(q-q')\,\delta(p-p') \tag{6.39}$$

- Moreover, a simple calculation shows that:

$$\iint \frac{dqdp}{2\pi\hbar}\widehat{A}(q,p) = \widehat{W}(0,0) = \widehat{\mathbb{I}} \tag{6.40}$$

with $\widehat{\mathbb{I}}$ the identity operator.

All this proves that the phase-point operators are a *complete (trace) orthonormal set of Hermitian operators.* In particular, substituting the Wigner function $\Omega^{-1}\left(\widehat{O}\right)$ for the function f in Eq.(6.35) and as: $\Omega\left(\Omega^{-1}\left(\widehat{O}\right)\right) = \widehat{O}$, we obtain at once the reconstruction:

$$\widehat{O} = \iint \frac{dqdp}{2\pi\hbar}\Omega^{-1}\left(\widehat{O}\right)(q,p)\,\widehat{A}(q,p) \tag{6.41}$$

in terms of the Wigner function and the phase-point operators, as well as, using Eq.(6.39):

$$\Omega^{-1}\left(\widehat{O}\right)(q,p) = Tr\left\{\widehat{O}\widehat{A}(q,p)\right\} \tag{6.42}$$

An explicit representation of phase-point operators satisfying all of the above properties is:

$$\widehat{A}(q,p) = \int dq'|q+q'/2\rangle \exp\left(iq'p/\hbar\right)\langle q-q'/2| \tag{6.43}$$

with matrix elements:

$$\left\langle x\left|\widehat{A}(q,p)\right|x'\right\rangle = 2\delta\left(x+x'-2q\right)\exp\left\{ip\left(x-x'\right)/\hbar\right\} \tag{6.44}$$

6.3 More on the Wigner Map

It is useful to have an expression for the Wigner map directly in terms of the matrix elements of the operators. Working again for simplicity in $\mathbb{R}^2$ and introducing resolutions of the identity in terms of plane waves: $\{|m\rangle\}$[92]:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p) = \int \frac{dxd\pi dldm}{2\pi\hbar}\exp\left\{-i\left(xp-\pi q\right)/\hbar\right\}\left\langle l|\widehat{O}|m\right\rangle\left\langle m|\widehat{W}^{\dagger}(x,\pi)|l\right\rangle \tag{6.45}$$

or $(\pi = \hbar k)$:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p) = \int \frac{dxdkdldm}{2\pi}\exp\left\{-i\left(xp/\hbar-kq\right)\right\}\left\langle l|\widehat{O}|m\right\rangle\left\langle m|\widehat{W}^{\dagger}(x,\hbar k)|l\right\rangle \tag{6.46}$$

and using:

$$\left\langle m|\widehat{W}^{\dagger}(x,\hbar k)|l\right\rangle = \exp\left\{-ix\left(m+l\right)/2\right\}\delta\left(l-m-k\right) \tag{6.47}$$

[92] $\langle x|m\rangle = \frac{1}{\sqrt{2\pi}}\exp\left\{imx\right\}$, and: $\int dm|m\rangle\langle m| = \mathbb{I}$.

one finds eventually:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int dke^{iqk}\left\langle -p/\hbar+k/2|\widehat{O}|-p/\hbar-k/2\right\rangle \tag{6.48}$$

with obvious generalizations to higher dimensions. As an example, if: $\widehat{A}=-\widehat{P}$, then, as: $\widehat{P}\,|m\rangle=\hbar m\,|m\rangle$:

$$\begin{aligned}\left\langle -p/\hbar+k/2|(-\widehat{P})|-p/\hbar-k/2\right\rangle &= (p\hbar+k/2)\left\langle -p/\hbar+k/2|-p/\hbar-k/2\right\rangle \\ &\equiv p\delta(k) \end{aligned}\tag{6.49}$$

and we find:

$$\Omega^{-1}\left((-\widehat{P})\right)(q,p)=p \tag{6.50}$$

as expected.

Also, it is easy to prove that:

$$\Omega^{-1}\left(\widehat{W}(q',,p')\right)(q,p)=\exp\left\{i\omega_0\left((q,p),(q',p')\right)/\hbar\right\} \tag{6.51}$$

Introducing now resolutions of the identity relative to the coordinates:

$$\Omega^{-1}\left(\widehat{A}\right)(q,p)=\int dkdxdx'e^{iqk}\left\langle -p/\hbar+k/2|\,x\right\rangle\left\langle x\,|\widehat{A}|\,x'\right\rangle\left\langle x'\,|-p/\hbar-k/2\right\rangle \tag{6.52}$$

the integration over k yields a delta-function, and we obtain, eventually, the celebrated *Wigner formula* [125, 228], or *Wigner transform* :

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int d\xi e^{ip\xi/\hbar}\left\langle q+\xi/2|\widehat{O}\,|q-\xi/2\right\rangle \tag{6.53}$$

Here too, setting: $\widehat{A}=\widehat{Q}$, we find at once: $\Omega^{-1}\left(\widehat{Q}\right)(q,p)=q$, as expected. As another example, consider, e.g.: $\widehat{A}=|\phi\rangle\langle\psi|$ (which is a prototype of a finite-rank operator). Then it is immediate to see that:

$$\Omega^{-1}\left(|\phi\rangle\langle\psi|\right)(q,p)=\int_{-\infty}^{\infty}d\xi e^{ip\xi/\hbar}\phi\left(q+\xi/2\right)\overline{\psi\left(q-\xi/2\right)} \tag{6.54}$$

Remark 49 *From Eq.(6.54) we obtain:*

$$\left|\Omega^{-1}\left(|\phi\rangle\langle\psi|\right)(q,p)\right|\leq 2\int_{-\infty}^{\infty}d\eta\left|\phi\left(q+\eta\right)\right|\left|\psi\left(q-\eta\right)\right| \tag{6.55}$$

and, using Schwartz's inequality:

$$\left|\Omega^{-1}\left(|\phi\rangle\langle\psi|\right)(q,p)\right|\leq 2\left\|\phi\right\|\left\|\psi\right\| \tag{6.56}$$

In particular, if $|\psi\rangle = |\phi\rangle$ and: $\langle\phi|\phi\rangle = 1$, i.e. for a one-dimensional projector: $P_\phi = |\phi\rangle\langle\phi|$:

$$\left|\Omega^{-1}\left(P_\phi\right)(q,p)\right| \leq 2 \tag{6.57}$$

As every density matrix can be written as a convex linear combination of one-dimensional projectors, we obtain eventually the uniform bound[93] *[68]:*

$$\left|\Omega^{-1}\left(\widehat{\rho}\right)(q,p)\right| \leq 2 \tag{6.58}$$

if $\widehat{\rho}$ is a density matrix.

Proceeding in a somewhat heuristic manner, let now $\widehat{O}$ be a self-adjoint operator with a completely discrete spectrum: $\widehat{O}|\phi_n\rangle = \lambda_n|\phi_n\rangle$, $\langle\phi_n|\phi_m\rangle = \delta_{nm}$ and: $\sum_n |\phi_n\rangle\langle\phi_n| = \mathbb{I}$. Then:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p) = \sum_n \lambda_n \int d\xi e^{ip\xi/\hbar}\phi_n\left(q+\xi/2\right)\overline{\phi_n\left(q+\xi/2\right)} \tag{6.59}$$

and hence, proceeding as before:

$$\left|\Omega^{-1}\left(\widehat{O}\right)(q,p)\right| \leq 2\sum_n |\lambda_n| = 2Tr\left|\widehat{O}\right| \tag{6.60}$$

where [201]: $\left|\widehat{O}\right| =: \sqrt{\widehat{O}^\dagger\widehat{O}}$. Trace-class operators[94] *are defined [201] by requiring finiteness of $Tr\left|\widehat{O}\right|$. Therefore:*

The Wigner function of any trace-class operator $\widehat{O}$ will be uniformly bounded by $2Tr\left|\widehat{O}\right|$.

It is easy to check that the Wigner transform inverts to:

$$\left\langle x|\widehat{O}|x'\right\rangle = \int \frac{dp}{2\pi\hbar}\exp\left\{-ip\left(x-x'\right)/\hbar\right\}\Omega^{-1}\left(\widehat{O}\right)\left(\frac{x+x'}{2},p\right) \tag{6.61}$$

As an example, let's consider the Wigner transform of $\widehat{O} = |\phi\rangle\langle\psi|$ as given by Eq.(6.54). Then it is immediate to check that, indeed:

$$\int \frac{dp}{2\pi\hbar}e^{\{-ip\left(x-x'\right)/\hbar\}}\Omega^{-1}\left(\widehat{O}\right)\left(\frac{x+x'}{2},p\right) = \phi(x)\overline{\psi(x')} = \langle x|\phi\rangle\langle\psi|x'\rangle \tag{6.62}$$

Example 50 *As a less simple example as compared to the previous ones, let us consider a $1D$ harmonic oscillator of mass m and proper frequency ω. The corresponding Hamiltonian is:*

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \frac{1}{2}m\omega^2\widehat{Q}^2 \tag{6.63}$$

[93] Note that we are using here a slightly different normalization than that used in Ref.[68].

[94] A class of operators comprising, in particular, finite-rank projection operators as well as density states.

with eigenvalues: $E_n = (n + 1/2)\hbar\omega$, $n \geq 0$ *and eigenfunctions:*

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\zeta^2/2\right) H_n(\zeta) \tag{6.64}$$

where ζ *is the dimensionless variable:* $\zeta = x\sqrt{m\omega/\hbar}$ *and the* H_n*'s are the Hermite polynomials [69]. We want to evaluate here the Wigner function (the Wigner map) associated with the "Boltzmann factor"* $\widehat{A} = \exp\left(-\beta\widehat{H}\right)$*, with* β *the inverse temperature . Of course:*

$$\left\langle x \left| e^{-\beta\widehat{H}} \right| x' \right\rangle = \sum_{n=0}^{\infty} e^{-\beta E_n} \psi_n(x) \psi_n(x') \tag{6.65}$$

Inserting the explicit form (6.64) of the eigenfunctions:

$$\left\langle x \left| e^{-\beta\widehat{H}} \right| x' \right\rangle = \sqrt{\frac{m\omega z}{\pi\hbar}} \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} e^{-\left(\zeta^2+\zeta'^2\right)/2} H_n(\zeta) H_n(\zeta'), \quad z = \exp(-\beta\hbar\omega) \tag{6.66}$$

Now, it turns out that[95] *[77, 128]:*

$$\sum_{n=0}^{\infty} \frac{z^n}{2^n n!} H_n(\zeta) H_n(\zeta') = \frac{1}{\sqrt{1-z^2}} \exp\left\{ \frac{2z\zeta\zeta' - z^2\left(\zeta^2+\zeta'^2\right)}{1-z^2} \right\}, \quad |z| < 1 \tag{6.67}$$

and therefore the matrix element (6.65)can be expressed in closed form as:

$$\left\langle x \left| e^{-\beta\widehat{H}} \right| x' \right\rangle = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\left(\zeta^2+\zeta'^2\right)/2} \sqrt{\frac{z}{1-z^2}} \exp\left[\frac{2z\zeta\zeta' - z^2\left(\zeta^2+\zeta'^2\right)}{1-z^2}\right] \tag{6.68}$$

Setting then: $x = q + \xi/2, x' = q - \xi/2$ *and inserting the result into Eq.(6.53) one finds eventually the Wigner function:*

$$\Omega^{-1}\left(e^{-\beta\widehat{H}}\right)(q,p) = \frac{1}{\cosh(\beta\hbar\omega/2)} \exp\left\{ -\tanh(\beta\hbar\omega/2) \left[\frac{m\omega}{\hbar} q^2 + \frac{p^2}{m\hbar\omega} \right] \right\} \tag{6.69}$$

Coming back now to the main object of this Section, an interesting consequence of Eq.(6.48) is the following. Let's calculate the $\mathcal{L}^2$ norm of $\Omega^{-1}\left(\widehat{A}\right)(q,p)$, i.e.:

$$\left\| \Omega^{-1}\left(\widehat{A}\right) \right\|^2 = \int \frac{dqdp}{2\pi\hbar} \left| \Omega^{-1}\left(\widehat{A}\right)(q,p) \right|^2 \tag{6.70}$$

Explicitly:

$$\begin{aligned} &\left\| \Omega^{-1}\left(\widehat{A}\right) \right\|^2 = \qquad (6.71) \\ &= \int \frac{dqdp}{2\pi\hbar} dkdk' e^{i(k'-k)q} \left\langle p/\hbar - k/2 | \widehat{A}^\dagger | p/\hbar + k/2 \right\rangle \left\langle p/\hbar + k'/2 | \widehat{A} | p/\hbar - k'/2 \right\rangle \end{aligned}$$

[95] This is known also as *Mehler's formula.*

Performing the integration over q , which produces a delta-function, and shifting variables: $p \to p + \hbar k/2$:

$$\left\| \Omega^{-1}\left(\widehat{A}\right) \right\|^2 = \int d\left(p/\hbar\right) dk \left\langle p/\hbar | \widehat{A}^{\dagger} | p/\hbar + k \right\rangle \left\langle p/\hbar + k | \widehat{A} | p/\hbar \right\rangle \tag{6.72}$$

The integration over k yields a resolution of the identity, and we end up with:

$$\left\| \Omega^{-1}\left(\widehat{A}\right) \right\|^2 = \int d\left(p/\hbar\right) \left\langle p/\hbar | \widehat{A}^{\dagger} \widehat{A} | p/\hbar \right\rangle \tag{6.73}$$

i.e., eventually:

$$\left\| \Omega^{-1}\left(\widehat{A}\right) \right\|^2 = Tr\left\{ \widehat{A}^{\dagger} \widehat{A} \right\} \tag{6.74}$$

and, if: $\widehat{A} = \Omega\left(f\right)$:

$$\left\| f \right\|^2 = Tr\left\{ \Omega\left(f\right)^{\dagger} \Omega\left(f\right) \right\} \tag{6.75}$$

The condition of finiteness (positivity is obvious) of $Tr\left\{A^{\dagger}A\right\}$ characterizes A as a *Hilbert-Schmidt* [201] operator. Therefore [198]:

Theorem 51 f *will be square-integrable* ($f \in \mathcal{L}^2\left(\mathbb{R}^2\right)$) *if and only if* $\Omega\left(f\right)$ *is Hilbert-Schmidt.Quite similarly:* $\Omega^{-1}\left(\widehat{A}\right)$ *will be square-integrable if and only if* $\widehat{A}$ *is Hilbert-Schmidt.*

The Weyl and Wigner maps establish therefore a *bijection* [77, 93] between Hilbert-Schmidt operators and square-integrable functions on phase space. This is consistent with the fact that both spaces are Hilbert spaces. Moreover, Eqs.(6.74) and (6.75) prove that the bijection, being an isometry, is also (strongly) bicontinuous.

The fact that: $\overline{\mathcal{F}_s\left(\eta, \xi\right)} = \mathcal{F}_s\left(-\eta, -\xi\right)$ as well as that: $\widehat{W}^{\dagger}\left(\xi, \eta\right) = \widehat{W}\left(-\xi, -\eta\right)$ allows also to prove at once that the Weyl and Wigner maps *"preserve conjugation"*, i.e. that:

$$\Omega\left(\overline{f}\right) = \Omega\left(f\right)^{\dagger} \tag{6.76}$$

as well as:

$$\Omega^{-1}\left(\widehat{O}^{\dagger}\right) = \overline{\Omega^{-1}\left(\widehat{O}\right)} \tag{6.77}$$

Therefore, in particular, if f is real, then, as already mentioned, $\Omega\left(f\right)$ will be a *symmetric* operator.

As a final remark, we observe that Eq. (6.61) implies also:

$$Tr_x\left(\widehat{O}\right) =: \int dx \left\langle x \left| O \right| x \right\rangle = \int \frac{dqdp}{2\pi\hbar} \Omega^{-1}\left(\widehat{O}\right)\left(q, p\right) \tag{6.78}$$

(with the same result for the similarly defined $Tr_p\left(\widehat{A}\right)$) as well as, of course:

$$\int \frac{dqdp}{2\pi\hbar} f(q,p) = Tr(\Omega(f)) \tag{6.79}$$

and this defines formally a *"trace" operation* on phase space:

$$Tr(f) =: \int \frac{dqdp}{2\pi\hbar} f(q,p) \tag{6.80}$$

Of course, all these results will make sense when all the quantities in the previous equations are finite. For example, if: $\widehat{A} = P_\psi = |\psi\rangle\langle\psi|, \langle\psi|\psi\rangle = 1$ is a one-dimensional projector, then:

$$\Omega^{-1}(P_\psi)(q,p) = \int d\xi e^{ip\xi/\hbar} \langle q+\xi/2|\psi\rangle \langle\psi|q-\xi/2\rangle \tag{6.81}$$

and:

$$\int \frac{dqdp}{2\pi\hbar} \Omega^{-1}(P_\psi)(q,p) = \int dq \langle q|\psi\rangle \langle\psi|q\rangle = \|\psi\|^2 = 1 \tag{6.82}$$

As a less trivial example, in the case of the harmonic oscillator we find with some long but elementary algebra using Eq.(6.69):

$$Tr\left\{\Omega^{-1}\left(e^{-\beta\widehat{H}}\right)\right\} = \int \frac{dqdp}{2\pi\hbar} \Omega^{-1}\left(e^{-\beta\widehat{H}}\right) = \frac{1}{2\sinh(\beta\hbar\omega/2)} \tag{6.83}$$

which is the expected result [183] for the canonical partition function of a $1D$ harmonic oscillator.

Remark 52 *The mere existence of the phase-space trace of* $\Omega^{-1}\left(\widehat{O}\right)$*, i.e. finiteness of* $\int(dqdp/2\pi\hbar)\,\Omega^{-1}\left(\widehat{O}\right)(q,p)$ *does not however guarantee that* $\widehat{A}$ *be trace-class, as this requires, as already recalled [201], the more stringent condition that :*

$$Tr\left(\left|\widehat{O}\right|\right) < \infty, \quad \left|\widehat{O}\right| =: \sqrt{O^\dagger O} \tag{6.84}$$

and $\left|\widehat{O}\right|$ *is not connected to the Wigner function* $\Omega^{-1}\left(\widehat{O}\right)$ *in any simple manner.*

6.4 The Moyal Product

Working again for simplicity[96] in $\mathcal{S} \approx \mathbb{R}^2$, the Wigner map allows for the definition of a new algebra structure on the space of functions $\mathcal{F}\left(\mathbb{R}^2\right)$, the *Moyal "$*$"-product* [94, 189, 228], that is defined as:

$$f * g =: \Omega^{-1}\left(\widehat{\Omega}(f) \cdot \widehat{\Omega}(g)\right) \tag{6.85}$$

[96] We stress once again that extensions to higher dimensions are essentially straightforward.

(as, generically: $\widehat{\Omega}(f)\cdot\widehat{\Omega}(g)\neq\widehat{\Omega}(g)\cdot\widehat{\Omega}(f)$, it is clear that, again generically: $f*g\neq g*f$).

This product is *associative*[97] (as the algebra of operators is), it is *distributive* w.r.t. the sum[98] (as $\widehat{\Omega}(.)$ is linear), but it is *non-local* and *non-commutative.* Indeed:

$$(f*g)(q,p)=\iint\frac{dxdk}{2\pi\hbar}\exp\left\{-i\omega_0\left((x,k),(q,p)\right)/\hbar\right\}Tr\left[\widehat{\Omega}(f)\cdot\widehat{\Omega}(g)\widehat{W}^{\dagger}(x,k)\right] \tag{6.86}$$

and:

$$\begin{gathered}Tr\left[\widehat{\Omega}(f)\cdot\widehat{\Omega}(g)\widehat{W}^{\dagger}(x,k)\right]=\\=\int\tfrac{d\xi d\eta d\xi' d\eta'}{(2\pi)^2}\mathcal{F}_s(f)(\eta,\xi)\,\mathcal{F}_s(g)(\eta',\xi')\,Tr\left[\widehat{W}(\hbar\xi,\hbar\eta)\,\widehat{W}(\hbar\xi',\hbar\eta')\,\widehat{W}^{\dagger}(x,k)\right]\end{gathered} \tag{6.87}$$

Now:

$$\begin{gathered}Tr\left[\widehat{W}(\alpha,\beta)\,\widehat{W}(\sigma,\tau)\,\widehat{W}^{\dagger}(x,k)\right]=\\=2\pi\delta(\alpha+\sigma-x)\,\delta(\beta+\tau-k)\exp\left\{-i\left[\beta(\alpha+\sigma)+k(\sigma-x)\right]/2\hbar\right\}\end{gathered} \tag{6.88}$$

Hence:

$$\begin{gathered}Tr\left[\widehat{\Omega}(f)\cdot\widehat{\Omega}(g)\widehat{W}^{\dagger}(x,k)\right]=\\\int\tfrac{d\xi d\xi' d\eta d\eta'}{2\pi\hbar}\mathcal{F}_s(f)(\eta,\xi)\,\mathcal{F}_s(g)(\eta',\xi')\,e^{-i(\eta x-k\xi)/2}\delta(\xi+\xi'-x/\hbar)\,\delta(\eta+\eta'-k/\hbar)\end{gathered} \tag{6.89}$$

and, using the deltas to get rid of the ξ',η' integrations and the explicit form of the symplectic Fourier transforms:

$$\begin{gathered}Tr\left[\widehat{\Omega}(f)\cdot\widehat{\Omega}(g)\widehat{W}^{\dagger}(x,k)\right]=\\4\int\tfrac{dadbdsdt}{2\pi\hbar}f(a,b)\,g(s,t)\,e^{-i(sk-tx)/\hbar}\delta(k-2(t-b))\,\delta(x-2(s-a))\end{gathered} \tag{6.90}$$

Inserting this result into Eq.(6.86) we eventually obtain:

$$(f*g)(q,p)=4\int\frac{dadbdsdt}{(2\pi\hbar)^2}f(a,b)\,g(s,t)\exp\left\{-\frac{2i}{\hbar}\left[(a-q)(t-p)+(s-q)(p-b)\right]\right\} \tag{6.91}$$

or:

$$(f*g)(q,p)=4\int\frac{dadbdsdt}{(2\pi\hbar)^2}f(a,b)\,g(s,t)\exp\left\{2i\omega_0\left((q-a,p-b),(q-s,p-t)\right)/\hbar\right\} \tag{6.92}$$

and this exhibits explicitly the non-locality of the Moyal product.

It can be shown[99] that:

---

[97] $f*(g*h)=(f*g)*h$

[98] $f*(g+h)=f*g+f*h$

[99] See,e.g., Ref.[240] for details.

- The Moyal product can be recast in the form:

$$(f * g)(q,p) = \sum_{n,m=0}^{\infty} \left(\frac{i\hbar}{2}\right)^{n+m} \frac{(-1)^n}{n!m!} \left\{ \frac{\partial^{m+n} f(a,b)}{\partial a^m \partial b^n} \frac{\partial^{m+n} g(a,b)}{\partial a^n \partial b^m} \right\} |_{a=q,b=p} \tag{6.93}$$

and that:

- Eq.(6.93) can be rewritten in compact form as:

$$(f * g)(q,p) = f(q,p) \exp\left\{ \frac{i\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right] \right\} g(q,p) \tag{6.94}$$

Other equivalent forms of the Moyal product are:

$$(f * g)(q,p) = f\left(q + \frac{i\hbar}{2} \frac{\overrightarrow{\partial}}{\partial p}, p - \frac{i\hbar}{2} \frac{\overrightarrow{\partial}}{\partial q} \right) g(q,p) \tag{6.95}$$

or:

$$(f * g)(q,p) = f(q,p)\, g\left(q - \frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial p}, p + \frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial q} \right) \tag{6.96}$$

Remark 53 *All the above expressions for the Moyal product apply of course to functions that are regular enough for the right-hand side of the defining equations to make sense. In particular, they will hold when f, g are "Schwartzian" functions [203] in $S(\mathbb{R}^2)$, i.e. they are of class C^∞ and of fast decrease at infinity.*

The form (6.93) exhibits explicitly the Moyal product as a series expansion in powers of $\hbar$. To lowest order:

$$f * g = fg + \frac{i\hbar}{2}\{f,g\} + \mathcal{O}(\hbar^2) \tag{6.97}$$

where $\{.,.\}$ is the Poisson bracket. The Planck constant $\hbar$ acts then as a "deformation parameter" of the usual associative product structure on the algebra of functions, making the product non-commutative. Indeed, it can be seen, e.g., from the expansion of the exponential in Eq.(6.94), that terms proportional to even powers of $\hbar$ are symmetric under the interchange $f \leftrightarrow g$, but terms proportional to odd powers are *anti*symmetric, and this makes the product non-commutative.

Example 54

- *$f \equiv 1$ or $g \equiv 1$. Then:*

$$(1 * g)(q,p) = g(q,p), (f * 1)(q,p) = f(q,p) \tag{6.98}$$

- *$f = q$. Then, at least if $g \in S^{\infty}\left(\mathbb{R}^2\right)$:*

$$\begin{aligned}(q * g)(q,p) &= 4\int \tfrac{dadbdsdt}{(2\pi\hbar)^2} a g(s,t) \exp\left\{\tfrac{2i}{\hbar}\left[(a-q)(t-p)+(s-q)(p-b)\right]\right\} = \\ &= 4\int \tfrac{dadbdsdt}{(2\pi\hbar)^2} g(s,t)\left(q+\tfrac{i\hbar}{2}\tfrac{\partial}{\partial t}\right)\exp\left\{\tfrac{2i}{\hbar}\left[(a-q)(t-p)+(s-q)(p-b)\right]\right\}\end{aligned} \tag{6.99}$$

and, integrating by parts in the second integral and using the previous result:

$$(q * g)(q,p) = \left(q + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) g(q,p) \tag{6.100}$$

Then, in view of the symmetry properties of the various terms in the expansion of the Moyal product in powers of $\hbar$:

$$(g * q)(q,p) = \left(q - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) g(q,p) \tag{6.101}$$

- *In the same way, if $f = p$:*

$$(p * g)(q,p) = \left(p - \frac{i\hbar}{2}\frac{\partial}{\partial q}\right) g(q,p) \tag{6.102}$$

etc.

- *If $f = q, g = p$ (or viceversa), then, using, e.g., Eq. (6.95):*

$$(q * p)(q,p) = qp + \frac{i\hbar}{2}; (p * q)(q,p) = qp - \frac{i\hbar}{2} \tag{6.103}$$

Notice that Eq.(6.100) implies:

$$\widehat{\Omega}(q) \cdot \widehat{\Omega}(g) = \widehat{\Omega}(qg) + \frac{i\hbar}{2}\widehat{\Omega}\left(\frac{\partial g}{\partial p}\right) \tag{6.104}$$

and similarly for the others.

The generalization of these results, as well as of those of the following Subsections, to higher dimensions, i.e. to: $\mathcal{S} = \mathbb{R}^{2n}$ with $n > 1$, are straightforward, so we will omit details here.

6.5 The Moyal Bracket(s), "Moyal" Quantum Mechanics and the Quantum-Classical Transition

6.5.1 The Moyal Bracket

Using the Moyal product we can define the *Moyal Bracket* $\{.,.\}_M$ as:

$$\{.,.\}_M : \mathcal{F}\left(\mathbb{R}^2\right) \times \mathcal{F}\left(\mathbb{R}^2\right) \to \mathcal{F}\left(\mathbb{R}^2\right); \; \{f,g\}_M =: \frac{1}{i\hbar}\left(f * g - g * f\right) \tag{6.105}$$

Hence, in particular:

$$\{f,g\}_M = \{f,g\} + \mathcal{O}\left(\hbar^2\right) \tag{6.106}$$

where $\{.,.\}$ is the standard Poisson bracket[100].

Being defined in terms of an associative product, the Moyal bracket fulfills all the properties of a Poisson bracket (linearity, anti-symmetry and the Jacobi identity), and defines a new Poisson structure on the (non-commutative) algebra of functions with the Moyal product. In particular, just as for the ordinary Poisson brackets, the Jacobi identity implies:

$$\{f, g * h\}_M = \{f,g\}_M * h + g * \{f,h\}_M \tag{6.107}$$

i.e. that $\{f,.\}$ is a *derivation* (with respect to the $*$-product) on the algebra of functions. Writing down explicitly the second term in (6.106): $\{f,g\}_M = \{f,g\} + \hbar^2 \{f,g\}_2 + ...$, we obtain:

$$\{f,g\}_2(q,p) = \frac{1}{24}\left\{\frac{\partial^3 f}{\partial q^3}\frac{\partial^3 g}{\partial p^3} - 3\frac{\partial^3 f}{\partial p \partial q^2}\frac{\partial^3 g}{\partial q \partial p^2} + 3\frac{\partial^3 f}{\partial p^2 \partial q}\frac{\partial^3 g}{\partial q \partial q^2} - \frac{\partial^3 f}{\partial p^3}\frac{\partial^3 g}{\partial q^3}\right\} \tag{6.108}$$

Therefore, $\{f,g\}_M$ contains, besides first-order derivatives, third and higher-order derivatives, and, although it is a derivation on the algebra of functions with the " $*$ " product, it is *not* a vector field (while $\{f,.\}$ *is* a vector field). The reason for that is precisely that the Moyal bracket is non-local, and hence Willmore's theorem [231] connecting (inner) derivations with vector fields does not apply. It is only when f is *at most a quadratic polynomial* that $\{f,.\}_M$ becomes a derivation on the usual pointwise product. Indeed, if this is the case, the Moyal and Poisson brackets of f with other functions coincide. As a check, we see that, in simple cases, we obtain:

$$\{q,p\}_M = 1, \;\; \{q,g\}_M = \frac{\partial g}{\partial p}, \;\; \{p,g\}_M = -\frac{\partial g}{\partial q} \tag{6.109}$$

Using the definitions of the Weyl and Wigner maps we have, in general:

$$\{f,g\}_M = i\Omega^{-1}\left(\widehat{\Omega}(f) \cdot \widehat{\Omega}(g) - \widehat{\Omega}(g) \cdot \widehat{\Omega}(f)\right)/\hbar \tag{6.110}$$

[100] The difference between the Moyal and Poisson brackets is $\mathcal{O}(\hbar^2)$, and not $\mathcal{O}(\hbar)$ as one could expect, and that because the difference $f * g - g * f$ contains only *odd* powers of $\hbar$.

i.e.:

$$\left[\widehat{\Omega}\left(f\right),\widehat{\Omega}\left(g\right)\right]=-i\hbar\widehat{\Omega}\left(\{f,g\}_M\right) \tag{6.111}$$

In particular, using (6.109) (and: $\widehat{\Omega}\left(1\right)=\mathbb{I}$)[101]:

$$\left[\widehat{\Omega}\left(q\right),\widehat{\Omega}\left(p\right)\right]=-i\hbar\mathbb{I},\ \ \left[\widehat{\Omega}\left(q\right),\widehat{\Omega}\left(\mathcal{H}\right)\right]=-i\hbar\widehat{\Omega}\left(\partial\mathcal{H}/\partial p\right) \tag{6.112}$$

$$\left[\widehat{\Omega}\left(p\right),\widehat{\Omega}\left(\mathcal{H}\right)\right]=i\hbar\widehat{\Omega}\left(\partial\mathcal{H}/\partial q\right) \tag{6.113}$$

Unless f and/or g are *at most* quadratic, $\{f,g\}_M\neq\{f,g\}$. Therefore, the commutator of the quantum operators associated with observables on phases space is *not* (*"modulo"* a multiplicative constant) the quantum operator associated with the Poisson bracket [56]. Generically, it becomes so only to lowest order in $\hbar$, and reproduces the Ehrenfest theorem [69].

6.5.2 Quantum Mechanics in Phase Space

First of all, it is of some interest, in view of the relevant rôle they play in Quantum Mechanics, to see here which phase-space functions correspond to projection operators on the Hilbert space. The latter, that we will denote as $\widehat{\mathcal{P}}$, are completely characterized by:

- $$\widehat{\mathcal{P}}^2=\widehat{\mathcal{P}},\quad idempotency \tag{6.114}$$

- $$\widehat{\mathcal{P}}^\dagger=\widehat{\mathcal{P}},\quad self-adjointness \tag{6.115}$$

As to (6.115), this requires the associated Wigner function $\Omega^{-1}\left(\widehat{\mathcal{P}}\right)$ to be *real*. As to (6.114), this implies, in terms of the Moyal product (cfr. Eq. (6.85)):

$$\Omega^{-1}\left(\widehat{\mathcal{P}}^2\right)=\Omega^{-1}\left(\widehat{\mathcal{P}}\right)=\Omega^{-1}\left(\widehat{\mathcal{P}}\right)*\Omega^{-1}\left(\widehat{\mathcal{P}}\right) \tag{6.116}$$

Moreover:

$$Tr\left(\Omega^{-1}\left(\widehat{\mathcal{P}}\right)\right)=Tr(\widehat{\mathcal{P}}) \tag{6.117}$$

and: $Tr\left(\Omega^{-1}\left(\widehat{\mathcal{P}}\right)\right)$ will be finite iff $\widehat{\mathcal{P}}$ is a finite-rank projection operator.

Therefore:

<u>Projection operators</u> are represented in phase space by real, uniformly-bounded (cfr. Eq.(6.58)) functions satisfying:

$$f*f=f \tag{6.118}$$

[101] The minus sign in the first commutator stems from the fact that $\Omega\left(p\right)=-\widehat{P}$, i.e. ultimately from the fact that we are using the symplectic and not the ordinary Fourier transform.

and:

$$Tr\left(f\right) < +\infty \tag{6.119}$$

iff the associated projector is of finite rank. Density states will be represented in turn by real, again uniformly-bounded, phase-space functions $f\left(q,p\right)$ *satisfying:* $Tr\left(f\right) = 1$ and:

$$Tr\left(f * f\right) \leq 1 \tag{6.120}$$

As discussed in Chapt.4, Quantum Mechanics can (and should) be consistently described in the framework of the projective Hilbert space $P\mathcal{H}$. Once this is identified (via the Hermitian structure, see the discussion in Chapt.4) with the space of rank-one projectors, it is natural to pose eigenvalue problems not for vectors in the Hilbert space but for the associate rank-one projectors, i.e. in the form:

$$\widehat{O}\widehat{P} = \lambda\widehat{P};\ \widehat{P}^{\dagger} = \widehat{P},\ \widehat{P}^{2} = \widehat{P},\ Tr\widehat{P} = 1 \tag{6.121}$$

with $\widehat{O}$ an observable and $\lambda \in \mathbb{R}$ the corresponding eigenvalue[102]. Put in this form, the eigenvalue problem can be easily formulated on phase space. Indeed, denoting by simplicity as $f_{\widehat{O}} = \Omega^{-1}\left(\widehat{O}\right)$ the Wigner function associated with $\widehat{O}$, the equivalent phase-space formulation will be:

$$f_{\widehat{O}} * f = \lambda f;\ f * f = f,\ f \in L_2\left(T^{*}Q\right) \tag{6.122}$$

for a real (and uniformly-bounded) function f. This will qualify f as the Wigner function associated with a projection operator: $f = \Omega^{-1}\left(\widehat{P}\right)$, with: $Trf = 1$ if it corresponds to a pure state.

A superposition rule capturing also interference phenomena can be formulated in terms of Wigner functions [144, 145, 146, 147, 148] following the lines of the discussion of Sect.4.1. If we denote as f_0 the Wigner function associated with a reference (pure) state (see Sect.4.1 for more details) and as f_1, f_2 those associated with two orthogonal (i.e.: $f_1 * f_2 = 0$) pure states, then to the linear superposition with coefficients c_1 and c_2, $\left|c_1\right|^2 + \left|c_2\right|^2 = 1$, there corresponds the Wigner function associated with Eq.(4.15), namely:

$$f = \sum_{i,j=1}^{2} c_i c_j^{*} \frac{f_i * f_0 * f_j}{\sqrt{Tr\left(f_i * f_0 * f_j * f_0\right)}} \tag{6.123}$$

where the phase-space trace has been defined in Eq.(6.80).

Coming now to quantum evolution, an observable (a self-adjoint operator) $\widehat{O}$ will evolve in time as:

$$\widehat{O}\left(t\right) = \widehat{U}^{\dagger}\left(t\right) \cdot \widehat{O} \cdot \widehat{U}\left(t\right) \tag{6.124}$$

[102] To avoid unnecessary technical complications, we pose here the problem in the discrete spectrum. Also, the last condition in Eq.(6.121) can be relaxed in favor of P becoming then a not necessarily one-dimensional eigenprojector onto the subspace spanned by the eigenvalue λ.

where the evolution operator is given by:

$$\widehat{U}(t) = \exp\left(-it\widehat{H}/\hbar\right) \tag{6.125}$$

$\widehat{H}$ being the Hamiltonian operator. Denoting again the Wigner function associated with $\widehat{O}$ as $f_{\widehat{O}}$, and from the very definition of the Moyal product:

$$f_{\widehat{O}(t)} = f_{\widehat{U}^{\dagger}(t)\cdot\widehat{O}\cdot\widehat{U}(t)} = f_{\widehat{U}^{\dagger}(t)} * f_{\widehat{O}} * f_{\widehat{U}(t)} \tag{6.126}$$

Using the (formal) series expansion of the evolution operator (6.125) we can also write explicitly the evolution operator in phase space $f_{\widehat{U}(t)}$ as [12]:

$$f_{\widehat{U}(t)} = \exp_*\left(-itf_{\widehat{H}}/\hbar\right) =: \sum_{n=0}^{\infty}\frac{(-it/\hbar)^n}{n!}\left(f_{\widehat{H}}\right)_*^n \tag{6.127}$$

where $(.)_*^n$ stands for an n-fold star-product.

Now, to lowest order in t: $f_{\widehat{U}(t)} \approx 1 - (it/\hbar)\, f_{\widehat{H}}$ etc., and we obtain easily:

$$\frac{d}{dt}f_{\widehat{O}(t)} = \left\{f_{\widehat{O}(t)}, f_{\widehat{H}}\right\}_M \tag{6.128}$$

or, more generally:

$$\frac{d}{dt}f(t) = \left\{f(t), f_{\widehat{H}}\right\}_M; \quad f(0) = f \tag{6.129}$$

with f any suitable function (e.g., a square-integrable function) on phase space, leading to:

$$f(t) = \exp_*\left(itf_{\widehat{H}}/\hbar\right) * f * \exp_*\left(-itf_{\widehat{H}}/\hbar\right) \tag{6.130}$$

and this is the phase-space description of quantum dynamics. As the classical ($\hbar \to 0$) limit of the Moyal bracket is the Poisson bracket, Eqs.(6.128) and/or (6.129) reduce, in the classical limit, to the description of the dynamics in terms of Poisson brackets.

6.6 "Alternative" Quantum Mechanics and Their Classical Counterparts

We can begin by recalling a theorem due to Dirac (see [56] and [89] for a more general discussion) which states that, given an associative, non-Abelian and maximally non-commutative[103] algebra $\mathcal{A}$ with identity over $\mathbb{R}$ or $\mathbb{C}$, and defining a "Poisson bracket"[104] on $\mathcal{A}$ as a map:

$$\{.,.\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \tag{6.131}$$

[103] That is, such that [89] the derived algebra: $\mathcal{A}' = Span\{[a,b]\}; a, b \in \mathcal{A}$, together with the identity, spans the whole of $\mathcal{A}$.

[104] Having in mind the algebra of operators on a Hilbert space, Dirac [56] calls it a "Quantum Poisson bracket".

that is bilinear, antisymmetric, satisfies the Jacobi identity:

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \ \forall a, b, c \in \mathcal{A} \tag{6.132}$$

and acts as a derivation on the product on the algebra, i.e.:

$$\{a, bc\} = \{a, b\} \, c + b \, \{a, c\} \ \ \forall a, b, c \in \mathcal{A} \tag{6.133}$$

then[105] the Poisson bracket $\{a, b\}$ is necessarily proportional to the "standard" commutator $ab - ba$.

This theorem was actually one of the main motivations why, in Chapt.1, we discussed alternative approaches to Quantum Mechanics involving modifications of the Hermitian product or, equivalently, of the associative product between operators.

Sticking to this last approach, we consider now a "deformed" associative product between operators defined as:

$$\widehat{A} \underset{(\widehat{K})}{\cdot} \widehat{B} =: \widehat{A} \cdot \widehat{K} \cdot \widehat{B} \tag{6.134}$$

where $\widehat{A}, \widehat{B}$ are linear operators and $\widehat{K}$ is a fixed, positive operator which is also a constant of the motion. This leads to the definition of the "deformed" commutator:

$$\left[\widehat{A}, \widehat{B}\right]_{(\widehat{K})} =: \widehat{A} \underset{(\widehat{K})}{\cdot} \widehat{B} - \widehat{B} \underset{(\widehat{K})}{\cdot} \widehat{A} \tag{6.135}$$

which satisfies again the Jacobi identity[106].

Given then two phase-space functions f and g, Eq.(6.134) leads to the "deformed" Moyal product:

$$f \underset{(k)}{*} g = f * k * g \tag{6.136}$$

where:

$$k =: \Omega^{-1} \left(\widehat{K}\right) \tag{6.137}$$

is the Wigner function associated with the operator $\widehat{K}$, and to the "deformed" Moyal bracket:

$$\{f, g\}_{M,k} =: \frac{1}{i\hbar}(f \underset{(k)}{*} g - g \underset{(k)}{*} f) \equiv \frac{1}{i\hbar}(f * k * g - g * k * f) \tag{6.138}$$

and, of course:

$$\{f, g\}_{M,1} \equiv \{f, g\}_M \tag{6.139}$$

Remark 55 *Requiring the operator $\widehat{K}$ to be strictly positive is a necessary condition*[107] *for the definition of a sensible "deformed" Hermitian product on the*

[105]See, e.g., Ref.[240] for details of the proof.
[106]See Sect.1.2.2 for further details.
[107]See again Sect.1.2.2.

Hilbert space. If this is the case, then $\widehat{K}$ *is invertible and the new (associative) algebra structure defined by Eq.(6.136) will have an identity* e*, given now by the "$*$-inverse" of* k*:* $e = k^{*-1}$*, where:* $k^{*-1} =: \Omega^{-1}\left(\widehat{K}^{-1}\right)$ *(i.e.:* $k * k^{*-1} = k^{*-1} * k = 1$*). This is of course the counterpart of the fact that the inverse of* $\widehat{K}$*,* $\widehat{K}^{-1}$*, plays the rôle of the identity for the deformed associative product (6.134) on the algebra of operators.*

Again with reference to the discussion in Sect.1.2.2, and in particular to Eq.(1.71), we see that now the dynamics will be described, in phase space, by the equation:

$$\frac{d}{dt}f = \left\{f, f_{\widehat{H}'}\right\}_{M,k} \tag{6.140}$$

in such a way that (cfr. Eq.(6.129)):

$$\left\{f, f_{\widehat{H}'}\right\}_{M,k} = \left\{f, f_{\widehat{H}}\right\}_{M} \tag{6.141}$$

where the new Hamiltonian function will be given by:

$$f_{\widehat{H}'} = \Omega^{-1}\left(\widehat{H}\cdot\widehat{K}^{-1}\right) = \Omega^{-1}\left(\widehat{K}^{-1}\cdot\widehat{H}\right) = f_{\widehat{H}} * f_{\widehat{K}^{-1}} \tag{6.142}$$

Moreover (cfr. Eq.(1.74)), time evolution will act again as a derivation on the deformed algebra of functions, i.e.:

$$\frac{d}{dt}\left(f \underset{(k)}{*} g\right) = \frac{df}{dt} \underset{(k)}{*} g + f \underset{(k)}{*} \frac{dg}{dt}, \quad \forall f, g \tag{6.143}$$

Turning now to the classical limit and using Eq.(6.97), a simple computation shows that, for $\hbar \to 0$, Eq.(6.136) becomes:

$$f \underset{(k)}{*} g \simeq fkg + \frac{i\hbar}{2}\left\{f, g\right\}_k + \mathcal{O}\left(\hbar^2\right) \tag{6.144}$$

with a "deformed" bracket is given now by:

$$\left\{f, g\right\}_k = \lim_{\hbar\to 0}\left\{f, g\right\}_{M,k} \tag{6.145}$$

and, explicitly:

$$\left\{f, g\right\}_k = k\left\{f, g\right\} + f\left\{k, g\right\} - g\left\{k, f\right\} \tag{6.146}$$

(once again: $\left\{f, g\right\}_1 \equiv \left\{f, g\right\}$). Being defined in terms of an associative product, this new bracket[108] satisfies the Jacobi identity, but, at variance with the Poisson bracket and as it is clear from Eq.(6.146), $\left\{f, .\right\}_k$ fails to be (for fixed f) a derivation on the algebra of functions (it is not even zero on constant functions).

[108] Also called [129] a *Jacobi bracket.*

6.6.1 Alternative Moyal-like brackets

In Section we go back to the GNS construction for the finite-dimensional $\mathbb{C}^*$-algebra $\mathcal{B}(\mathbb{C}^n)$ we have discussed in 4.3. Recall that different states over $\mathcal{B}(\mathbb{C}^n)$ give rise to different representations and hence to different realizations of the corresponding Hilbert space. We have already noticed that any such state is represented by a positive $n \times n$ matrix K which can be used to define an alternative scalar product on $\mathbb{C}^n$ of the form

$$z \cdot_K w := \sum_{j,k=1}^{n} \bar{z}_j K_{jk} w_k \tag{6.147}$$

for any $z, w \in \mathbb{C}^n$. In turn, we can define a different multiplication rule in $\mathcal{B}(\mathbb{C}^n)$ by means of:

$$A \cdot_K B = A \cdot K \cdot B \tag{6.148}$$

for any $A, B \in \mathcal{B}(\mathbb{C}^n)$. This product is associative, so that $(\mathcal{B}(\mathbb{C}^n), \cdot_K)$ is a $\mathbb{C}^*$-algebra. Accordingly, we can define alternative Lie algebra and Jordan algebra structures via:

$$[A, B]_K := \frac{i}{2}(A \cdot_K B - B \cdot_K A) \tag{6.149}$$

$$A \circ_K B := \frac{1}{2}(A \cdot_K B + B \cdot_K A) \tag{6.150}$$

Let us consider now a quantum system whose dynamics is specified by a Hamiltonian H, yielding the standard Heisenberg equation:

$$i\hbar \dot{A} = [A, H] \tag{6.151}$$

Suppose that $[H, K] = H \cdot K - K\dot{H} = 0$. By setting $H_K = K^{-1} \cdot H$, one can easily verify that, for any for any $A \in \mathcal{B}(\mathbb{C}^n)$:

$$[A, H] = A \cdot_K H_K - H_K \cdot A = [A, H_K]_K \tag{6.152}$$

Hence we have an alternative Hesienberg-like description which makes use of the alternative product (6.148):

$$i\hbar \dot{A} = [A, H_K]_K \tag{6.153}$$

These alternative structures are therefore analogue to those we have examined in classical dynamics when we have studied bi-Hamiltonian systems.

We can analyze these structures also in terms of the Wigner-Weyl formalism introduced in the previous paragraphs. We already know (see Sect. 4.2.4) that on the space of Kähler functions on the projective space, $\mathcal{F}^{\mathbb{C}}(P\mathcal{H})$, we can define a star-product, that of formula (4.98), such that:

$$f_A \star f_B = f_{AB} \tag{6.154}$$

We can then define an antisymmetric star-bracket according to:

$$\{f,g\}_\star := \frac{1}{2i}(f \star g - g \star f) \tag{6.155}$$

for any $f, g \in \mathcal{F}^{\mathbb{C}}(P\mathcal{H})$, which yields the standard Poisson bracket in the classical limit. Now, it is known [206] that any associative local product in $\mathcal{F}^{\mathbb{C}}(P\mathcal{H})$ is of the form:

$$f \cdot_k g := f \cdot k \cdot g \tag{6.156}$$

for some $k \in \mathcal{F}^{\mathbb{C}}(P\mathcal{H})$, $k > 0$. With this product, we can now define an alternative $\star_k$-product and $\star_k$ Lie and Jordan brackets:

$$f_A \star_k f_B = f_A \star k \star f_B \tag{6.157}$$

$$\{f_A, f_B\}_{\star k} = \frac{1}{2i}(f \star_k g - g \star_k f) \tag{6.158}$$

$$f_A \circ_k f_B = \frac{1}{2}(f \star_k g + g \star_k f) \tag{6.159}$$

We are back here to the construction of "deformed" Moyal brackets we have discussed in the previous paragraph. We have already seen that, in the classical limit, we get:

$$\lim_{\hbar \to 0} \frac{1}{\hbar}\{f_A, f_B\}_{\star_k} = \{f,g\} + f\{k,g\} - g\{k,f\} := \{f,g\}_k \tag{6.160}$$

obtaining the standard Poisson bracket only if $k = 1$. In a similar way, we see that:

$$\lim_{\hbar \to 0} f \circ_k g = f \cdot k \cdot := f \cdot_k g \tag{6.161}$$

This shows that the alternative quantization schemes we have introduced in the previous paragraph depend on the associative products $\star_k$ one can define on the originally commutative algebra $k \in \mathcal{F}^{\mathbb{C}}(P\mathcal{H})$.

6.6.2 "Conformal" Poisson Tensors Associated with Deformed Moyal Products

From now on we will consider the case $\mathcal{S} = \mathbb{R}^{2n}$ for generic $n > 1$, the main reason being that most of what will be said becomes trivial for $n = 1$.

As discussed in previous Sections, assigning a Poisson bracket is equivalent to assigning a *bi-vector field,* i.e. a totally antisymmetric tensor of type $(2,0)$, the *Poisson tensor*, given, in local collective coordinates, as:

$$\Lambda = \frac{1}{2}\Lambda^{ij}\frac{\partial}{\partial \xi^i} \wedge \frac{\partial}{\partial \xi^j};\ \Lambda^{ij} + \Lambda^{ji} = 0 \tag{6.162}$$

and such that:

$$\{f,g\} = \Lambda\left(df, dg\right) \tag{6.163}$$

In general, on can define, on multivectors, a bracket, the *Schouten bracket* [194, 209], that associates to every pair X, Y of multivectors of ranks n and m respectively a multivector $[X, Y]_S$ of rank $n + m - 1$. Limiting ourselves to bi-vectors, if X and Y are monomials:

$$X = \chi_1 \wedge \chi_2, \quad Y = \eta_1 \wedge \eta_2 \tag{6.164}$$

(with the χ's and η's vector fields), then:

$$[X, Y]_S = [\chi_1, \eta_1] \wedge \chi_2 \wedge \eta_2 - [\chi_1, \eta_2] \wedge \chi_2 \wedge \eta_1 - [\chi_2, \eta_1] \wedge \chi_1 \wedge \eta_2 + [\chi_2, \eta_2] \wedge \chi_1 \wedge \eta_1 \tag{6.165}$$

It follows that, if f, g are functions:

$$\begin{aligned} [fX, gY]_S &= fg\,[X, Y]_S + \\ &+ f\,(L_{\chi_2} g)\,\chi_1 \wedge \eta_1 \wedge \eta_2 - f\,(L_{\chi_1} g)\,\chi_2 \wedge \eta_1 \wedge \eta_2 + \\ &+ g\,(L_{\eta_2} f)\,\chi_1 \wedge \chi_2 \wedge \eta_1 - g\,(L_{\eta_1} f)\,\chi_1 \wedge \chi_2 \wedge \eta_2 \end{aligned} \tag{6.166}$$

and then the Schouten bracket can be extended by linearity to arbitrary bi-vectors.

The Jacobi identity can be expressed in terms of the Poisson tensor as:

$$[\Lambda, \Lambda]_S = 0 \tag{6.167}$$

and this is equivalent, whenever the Poisson tensor is not degenerate and allows then for the definition of a symplectic two-form ω, to the closure of the latter.

Remark 56 *As, in dimension two, there are no non-vanishing tri-vector fields (and all two-forms are closed), it is clear why what we are saying here becomes essentially void in dimension two. There, every pair of bi-vector fields has a vanishing Schouten bracket.*

The "deformed" bracket (6.146) can be rewritten as:

$$\{f, g\}_k = \Lambda'\,(df, dg) + f L_{X_k} g - g L_{X_k} f \tag{6.168}$$

where: $X_k =: \{k, .\}$ is the Hamiltonian vector field associated with the function k, and:

$$\Lambda' = k\Lambda \tag{6.169}$$

is what is called [12, 13] a *conformal Poisson tensor* with *conformal factor* k. Equivalently:

$$\{f, g\}_k = \Lambda'\,(df, dg) + f\,\{k, g\} - g\,\{k, f\} \tag{6.170}$$

Due to the presence of the conformal factor, the Schouten bracket of the conformal Poisson tensor with itself does not vanish anymore. Instead [12]:

$$[\Lambda', \Lambda']_S = -2X_k \wedge \Lambda' \tag{6.171}$$

and also, as X_k is a Hamiltonian vector field:

$$L_{X_k} \Lambda' \equiv k L_{X_k} \Lambda = 0 \tag{6.172}$$

Remark 57 *The bracket (6.146) is $\mathbb{R}$-linear homogeneous in the conformal factor k. So, any two such brackets with conformal factors, say, k_1 and k_2, will give rise to a bracket of the same form (a "compatible" bracket, in this sense) with conformal factor: $k = k_1 + k_2$. This seems to imply that, in order to obtain non-compatible classical limits, one should introduce some amount of non-linearity. This can be done by using non-linearly related Poisson structures.*

Remark 58 *Extrapolating now the Jacobi bracket (6.168) to dimension one, one finds nonetheless an interesting consequence. In this case, and "a fortiori"', $\Lambda = \Lambda' \equiv 0$, and hence:*

$$\{f, g\}_k = f L_{X_k} g - g L_{X_k} f \tag{6.173}$$

If we consider a circle S^1 with angular coordinate $\varphi \in [0, 2\pi]$ and measure $d\varphi/2\pi$,consider periodic functions that can be expanded in Fourier series on the O.N. basis:

$$f_n = e^{in\varphi}, \quad n \in \mathbb{Z} \tag{6.174}$$

and take:

$$X_k = i\frac{\partial}{\partial\varphi} \tag{6.175}$$

then Eq.(6.173) yields at once:

$$\{f_n, f_m\}_k = (n - m)\, f_{n+m} \tag{6.176}$$

which is nothing but the classical conformal algebra [55] (i.e. the Virasoro algebra without central charge).

6.6.3 Conformal Poisson Brackets and the *KMS* Condition in Phase Space

We will consider here the algebra $\mathcal{A}$ of functions on phase space equipped with the $*$-product (the Moyal product for the time being) and with the associated bracket.

Evolution in time on this algebra is an automorphism of $\mathcal{A}$ described by Eqs.(6.129) and (6.130). In particular, the latter states that:

$$\mathcal{A} \ni f \to f(t) = \exp_*\left(itf_{\widehat{H}}/\hbar\right) * f * \exp_*\left(-itf_{\widehat{H}}/\hbar\right) \tag{6.177}$$

Let now ω be a state[109] on the algebra. *Correlation functions* will be in general of the form: $\omega(f(t) * g(t'))$, $f, g \in \mathcal{A}$. Time-translational invariance will be assumed [187] for equilibrium states [95]. Hence:

$$\omega(f(t) * g(t')) = \omega(f(t - t') * g) = \omega(f * g(t' - t)) \tag{6.178}$$

[109] i.e. [95] a linear functional that is *real, positive* and *normalized*, the latter condition being equivalent [95] to: $\omega(1) = 1$.

will be assumed throughout. In particular, setting $g = 1$ in Eq.(6.178), we obtain:

$$\omega\left(f\left(t\right)\right) \equiv \omega\left(f\right) \forall f, t \tag{6.179}$$

With any pair $f, g \in \mathcal{A}$ we can associate the correlation functions [187]:

$$\mathcal{G}_{fg}\left(t\right) = \omega\left(f\left(t\right) * g\right) \tag{6.180}$$

and:

$$\mathcal{F}_{fg}\left(t\right) = \omega\left(g * f\left(t\right)\right) \tag{6.181}$$

Making t into a complex variable, the state ω will be said to be a (Kubo, Martin, Schwinger) *KMS state* at (inverse) temperature β [3, 95, 96, 103, 117, 181, 187, 193] if:

- $\mathcal{G}_{fg}\left(t\right)$ is bounded and continuous in the strip: $-\hbar\beta \leq \operatorname{Im} t \leq 0$ and analytic inside the strip.
- The same for $\mathcal{F}_{fg}\left(t\right)$ but in the strip $0 \leq \operatorname{Im} t \leq \hbar\beta$ and:
- The two are connected by:

$$\mathcal{G}_{fg}\left(t\right) = \mathcal{F}_{fg}\left(t + i\hbar\beta\right), \quad -\hbar\beta < \operatorname{Im} t < 0 \tag{6.182}$$

Taking then boundary values on the real axis, we obtain the *KMS condition:*

$$\omega\left(f\left(t\right) * g\right) = \omega\left(g * f\left(t + i\hbar\beta\right)\right) \tag{6.183}$$

Remark 59 *In the operator language, the KMScondition is usually proved (at least for bounded operators), using the cyclic invariance of the trace [110, 187] for systems whose (thermodynamic) equilibrium states are described by the canonical ensemble or (with minor modifications) by the grand-canonical ensemble.*

Remark 60 *Although the KMS condition is usually stated for equilibrium states at non-zero temperature, there is a similar condition [103] characterizing the ground state(s) at zero temperature, namely that* $\mathcal{G}_{fg}\left(t\right)$ *be, for real times, the boundary value on the real axis of an entire function that is uniformly bounded for* $\operatorname{Im} t \leq 0$.

Noticing that:

$$\begin{aligned} f\left(t + i\hbar\beta\right) &= \exp_*\left(i(t + i\hbar\beta) f_{\widehat{H}}/\hbar\right) * f * \exp_*\left(-i(t + i\hbar\beta) f_{\widehat{H}}/\hbar\right) = \\ &= \exp_*\left(-\beta f_{\widehat{H}}\right) * f\left(t\right) * \exp_*\left(\beta f_{\widehat{H}}\right) \end{aligned} \tag{6.184}$$

and expanding the exponentials in the last expression::

$$f\left(t + i\hbar\beta\right) \simeq f\left(t\right) + i\hbar\beta\left\{f\left(t\right), f_{\widehat{H}}\right\}_M + \mathcal{O}\left(\hbar^2\right) \tag{6.185}$$

and, as: $\{.,.\}_M = \{.,.\}$ (the classical Poisson bracket) to lowest order in $\hbar$, we obtain the (correct) expansion:

$$f(t + i\hbar\beta) \simeq f(t) + i\hbar\beta \left\{ f(t), f_{\widehat{H}} \right\} + \mathcal{O}\left(\hbar^2\right) \tag{6.186}$$

and hence the *classical KMS condition* [3, 12, 13]:

$$\omega\left(\{f(t), g\}\right) = \beta\omega\left(g\left\{f(t), f_{\widehat{H}}\right\}\right) \tag{6.187}$$

Interchanging the rôles of f and g and taking differences, we obtain also:

$$\omega\left(\{f(t), g\}\right) = \frac{1}{2}\beta\omega\left(f(t)\left\{f_{\widehat{H}}, g\right\} - g\left\{f_{\widehat{H}}, f(t)\right\}\right) \tag{6.188}$$

Remark 61 *Setting $g = 1$ in Eq.(6.188) we obtain: $\omega\left(\left\{f_{\widehat{H}}, f(t)\right\}\right) = 0\ \forall f \in \mathcal{A}$. Adding then $(-1/2)\,\beta\omega\left(\left\{f_{\widehat{H}}, f(t)\,g\right\}\right) = 0$ to the r.h.s. of Eq.(6.188) we re-obtain Eq.(6.187), and the two are therefore equivalent.*

Noticing further that:

$$\frac{1}{2}\beta\left\{f_{\widehat{H}}, .\right\} \equiv -e^{(1/2)\beta f_{\widehat{H}}}\left\{e^{-(1/2)\beta f_{\widehat{H}}}, .\right\} \tag{6.189}$$

we can rewrite Eq.(6.188) in the form:

$$\begin{aligned}\omega\Big(e^{(1/2)\beta f_{\widehat{H}}}\Big[e^{-(1/2)\beta f_{\widehat{H}}}\{f(t), g\} \quad &+ \quad f(t)\left\{e^{-(1/2)\beta f_{\widehat{H}}}, g\right\} \\ &- \quad g\left\{e^{-(1/2)\beta f_{\widehat{H}}}, f(t)\right\}\Big]\Big) = 0\end{aligned} \tag{6.190}$$

Comparison with Eq.(6.170) shows then that:

The classical KMS condition (6.187) is equivalent to the condition

$$\omega\left(e^{(1/2)\beta f_{\widehat{H}}}\{f(t), g\}_k\right) = 0 \ \ \forall f, g \in \mathcal{A} \tag{6.191}$$

where the bracket on the l.h.s. of Eq.(6.191) is the conformal bracket (6.170) with conformal factor

$$k = \exp\left(-(1/2)\,\beta f_{\widehat{H}}\right) \tag{6.192}$$

We turn now to the full quantum case (i.e. away from the limit $\hbar \to 0$). Define (cfr. Eqs.(6.192)and (6.130)):

$$k_\beta =: \exp_*\left(-(1/2)\,\beta f_{\widehat{H}}\right) \tag{6.193}$$

where:

$$\exp_* f =: 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{f * f * ... * f}_{n\ times} \tag{6.194}$$

whose $*$-inverse is $k_{-\beta}$. This defines the automorphism:

$$\sigma : f \rightarrow \sigma(f) = f(i\hbar\beta/2) = k_\beta * f * k_{-\beta} \tag{6.195}$$

(notice that: $\sigma(f * g) = \sigma(f) * \sigma(g)\, \forall f, g$) and the KMS condition (6.183) can be written as:

$$\omega(f(t) * g) = \omega\left(g * \sigma^2(f(t))\right) \tag{6.196}$$

Substituting now $\sigma(g)$ for g in Eq.(6.196) we find:

$$\omega\left(\sigma(g) * \sigma^2(f(t))\right) = \omega(\sigma(g * \sigma(f(t)))) = \omega(g * \sigma(f(t))) \tag{6.197}$$

the last passage following from time-translational invariance[110] (Eq.(6.179)) and, eventually:

$$\omega(f(t) * \sigma(g)) = \omega(g * \sigma(f(t))) \tag{6.198}$$

or, in terms of the deformed Moyal bracket (6.138) with deformation factor $k = k_\beta$:

$$\omega\left(\{f(t), g\}_{M,k_\beta} * k_\beta^{-1}\right) = 0 \tag{6.199}$$

But:

$$\{f(t), g\}_{M,k_\beta} * k_\beta^{-1} = \sigma\left[k_{-\beta} * \{f(t), g\}_{M,k_\beta}\right] \tag{6.200}$$

and, using again Eq.(6.179) , we obtain eventually [12, 13]:

$$\omega\left(k_{-\beta} * \{f(t), g\}_{M,k_\beta}\right) \equiv \omega\left(\exp_*\left((1/2)\,\beta f_{\widehat{H}}\right) * \{f(t), g\}_{M,k_\beta}\right) = 0 \tag{6.201}$$

which is the quantum version of the classical KMS condition, with exponentials replaced by "$*$-exponentials" and (deformed) Poisson brackets replaced by (deformed) Moyal brackets.

[110] If time-translational invariance is not assumed, then Eq.(6.196) leads, setting $g = 1$, to: $\omega((\sigma^2 - 1)f) = 0$. As what is needed to complete the argument is instead the condition (see below): $\omega((\sigma - 1)f) = 0$, one has then to assume [12] the mapping $\sigma + 1$ to be invertible.

7 Additional Topics and Concluding Remarks

7.1 Some Generalizations

Weyl systems, the way we have presented them, have been built with the use of a specific prescription (whose basic ingredients (see Chapt.5) are a vector spaces E and a symplectic structure over E) to deal with a specific prescription for the ordering problem that arises in the quantization procedure, one that is known as the "Weyl ordering" prescription (see Sect.6.1).

To deal with other ordering prescriptions that are available in the literature (say, normal, antinormal or other "s-ordering" prescriptions (see, e.g. Ref.[121]) one has to enlarge slightly the setting of Weyl systems.

Consider then a symplectic vector space with symplectic form $\omega(.,.)$, equipped however with an additional complex structure and therefore (see Chapt.4) with an Hermitian structure $\langle .|. \rangle$. In this way, having the Hermitian structure at hand, one can replace the (conventional) Weyl map, i.e.:

$$\widehat{W}(\mathbf{v}_1)\,\widehat{W}(\mathbf{v}_2)\,\widehat{W}^{-1}(\mathbf{v}_1)\,\widehat{W}^{-1}(\mathbf{v}_2) = e^{-i\omega(\mathbf{v}_1,\mathbf{v}_2)}\mathbb{I} \tag{7.1}$$

with the following one:

$$\widehat{W}(\mathbf{v}_1)\,\widehat{W}(\mathbf{v}_2)\,\widehat{W}^{-1}(\mathbf{v}_1)\,\widehat{W}^{-1}(\mathbf{v}_2) = e^{-\langle \mathbf{v}_1|\mathbf{v}_2\rangle}\mathbb{I} \tag{7.2}$$

Here the r.h.s. is no more a unitary transformation, i.e. an element of $U(1)$, but it is instead an element of $\mathbb{C}_0 \equiv U(1) \times \mathbb{R}_+$.

More generally, by splitting, as we have done repeatedly, the Hermitian structure into its real and imaginary parts: $\langle .|. \rangle = g(.,.) + i\omega(.,.)$, it is possible to consider a further generalization by setting:

$$\widehat{W}(\mathbf{v}_1)\,\widehat{W}(\mathbf{v}_2)\,\widehat{W}^{-1}(\mathbf{v}_1)\,\widehat{W}^{-1}(\mathbf{v}_2) = e^{-sg(\mathbf{v}_1,\mathbf{v}_2)-i\omega(\mathbf{v}_1,\mathbf{v}_2)}\mathbb{I} \tag{7.3}$$

with the "deformation parameter" s taking values in $[-1,1]$. This kind of generalization can become quite useful in dealing with the problem of second quantization (see below Sect.7.4).

Remark 62 *Notice that, the metric tensor g having been replaced by sg, the link between the real and imaginary parts of the Hermitian structure and the complex structure gets lost here for all $s \neq \pm 1$.*

From our point of view, this kind of generalization raises a new problem concerning the Moyal product. Namely, besides the bi-differential operator

$$\exp\left[i\left(\overleftarrow{\frac{\partial}{\partial x^\mu}} \wedge \overrightarrow{\frac{\partial}{\partial p_\mu}}\right)\right] \tag{7.4}$$

we will be forced to consider in addition also the bi-differential operator

$$\exp\left[s\left(\delta^{\mu\nu}\frac{\overleftarrow{\partial}}{\partial x^{\mu}}\otimes\frac{\overrightarrow{\partial}}{\partial x^{\nu}}+\delta_{\mu\nu}\frac{\overleftarrow{\partial}}{\partial p_{\mu}}\otimes\frac{\overrightarrow{\partial}}{\partial p_{\nu}}\right)\right] \tag{7.5}$$

In the framework of our "deformation" construction, and with reference to the discussion of Nijenhuis operators and of the Hochschild cohomology that is summarized in App.A, it is possible however to show that these additional terms do not change the cohomology class of the algebra we obtain by using only the Poisson tensor, i.e. the bi-differential operator (7.4), as the following example shows.

Example 63 *To illustrate the situation, it will be enough to consider the new product on functions defined on R along with the deformation of the usual pointwise product. We can consider then the bilinear map:*

$$(f,g)\to\text{"}f*g\text{"}:=f\exp\left\{-s\frac{\overleftarrow{\partial}}{\partial x}\otimes\frac{\overrightarrow{\partial}}{\partial x}\right\}g \tag{7.6}$$

Now, it is possible to show that the linear map T defined by:

$$T=\exp\left\{-\frac{s}{2}\frac{\partial^2}{\partial x^2}\right\} \tag{7.7}$$

is such that:

$$\text{"}f*g\text{"}=T\left(f\cdot g\right)-T\left(f\right)\cdot g-f\cdot T\left(g\right) \tag{7.8}$$

(with the dot denoting the usual pointwise product), thus proving (see again App.A) that the bilinear map (7.6) is indeed a coboundary in the Hochschild cohomology of the algebra of functions with the pointwise product.

7.2 Pseudo-Hermitian Quantum Mechanics

It is appropriate at this point of our exposition to mention that many aspects of our mathematical considerations have also appeared in a setting that has a completely different origin, namely the field of pseudo-Hermitian Quantum Mechanics. Pseudo-Hermitian Quantum Mechanics (PHQM) is an attempt to generalize Quantum Mechanics due mainly to C.M.Bender and collaborators (see, e.g., [15] and references therein). One starts with a Hilbert space equipped with an Hermitian product $\langle\cdot,,\cdot\rangle$ and a Hamiltonian H which is diagonalizable but is not Hermitian, i.e., in general:

$$\langle\psi,H\phi\rangle\neq\langle H\psi,\phi\rangle \tag{7.9}$$

We shall assume for simplicity the spectrum of H to be entirely discrete, this meaning that the eigenvalue equation

$$H|\psi_n\rangle=\lambda_n|\psi_n\rangle \tag{7.10}$$

admits of a complete set $\{|\psi_n\rangle\}_n$ of eigenfunctions which cannot, in general, be chosen to be orthonormal. Suppose in addition that $\{|\psi_n\rangle\}_n$ admits of a bi-orthonormal extension $\{|\psi_n\rangle\}_n, |\phi_n\rangle\}_n$, i.e. that there exists another complete set $\{|\phi_n\rangle\}_n$ such that[111]:

$$\langle\phi_m|\psi_n\rangle = \delta_{mn} \tag{7.11}$$

Notice that this implies:

$$\langle\phi_m|H\psi_n\rangle = \lambda_n\delta_{mn} \tag{7.12}$$

which implies in turn:

$$(\lambda_m\langle\phi_m| - \langle\phi_m|H)\,|\psi_n\rangle = 0\ \forall n \tag{7.13}$$

and hence:

$$\langle\phi_m|H = \lambda_m\langle\phi_m| \tag{7.14}$$

i.e. that the $\langle\phi_m|$'s are (a complete set of) *left* eigenvectors of H.

Then one has a resolution of identity:

$$\mathbb{I} = \sum_n |\psi_n\rangle\langle\phi_n| = \sum_n |\phi_n\rangle\langle\psi_n| \tag{7.15}$$

Now one defines a new operator η:

$$\eta = \sum |\phi_n\rangle\langle\phi_n| \tag{7.16}$$

which can be easily shown [188] to be invertible, with inverse

$$\eta^{-1} = \sum |\psi_n\rangle\langle\psi_n| \tag{7.17}$$

and positive. Thus one can define a new new Hermitian product that will be related to the original one by:

$$h(\cdot,\cdot) = \langle\cdot,,\eta\cdot\rangle \tag{7.18}$$

In other words, η is a positive operator that behaves as a (1, 1)-type tensor connecting the new and the old metrics. The latter is then used only to identify the topology of the vector space of states, which turns out to be equivalent [188] to the one defined by the new scalar product.

It is immediate to see that: (i) the complete set of eigenfunctions $\{|\psi_n\rangle\}_n$ becomes orthonormal w.r.t. $h(\cdot,\cdot)$, $h(\psi_n,\psi_m) = \delta_{nm}$ and: (ii) the Hamiltonian H becomes Hermitian, i.e.:

$$h(\psi, H\phi) = h(H\psi,\phi) \tag{7.19}$$

[111] Such a set always exists provided $\{|\psi_n\rangle\}_n$ is a Riesz basis, i.e. provided one can find a bounded invertible operator A and an orthonormal basis $\{|\chi_n\rangle\}_n$ such that $|\psi_n\rangle = A|\chi_n\rangle$. Indeed in this case one has: $|\psi_n\rangle = \sum_m A_{mn}|\chi_n\rangle$ with $A_{mn} = \langle\chi_m|A\chi_n\rangle$ and can set: $|\phi_m\rangle = \sum_j (A^{-1})^*_{jm}|\chi_j\rangle$.

provided that

$$H^\dagger = \eta H \eta^{-1} \tag{7.20}$$

which is true iff the spectrum of H is real, as one can easily find after checking that: $H = \sum_m \lambda_m |\psi_m\rangle\langle\phi_m|$, while: $\eta H \eta^{-1} = \sum_m \lambda_m |\phi_m\rangle\langle\psi_m|$ and: $H^\dagger = \sum_m \lambda_m^* |\phi_m\rangle\langle\psi_m|$. Hermiticity of H w.r.t. to the new Hermitian product implies of course that $h(\cdot,\cdot)$ is preserved by the dynamical evolution (while $\langle\cdot,,\cdot\rangle$ is not). It is clear that, from our point of view, the problem appears as a sort of inverse problem, i.e. the problem of determining all Hermitian products which are preserved by the flow defined by the Hamiltonian H. Clearly, once a solution has been found, there exist many others that can be found by using appropriate operators in the commutant of H. Indeed, if A is such that $[A,H]=0$, then:

$$(\eta A) H (\eta A)^{-1} \equiv \eta H \eta^{-1} = H^\dagger \tag{7.21}$$

and this defines the new Hermitian product:

$$h_A(\cdot,\cdot) = h(\cdot, A\cdot) = \langle \cdot,, \eta A \cdot\rangle \tag{7.22}$$

The appropriate conditions on A will be that it be invertible and that ηA be still a positive operator, and the conditions on H, namely that it be diagonalizable with a real and discrete spectrum, appear simply as conditions for the inverse problem to have a solution (and hence in general many others). Thus, while in the usual approach one fixes a Hilbert space (and hence an Hermitian product) and looks for observables and unitary evolution, in PHQM it is the dynamical evolution that is given, and one looks for the Hermitian products that are preserved by the evolution. Recalling our discussion of Sect. 1.2, we notice also that the new scalar product $h(\cdot,\cdot)$ induces a new associative product between operators:

$$A \cdot_\eta B = A \eta B \tag{7.23}$$

It is clear that even if $[A,B]=0$ then $[A,B]_\eta = A\eta B - B\eta A \neq 0$ in general. For example, if both A and B admits the following decomposition in term of the bi-orthonormal system:

$$A = \sum_n a_n |\psi_n\rangle\langle\phi_n| \ \ , \ \ B = \sum_n b_n |\psi_n\rangle\langle\phi_n| \tag{7.24}$$

so that $[A,B]=0$, one has:

$$[A,B]_\eta = \sum_{mn} (a_m b_n - a_n b_m) \langle\phi_m|\phi_n\rangle |\psi_m\rangle\langle\phi_n| \tag{7.25}$$

which is not zero since not all $\langle\phi_m|\phi_n\rangle$ are necessarily zero. When operators with continuous spectra are involved, it may be the case that the Hermitian products rendering the Hamiltonian Hermitian need not induce commutation relations for which the operator is localizable. By this we mean that the position operators need not commute w.r.t. the new associative product that has been induced on the operators.

Let us end this section by giving a simple example of a pseudo-hermitian operator [116] . We consider the Hilbert space $L^2([0,d])$ with the standard scalar product $\langle \cdot, \cdot \rangle$ and an operator H_α defined on twice (weakly) differentiable functions in $L^2([0,d])$ given by the quadratic form:

$$h_\alpha(\phi, \psi) = \langle \phi', \psi' \rangle + i\alpha\phi(d)^*\psi(d) - i\alpha\phi(0)^*\psi(0) \tag{7.26}$$

where α is any real number. Some straightforward algebra shows that the eigenvalue problem admits the following solutions:

$$\psi_0(x) = \quad A_0 \exp(-i\alpha x) \qquad \lambda_0 = \alpha^2 \tag{7.27}$$

$$\psi_j(x) = \quad A_j \left[\cos(k_j x) - i\tfrac{\alpha}{k_j}\sin(k_j x)\right] \quad \lambda_j = k_j^2 \tag{7.28}$$

$$\lambda_j = k_j^2\ ,\ k_j = j\frac{\pi}{d}\ ,\ j = 1, 2, \cdots$$

provided that $\alpha d/\pi \notin \mathbb{Z} - \{0\}$. It also easy to see that $H_\alpha^\dagger = H_{-\alpha}$ and its eigenfunctions and eigenvalues are given by:

$$\phi_0(x) = \quad B_0 \exp(i\alpha x) \qquad \lambda_0 = \alpha^2 \tag{7.29}$$

$$\phi_j(x) = \quad B_j \left[\cos(k_j x) + i\tfrac{\alpha}{k_j}\sin(k_j x)\right] \quad \lambda_j = k_j^2 \tag{7.30}$$

$$\lambda_j = k_j^2\ ,\ k_j = j\frac{\pi}{d}\ ,\ j = 1, 2, \cdots$$

Both the sets $\{|\psi_n\rangle\}_{n=0}^\infty$ and $\{|\phi_n\rangle\}_{n=0}^\infty$ are complete [116] and the coefficients A_n, B_n can be chosen so that

$$\langle \phi_j | \psi_k \rangle = \delta_{jk} \tag{7.31}$$

which shows that $\{|\psi_n\rangle, |\phi_n\rangle\}_{n=0}^\infty$ is a bi-orthonormal basis. Thus the invertible positive operator η_α, that can now be used to define a new scalar product w.r.t. which H_α becomes hermitian, assumes the form:

$$\eta_\alpha = \sum_{j=0}^\infty \langle \phi_j, \cdot \rangle \phi_j \tag{7.32}$$

In ref. [116] it is shown that it can be recast in the following form:

$$\eta_\alpha = \mathbb{I} + \langle \phi_0, \cdot \rangle \phi_0 + \theta_0 + i\alpha\theta_1 + \alpha^2\theta_2 \tag{7.33}$$

where, for any $\psi(x) \in L^2([0,d])$:

$$(\theta_0\psi)(x) := -\frac{1}{d}(J\psi)(d)$$

$$(\theta_1\psi)(x) := 2(J\psi)(x) - \frac{x}{d}(J\psi)(d) - \frac{1}{d}(J^2\psi)(d)$$

$$(\theta_2\psi)(x) := -(J^2\psi)(x) + \frac{x}{d}(J^2\psi)(d)$$

with

$$(J\psi)(x) := \int_0^x dx\psi(x) \tag{7.34}$$

which allows to prove explicitly that indeed η_α is bounded, invertible and positive.

7.3 The Rôle of Linear Structures in Statistical and Quantum Mechanics

7.3.1 "Reformulating" the Von Neumann Theorem

In Sects.3.3.2 and 3.3.3 we have examined the situation in which it is possible to define alternative linear structures at the classical level. We will examine now the quantum case.

In general, if two non-linearly related linear structures (and associated symplectic forms) are available for a classical system, then one can set up two different Weyl systems realized on two different Hilbert space structures made of functions defined on the same Lagrangian subspace (see the example below) but anyhow with different Lebesgue measures. These two Lebesgue measures, call them $d\mu$ and $d\mu'$, will be associated with different actions of the Abelian vector group of translations that are not linearly related. When compared by writing both in the same coordinate system they will not be simply proportional with a constant proportionality factor. Functions that are square-integrable in one setting need not be such in the other. Moreover, a necessary ingredient in the Weyl quantization program is the use of the (standard or symplectic) Fourier transform. For the same reasons as outlined above, it is clear that the two different linear structures will define genuinely different Fourier transforms.

In this way one can "evade" the uniqueness part of von Neumann's theorem. What the present discussion is actually meant at showing is that there are assumptions, namely that the linear structure (and symplectic form) are given once and for all and are unique, that are implicitly assumed but not explicitly stated in the usual formulations of the theorem, and that, whenever more structures are available, the situation can be much richer and lead to genuinely and non-equivalent (in the unitary sense) formulations of Quantum Mechanics.

Let us illustrate these considerations by going back to the example of the $1D$ harmonic oscillator that has been discussed in Sect.3.3.2. To quantize this system according to the Weyl scheme we have first of all to select a Lagrangian subspace $\mathcal{L}$ of $\mathbb{R}^2$ and a Lebesgue measure $d\mu$ on it defining then $L^2(\mathcal{L}, d\mu)$. When we endow $\mathbb{R}^2$ with the standard linear structure $\Delta = q\partial/\partial q + p\partial/\partial p$, we can choose $\mathcal{L} = \{(q,0)\}$ and $d\mu = dq$. Consider now, e.g., the change of coordinates: $\phi : (q,p) \leftrightarrow (Q,P)$ defined by [66]:

$$q = Q\left(1 + \lambda R^2\right),\ p = P\left(1 + \lambda R^2\right) \tag{7.35}$$

parametrized by: $\lambda \geq 0$ and where: $R^2 = Q^2 + P^2$. Eqs.(7.35) invert to $\left(r = \sqrt{q^2 + p^2}\right)$:

$$Q = qK\left(r\right),\ P = pK\left(r\right) \tag{7.36}$$

where K is a positive function, the (unique) real solution of the equation[112]:

$$\lambda r^2 K^3 + K - 1 = 0 \tag{7.37}$$

Now we can consider the linear structure defined by: $\Delta' = Q\partial/\partial Q + P\partial/\partial P$ and take: $\mathcal{L}' = \{(Q, 0)\}$ and: $d\mu' = dQ$.

Notice that $\mathcal{L}$ and $\mathcal{L}'$ are the same subset of $\mathbb{R}^2$, defined by the conditions $P = p = 0$ and with the coordinates related by the relation $Q = qK(r = |q|)$. Nevertheless the two Hilbert spaces $L^2(\mathcal{L}, d\mu)$ and $L^2(\mathcal{L}', d\mu')$ are not related via a unitary map since the Jacobian of the coordinate transformations is not constant[113]

As a second step in the Weyl scheme, we construct in $L^2(\mathcal{L}, d\mu)$ the operator $\hat{U}(\alpha)$:

$$\left(\hat{U}(\alpha)\psi\right)(q) = e^{i\alpha q/\hbar}\psi(q)\ ,\ \psi(q) \in L^2(\mathcal{L}, d\mu), \tag{7.38}$$

whose generator is $\hat{x} = q$, and the operator $\hat{V}(h)$:

$$\left(\hat{V}(h)\psi\right)(q) = \psi(q+h)\ \psi(q) \in L^2(\mathcal{L}, d\mu), \tag{7.39}$$

which is generated by $\hat{\pi} = -i\hbar\partial/\partial q$. The quantum Hamiltonian can be written as $H = \hbar\left(a^\dagger a + \frac{1}{2}\right)$ where $a = (\hat{x} + i\hat{\pi})/\sqrt{2}\hbar$ (here the adjoint is taken with respect to the complex structure compatible with the Lebesgue measure $d\mu$). Similar expressions hold in $L^2(\mathcal{L}', d\mu')$, and we will obtain unitary operators $\hat{U}'(\alpha)$, $\hat{V}'(h)$ with infinitesimal generators: $\widehat{X} = Q$ and: $\widehat{\Pi} = -i\hbar\partial/\partial Q$. Notice that, when seen as an operator in the previous Hilbert space, $\hat{V}'(h)$ implements [66] translations with respect to the linear structure defined, in the notation of Sect.3.3.2 by:

$$(\hat{V}'(h)\psi)(q) = \psi(q +_{(\phi)} h). \tag{7.40}$$

Denoting as usual with a dagger but also with an additional prime the adjoints taken with respect to the complex structure compatible with the Lebesgue measure $d\mu'$, the quantum Hamiltonian will be now: $H' = \hbar\left(A^{\dagger\prime}A + \frac{1}{2}\right)$ with $A = (\widehat{X} + i\widehat{\Pi})/\sqrt{2}\hbar$.

It is interesting to notice that, in the respective Hilbert spaces: $[a, a^\dagger] = \mathbb{I}$ as well as: $[A, A^{\dagger\prime}] = \mathbb{I}$, so that we obtain two different and not linearly related realizations of the Heisenberg algebra.

In terms of the "uppercase" variables, we obtain [66] with some algebra:

$$\widehat{x} = (1 + \lambda\widehat{X}^2)\widehat{X} \tag{7.41}$$

and:

$$\widehat{\pi} = (1 + 3\lambda\widehat{X}^2)^{-1}\widehat{\Pi} \tag{7.42}$$

[112] Eq.(7.37) below shows that, actually: $K = K\left(\lambda r^2\right)$. K is monotonically decreasing for $\lambda \geq 0$ and: $\lambda = 0 \leftrightarrow K \equiv 1$, while: $K \underset{\lambda\to\infty}{\approx} \left(\lambda r^2\right)^{-1/3}$.

[113] In fact: $d\mu = (1 + 3\lambda Q^2)d\mu'$.

so, while the position operator $\widehat{x}$ will be self-adjoint with respect to both measures, the conjugate momentum operator will be not, and indeed, while: $\widehat{x}^\dagger = \widehat{x}^{\dagger\prime} = \widehat{x}$ and: $\widehat{\pi}^\dagger = \widehat{\pi}$, we obtain instead [66]:

$$\widehat{\pi}^{\dagger\prime} = \widehat{\pi} - 6i\lambda\widehat{X}(1 + 3\lambda\widehat{X}^2)^{-2} \tag{7.43}$$

Thus, the C^*-algebra generated by $\hat{x}, \hat{\pi}, \mathbf{I}$ seen as operators acting on $L^2(\mathcal{L}, d\mu)$ is closed, whereas the one generated by $\hat{x}, \hat{\pi}, \mathbf{I}$ and their adjoints $\hat{x}^{\dagger\prime}, \hat{\pi}^{\dagger\prime}, \mathbf{I}^{\dagger\prime}$ acting on $L^2(\mathcal{L}', d\mu')$ does not close because we generate new operators whenever we consider the commutator between $\hat{\pi}$ and $\hat{\pi}^{\dagger\prime}$. As a consequence, the operators $\hat{x}, \hat{\pi}$ and $\hat{x}', \hat{\pi}'$ close on the Heisenberg algebra only if we let them act on two different Hilbert spaces generated, respectively, by the sets of the Fock states

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \tag{7.44}$$

$$|N\rangle = \frac{1}{\sqrt{N!}}(A^{\dagger\prime})^N|0\rangle. \tag{7.45}$$

7.3.2 Alternative Descriptions and Statistical Mechanics

By further considering the example of the $1D$ harmonic oscillator, we would like to examine whether alternative Hamiltonian descriptions do lead to the same thermodynamical description of a given system.

Let us start from the classical case, when the symplectic form can be rewritten on $\mathbf{R^2} - \{\mathbf{0}\}$ as:

$$\omega = dp \wedge dq = dH \wedge \xi \tag{7.46}$$

with:

$$\xi = dt = \frac{pdq - qdp}{2H} \tag{7.47}$$

and the "time function" t will be given by: $t = (1/\omega)\tan^{-1}\{m\omega q/p\}$, which emphasizes its local character. Thus $\mathbf{R^2} - \{\mathbf{0}\}$ can be identified with $\mathbf{S^1} \times \mathbf{R}^+$ parametrized by dH and dt. The associated canonical[114] partition function is easily evaluated, and the well-known result [187] is:

$$\mathcal{Z} = h^{-1}\int_{\mathbf{R^2}} \exp\{-\beta H\}\omega = h^{-1}\int_0^\infty dE\exp\{-\beta E)\int_{\Sigma(E)} dt = \frac{1}{\beta\hbar\omega} \tag{7.48}$$

Here $\Sigma(E)$ denotes the one-dimensional "surface" of constant energy E, $\beta = 1/k_B T$ with T the (absolute) temperature and k_B the Boltzmann constant,

[114]We will restrict here to the canonical ensemble of (both classical and quantum) Statistical Mechanics.

while h (and: $\hbar = h/2\pi$) is a numerically undetermined constant with the dimension of an action[115].

In order to keep track of the correct dimensions of the various physical quantities involved, let's consider a new Hamiltonian of the form:

$$H_f = \beta_0^{-1} f(\beta_0 H) \tag{7.49}$$

where β_0 is a "fiducial" quantity, fixed once and for all and having dimension $[energy]^{-1}$, and $f(.)$ is a real function[116]. It is easy to prove that if Γ is Hamiltonian w.r.t. (H, ω), then it will be Hamiltonian as well w.r.t. (H_f, ω_f), where ω_f is defined as:

$$\omega_f = dH_f \wedge dt \tag{7.50}$$

Having redefined (through the new symplectic form) the volume element in phase space, it is natural to redefine the partition function as:

$$\mathcal{Z}_f = h^{-1} \int_{\mathbf{R^2}} \exp\{-\beta H_f\} \omega_f \tag{7.51}$$

But then:

$$\mathcal{Z}_f = h^{-1} \int dE_f \exp\{-\beta E_f\} \int_{\Sigma(E_f)} dt \tag{7.52}$$

We notice that the nonlinear change of coordinates (3.93) defines such a transformation on the Hamiltonian if we set: $f(\beta_0 H) \equiv \phi(H)$.

We come now to the analogous problem in the context of Quantum Mechanics. In terms of the creation and annihilation operators a and $a^\dagger$, with the standard commutation relations:

$$[a, a^\dagger] = 1 \tag{7.53}$$

one constructs a basis in the Fock space as:

$$|n\rangle_1 = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{7.54}$$

with $|0\rangle$ the Fock vacuum and the standard scalar product, that we will denote as $\langle .|.\rangle_1$:

$$\langle n|m\rangle_1 = \delta_{nm} \tag{7.55}$$

[115]It is well known that one is forced [187] to introduce it in the context of classical Statistical Mechanics in order to obtain a dimensionless expression for the partition function, so as to make sense of expressions such as : $\mathcal{F} = -\beta^{-1} \ln \mathcal{Z}$ for the (Helmoltz) free energy. The value of h is fixed unambiguously at that of Planck's constant at the quantum level of Statistical Mechanics.

[116]We will assume $f' > 0$ throughout, and that in order: *i*) to give a sensible meaning to integrals (see below) over phase space and: *ii*) not to change the number of critical points. The original Hamiltonian will correspond of course to $f(x) = x$.

We need to define for any (trace-class) linear operators the trace as:

$$Tr_1 \hat{O} = \sum_{n=0}^{\infty} \left\langle n|\hat{O}|n \right\rangle_1 \tag{7.56}$$

in order to be able to calculate the partition function at the quantum level as:

$$Z \equiv Tr \exp\{-\beta H\} = \sum_n \langle n| \exp\{-\beta H\} |n\rangle_1 \tag{7.57}$$

Now, we perform a "nonlinear change of variables" by defining [150, 67] new operators as:

$$A = f(\widehat{n})a \tag{7.58}$$

with $f(\widehat{n})$ a positive, monotonically increasing and nowhere vanishing function of the number operator $\widehat{n} = a^\dagger a$.

At this point, a little care is required when defining the adjoint of any operator: with the scalar product $\langle .|. \rangle_1$, with which $a^\dagger$ is the adjoint of a, the adjoint of A is of course: $A^\dagger = a^\dagger f(\hat{n})$.

It is pretty clear that, $\widehat{n}$ being a constant of the motion, the equations of motion for A and $A^\dagger$ will be the same as before. We can however reconstruct a different Fock space by assuming the same vacuum and defining new states[117] as:

$$|n\rangle_2 = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{7.59}$$

with a new scalar product defined as:

$$\langle n|m\rangle_2 = \delta_{nm} \tag{7.60}$$

The nonlinearity of the transformation reflects itself in the fact that, despite the fact that $|n\rangle_1$ and $|n\rangle_2$ are proportional, the linear structure in the Fock space labeled by "1" does not carry over to the linear structure of space "2". This has to do with the fact that the proportionality factors between the $|n\rangle_1$'s and the $|n\rangle_2$'s depend on n. In other words, if we try to induce on space "2" a linear structure modeled on that of space "1", the latter will not be compatible with the bilinearity of the scalar product $\langle .|. \rangle_2$ that we have just defined.

Now, $A^\dagger$ is no more the adjoint of A w.r.t. the new Hermitian structure we have introduced. If we denote by $(.)_2^\dagger$ the adjoint of any operator w.r.t. the second Hermitian structure, then we find:

$$(A^\dagger)_2^\dagger = \frac{1}{f(\hat{n})} a \tag{7.61}$$

which is quite different from A. The pair $\{(A^\dagger)_2^\dagger, A^\dagger\}$ will yield a new ("nonlinear") realization of the Heisenberg algebra, and indeed it is immediate to see that:

$$[(A^\dagger)_2^\dagger, A^\dagger] = 1 \tag{7.62}$$

[117] Note that, with this definition: $|n\rangle_2 = \{\prod_{k=0}^{N-1} f(k)\}|n\rangle_1$

Now, $(A^\dagger)_2^\dagger$ and $A^\dagger$ will obey the same equations of motion as a and $a^\dagger$, that can be derived from the previous commutation relations and from the Hamiltonian: $\widetilde{H} = A^\dagger (A^\dagger)_2^\dagger + 1/2$ (which turns out actually to coincide with the old one when written in terms of the original creation and annihilation operators) and that will have therefore the same spectrum. Defining then consistently the trace of any operator $\hat{O}$ as:

$$Tr_2 \hat{O} = \sum_{n=0}^{\infty} \left\langle n | \hat{O} | n \right\rangle_2 \tag{7.63}$$

will lead to the same partition function.

7.4 Weyl Systems and Second Quantization

7.4.1 Some Preliminaries

We recall here, mainly to fix the notation, what are the main ingredients for the construction of a Weyl system that were discussed at the beginning of this Chapter. What we need is:

- A real, symplectic vector space $\mathcal{S}$ whose symplectic form (skew-symmetric and non-degenerate) will be denotes as $\omega(.,.)$. If $\mathcal{S}$ is finite-dimensional, then: $\dim \mathcal{S} = 2n$ for some integer n. $\mathcal{S}$ will be required (see Sect.3.5.1 for more details) to possess also a complex structure J, i.e. a $(1,1)$-tensor satisfying: $J^2 = -\mathbb{I}_{2n\times 2n}$ and compatible with ω, which means:

  $$\omega(z, Jz') + \omega(Jz, z') = 0 \ \forall z, z' \in \mathcal{S} \tag{7.64}$$

  and implies that:

  $$g(.,.) =: \omega(., J(.)) \tag{7.65}$$

  $(g(z,z') = \omega(z, Jz'))$ will be symmetric and nondegenerate, hence a metric and a positive one iff:

  $$\omega(z, J.z) > 0, \forall z \neq 0 \tag{7.66}$$

  It is always possible to decompose S into the direct sum of two Lagrangian subspaces S_1 and S_2, $S = S_1 \oplus S_2$, in such a way that, writing (in an unique way): $z = (z_1, z_2) = (z_1, 0) + (z_2, 0), z_1 \in S_{\ 1}, z_2 \in S_{\ 2}$, ω can be written "in Darboux form", being represented by the matrix:

  $$\|\omega_{ij}\| = \begin{vmatrix} \mathbf{0}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0}_{n\times n} \end{vmatrix} \tag{7.67}$$

  i.e.:

  $$\omega(z, z') = z_1 \cdot z_2' - z_2 \cdot z_1' \tag{7.68}$$

the dot denoting the standard Euclidean scalar product. The (compatible) complex structure J will act as[118]:

$$J : (z_1, z_2) \mapsto (-z_2, z_1) \tag{7.69}$$

The vector space $\mathcal{S}$ can be viewed either as the cotangent space of either $\mathcal{S}_1$ or $\mathcal{S}_2$ or, alternatively, as the realification [5] of a complex vector space of complex dimension n, in which case, writing, e.g.: $z = z_1 + iz_2$, the complex structure will act as multiplication by the imaginary unit i. A Weyl system will consist then of:

- A map: $W : \mathcal{S} \to \mathcal{U}(\mathcal{H}); \mathcal{S} \ni z \mapsto \widehat{W}(z) \in \mathcal{U}(\mathcal{H})$ into the set $\mathcal{U}(\mathcal{H})$ of the unitary operators over a Hilbert space $\mathcal{H}$ which is strongly continuous and satisfies:

$$\widehat{W}(z)\,\widehat{W}(z') = \widehat{W}(z+z')\exp\{i\omega(z,z')/2\} \;\; ,\forall z, z' \in \mathcal{S} \tag{7.70}$$

where (here and in the following) we have set for simplicity $\hbar = 1$. We have already discusses how, using Stone's theorem [201], one can represent $\widehat{W}(z)$ as:

$$\widehat{W}(z) = \exp\left\{i\widehat{G}(z)\right\} \tag{7.71}$$

with $\widehat{G}(z)$ (essentially) self-adjoint, $\widehat{G}(tz) = t\widehat{G}(z)$ and:

$$\left[\widehat{G}(z), \widehat{G}(z')\right] = -i\omega(z, z') \tag{7.72}$$

Remark 64 *Using the truncated Baker-Campbell-Hausdorff formula*[119] *one can also write:*

$$\begin{aligned} \exp\left\{it\widehat{G}(z)\right\} \cdot \exp\left\{it\widehat{G}(z')\right\} &= \exp\left\{it\left[\widehat{G}(z) + \widehat{G}(z')\right]\right\} \cdot \\ &\cdot \exp\left\{-\frac{1}{2}t^2\left[\widehat{G}(z), \widehat{G}(z')\right]\right\} \end{aligned} \tag{7.73}$$

whence, comparing with Eqs.(7.70) and (7.72) and expanding in t:

$$\widehat{G}(z) + \widehat{G}(z') = \widehat{G}(z+z'), \forall z, z' \tag{7.74}$$

Remark 65 *To be more precise, the l.h.s.'s of both Eqs.(7.72) and (7.74) should be properly understood [24] as the closures of the commutator and of the sum respectively.*

We know also from Sect.5.2 that, via the von Neumann theorem [223], one can realize concretely $\mathcal{H}$ as the Hilbert space of square-integrable functions over a Lagrangian submanifold $Q \subset \mathcal{S}$, and how[120] different realizations of $\mathcal{H}$ are mutually unitarily related.

---

[118] Notice that J is not unique. For example [24], if J is a complex structure, then also: $J' = S^{-1}JS$ will be such if S is any symplectic transformation.

[119] $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ whenever: $[A,[A,B]] = [B[A,B]] = 0$.

[120] As long as we do not alter (see Sect.7.3.1) the linear structure in a non-linear way.

7.4.2 Weyl Systems over a Hilbert Space. Second Quantization

Following the scheme set up in Sect.5.2, assume that we have realized the Hilbert space $\mathcal{H}$ as the (complete) Hilbert space $L_2(Q)$, with Q a Lagrangian submanifold of the original (real) vector space $\mathcal{S}$. To fix the ideas, and in the notation of the previous Subsection, we can take, e.g.: $Q = \mathcal{S}_1$ and, writing now: $z = (\mathbf{q}, \mathbf{p})$ and: $\widehat{W}(z) = \widehat{W}(\mathbf{q}, \mathbf{p})$, we have then, with: $\psi \in L_2(\mathcal{S}_1)$ and: $\mathbf{x} \in \mathcal{S}_1$:

$$\left(\widehat{W}(\mathbf{q}, 0)\psi\right)(\mathbf{x}) =: \left(\widehat{U}(\mathbf{q})\psi\right)(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{q}) \tag{7.75}$$

and:

$$\left(\widehat{W}(0, \mathbf{p})\psi\right)(\mathbf{x}) =: \left(\widehat{V}(\mathbf{p})\psi\right)(\mathbf{x}) = \exp\{i\mathbf{p}\cdot\mathbf{x}\}\psi(\mathbf{x}) \tag{7.76}$$

(here too we are setting: $\hbar = 1$).

We will consider here $\mathcal{H} \simeq L_2(Q)$ as a "single-particle Hilbert space", and we will proceed to setting up a description of an assembly of identical particles, fixing our attention, for the sake of illustration, on the case of particles obeying Bose statistics.

We turn now explicitly to the Hilbert space $L_2(Q)$, which is endowed with the Hermitian (linear in the second factor) scalar product ($d\mathbf{x}$ standing for the Lebesgue measure):

$$h(\psi, \psi') =: \int d\mathbf{x}\overline{\psi}(\mathbf{x})\psi'(\mathbf{x}) \tag{7.77}$$

Writing: $\psi = u + iv$ for every $\psi \in \mathcal{H}$, the complex Hilbert space $\mathcal{H}$ can be realified [5] into the real linear vector space of pairs (u, v), equipped with both a (positive) metric:

$$g((u, v), (u', v')) = \int d\mathbf{x}[uu' + vv'] = \operatorname{Re} h(\psi, \psi') \tag{7.78}$$

and a symplectic form:

$$\omega((u, v), (u', v')) = \int d\mathbf{x}[uv' - vu'] = \operatorname{Im} h(\psi, \psi') \tag{7.79}$$

i.e.:

$$h(.,.) = g(.,.) + i\omega(.,.) \tag{7.80}$$

with the complex structure (see the previous Subsection) acting as:

$$J : (u, v) \mapsto (-v, u) \tag{7.81}$$

(and hence: $g((u, v), (u', v')) \equiv \omega((u, v), J(u', v'))$).

One can set up now a Weyl system in the form:

$$\begin{aligned} &(u, v) \mapsto \widehat{W}(u, v) \\ &\widehat{W}(u, v)\widehat{W}(u', v') = \widehat{W}(u + u', v + v')\exp\{i\omega((u, v), (u', v'))/2\} \end{aligned} \tag{7.82}$$

Representing $\widehat{W}(u,v)$ as:

$$\widehat{W}(u,v) = \exp\left\{i\widehat{G}(u,v)\right\} \tag{7.83}$$

with a self-adjoint generator $\widehat{G}$, we have (cfr. Eqs.(7.72) and (7.74)):

$$\left[\widehat{G}(u,v), \widehat{G}(u',v')\right] = -i\omega\left((u,v),(u',v')\right) \tag{7.84}$$

as well as:

$$\widehat{G}(u,v) = \widehat{\Pi}(u) + \widehat{\Psi}(v) \;\; ; \widehat{\Pi}(u) =: \widehat{G}(u,0), \widehat{\Psi}(v) =: \widehat{G}(0,v) \tag{7.85}$$

with the commutation relations[121]:

$$\left[\widehat{\Psi}(v), \widehat{\Pi}(u)\right] = i\int d\mathbf{x}\, u(\mathbf{x})\, v(\mathbf{x}) \tag{7.86}$$

as well as:

$$\left[\widehat{\Psi}(v), \widehat{\Psi}(v')\right] = \left[\widehat{\Pi}(u), \widehat{\Pi}(u')\right] = 0 \tag{7.87}$$

Being $\mathbb{R}$-linear in their arguments, it is customary to represent both operators $\widehat{\Psi}$ and $\widehat{\Pi}$ in the form [24, 213]:

$$\widehat{\Psi}(v) = \int d\mathbf{x}\widehat{\Psi}(\mathbf{x})\, v(\mathbf{x})\, ; \;\; \widehat{\Pi}(u) = \int d\mathbf{x}\widehat{\Pi}(\mathbf{x})\, u(\mathbf{x}) \tag{7.88}$$

i.e. in terms of the distribution-valued (Hermitian) *field operator* $\widehat{\Psi}(\mathbf{x})$ and of its *conjugate momentum* $\widehat{\Pi}(\mathbf{x})$ obeying, as a consequence of Eqs.(7.86) and (7.87), the (equal-time) commutation relations:

$$\left[\widehat{\Psi}(\mathbf{x}), \widehat{\Pi}(\mathbf{x}')\right] = i\delta(\mathbf{x}-\mathbf{x}') \tag{7.89}$$

as well as:

$$\left[\widehat{\Psi}(\mathbf{x}), \widehat{\Psi}(\mathbf{x}')\right] = \left[\widehat{\Pi}(\mathbf{x}), \widehat{\Pi}(\mathbf{x}')\right] = 0, \forall \mathbf{x},\mathbf{x}' \tag{7.90}$$

These operators are easily recognized to be appropriate for the description [213] of a bosonic field. Having constructed (admittedly in a partly heuristic way) the algebra of field operators, one should then proceed to construct the physical vacuum[122] and of the associated Hilbert space on which this algebra of operators acts via, e.g., the *GNS* construction [8, 95] or defining [24], in terms of the $\widehat{W}$'s, a generating functional for the Wightman functions [95], using the "reconstruction theorem" of Axiomatic Field Theory [216]. We shall outline here however a slightly different route that leads more directly to the usual Fock space description of (bosonic) quantum fields.

[121] See however the Remark following Eq.(7.74).
[122] We do not discuss here problems of uniqueness of the vacuum state.

Reinstating for brevity the notation: $\psi = (u, v)$ for the (real) pair (u, v), we can use the generators: $\widehat{G}(\psi) =: \widehat{G}(u, v)$ to define annihilation and creation operators $\widehat{a}(\psi)$ and $\widehat{a}^{\dagger}(\psi)$ associated with the state ψ as:

$$\widehat{a}(\psi) = \frac{1}{\sqrt{2}}\left[\widehat{G}(\psi) + i\widehat{G}(J\psi)\right]; \widehat{a}^{\dagger}(\psi) = \frac{1}{\sqrt{2}}\left[\widehat{G}(\psi) - i\widehat{G}(J\psi)\right] \tag{7.91}$$

A little algebra shows then that:

$$\left[\widehat{a}(\psi), \widehat{a}(\psi')\right] = \left[\widehat{a}^{\dagger}(\psi), \widehat{a}^{\dagger}(\psi')\right] = 0 \ \forall \psi, \psi' \tag{7.92}$$

while:

$$\left[\widehat{a}(\psi), \widehat{a}^{\dagger}(\psi')\right] = h(\psi, \psi') \tag{7.93}$$

If we consider in particular an ON basis[123] $\{\psi_n\}_0^{\infty}$ in the "single-particle" Hilbert space $\mathcal{H}$ $(h(\psi_n, \psi_m) = \delta_{nm})$ and define:

$$\widehat{a}_n =: \widehat{a}(\psi_n) \tag{7.94}$$

then:

$$\left[\widehat{a}_n, \widehat{a}_m^{\dagger}\right] = \delta_{nm} \tag{7.95}$$

and all the other commutators vanish. With these operators at hand, one can then proceed to the construction of the Fock space following, e.g., the approach discussed by J.M.Cook [45] already in the early Fifties.

Of course, one can also work directly with the exponential form (7.70) of a Weyl system, as we will see now. The possibility to do so relies on the following observations that can be easily verified if we work on a finite n-dimensional Hilbert space $\mathcal{H}$. We will denote with $\mathbb{K}$ the space of (complex) functions $f(z) = f(z_1, z_2, \cdots, z_n)$, $z_j \in \mathbb{C}$, on $\mathcal{H}$ which are square-integrable according to the (Gaussian) measure:

$$\|f\|^2 =: \int \left(\prod_{j=1}^{n} \frac{d\mathrm{Re}z_j \, d\mathrm{Im}z_j}{\pi}\right) e^{-\langle z,z\rangle}|f(z)|^2 < \infty \tag{7.96}$$

On such space, for any $z \in \mathcal{H}$ let us consider the operator:

$$W(z) : f(w) \mapsto f_z(w) =: f(w - z)\exp\left(\frac{\langle w, z\rangle}{2} - \frac{\langle z, z\rangle}{4}\right) \tag{7.97}$$

which: $i)$ conserves the norm: $\|f\|^2 = \|f_z\|^2$ and $ii)$ satisfies the relation

$$W(z)W(z') = W(z + z')\exp\left(\frac{i\mathrm{Im}\langle z, z'\rangle}{2}\right) \tag{7.98}$$

and hence allow for the definition of a Weyl system which is irreducible on the subspace of $\mathbb{K}$ of antiholomorphic functions, $\mathcal{F}_{\mathbb{K}}$, which can be seen [8] as the

[123] For example, if $\mathcal{H} = L_2(\mathbb{R})$, we could choose [8] the basis of the eigenfunctions of the $1D$ harmonic oscillator (the Hermite functions [77]).

closure w.r.t. the norm defined above of the space of antiholomorphic polynomials in the n variables $z_1, z_2, \cdots, z_n$. A straightforward calculation shows that setting $u^{(j)} = (0, \cdots, 0, u_j, 0, \cdots, 0)$ and $v^{(j)} = (0, \cdots, 0, iv_j, 0, \cdots, 0)$, with $u_j, v_j \in \mathbb{R}$, one has:

$$iG(u^{(j)}) = -\frac{\partial}{\partial \mathrm{Re} w_j} + \frac{\bar{w_j}}{2}$$
$$iG(v^{(j)}) = -\frac{\partial}{\partial \mathrm{Im} w_j} - i\frac{\bar{w_j}}{2}$$

so that the annihilation/creation operators are given by

$$\begin{aligned} \hat{a}_j &=: \frac{G(u^{(j)}) - iG(v^{(j)})}{\sqrt{2}} = \sqrt{2}i\partial_{w_j} - \frac{i}{\sqrt{2}}\bar{w}_j \\ &= -\frac{i}{\sqrt{2}}\bar{w}_j \text{ on } \mathcal{F}_{\mathbb{K}} \\ \hat{a}_j^\dagger &=: \frac{G(u^{(j)}) + iG(v^{(j)})}{\sqrt{2}} = \sqrt{2}i\partial_{\bar{w}_j} \end{aligned}$$

and clearly satisfy bosonic-like commutation relations. Then the vacuum (or cyclic vector) is given by the constant unit monomial $P_0(z) = 1$. Notice also that for any unitary operator $U \in \mathcal{U}(\mathcal{H})$ we may construct a unitary operator $\Gamma(U) \in \mathcal{U}(\mathcal{F}_{\mathbb{K}})$ via the map:

$$\Gamma(U) : W(z) \mapsto W(U^{-1}z) \tag{7.99}$$

A generalization of such results to an infinite dimensional Hilbert space $\mathcal{H}$ requires of course caution in the definition of domains of operators as well in the definition of the spaces $\mathbb{K}$ and $\mathcal{F}_{\mathbb{K}}$. This can be done by introducing the so called isonormal [8] distribution g, which determines a measure dg on the Hilbert space which, when restricted to finite dimensional subspaces looks like a Gaussian measure with variance σ, and defining the space $\mathbb{K}$ as the completion of the space of polynomials on $\mathcal{H}$ w.r.t. the inner product

$$\int_L \overline{P'(\psi)} P(\psi) dg(\psi) \tag{7.100}$$

L being any finite-dimensional subspace of $\mathcal{H}$ on which the polynomials P, P' have support. The space $\mathcal{F}_{\mathbb{K}}$ is now the subspace of those functions F on $\mathcal{H}$ such that their restrictions $F|_L$ on any finite-dimensional subspace L are antiholomorphic in the usual sense. Thus one gets a complex representation for the bosonic field in which the Weyl system is given by the operators [8]:

$$W(\psi) : F(\phi) \mapsto F(\phi - \psi) \exp\left(\frac{\langle \psi, \phi \rangle}{2\sigma} - \frac{\langle \phi, \phi \rangle}{4}\right), \ \forall \psi \in \mathcal{H} \tag{7.101}$$

For such representation, the cyclic vector is the function on $\mathcal{H}$ identically equal to one. Also, for any $U \in \mathcal{U}(\mathcal{H})$ we have a unitary operator $\Gamma(U) \in \mathcal{U}(\mathcal{F}_{\mathbb{K}})$ such that $W(\psi) \mapsto W(U^{-1}\psi)$

This completes the discussion of how Weyl's approach can lead, in a rather natural and elegant way, to the formalism of second quantization and hence of Field Theory. We have done that for bosons, and we refer to the literature (see in particular Refs. [8],[24] and [45]) for the parent construction for the case of fermions. Alternative Hilbert space structures will give rise also to additional ambiguities in the commutation relations for the fields.

7.5 Concluding Remarks

By using the geometrical formulation of Quantum Mechanics we have bee able to "export" from the classical to the quantum framework many problems that arise in the classical setting, and we have constructed a more direct "bridge" which realizes Dirac's demand [56]that problems arising in Classical Mechanics must be a suitable limit of analogous problems arising in Quantum Mechanics.

In particular, we have addressed the problem of the quantum interpretation of the bi-Hamiltonian description of completely integrable systems in the classical setting.

Alternative quantum Hamiltonian descriptions have been provided in various pictures of Quantum Mechanics, the Schrödinger, Heisenberg and Weyl-Wigner-Moyal pictures.

We have also shown that it is possible to deal with nonlinear transformations in Quantum Mechanics without giving up the superposition principle which is associated with quantum interference phenomena.

The rôle of dynamically determined structures versus pre-assigned mathematical structures in the formalization of Quantum Mechanics has been further elucidated.

One may wonder if, in analogy with what happens in General Relativity, where the metric is determined by solving the Einstein equations, one can conceive of some field equations whose solutions would provide the Hermitian tensor to be used in the description of quantum systems.

By mentioning how to deal with Second Quantization and Quantum Field Theories in this framework we have hinted at the idea that this approach may provide suggestions for the introduction of interactions in a pure quantum field-theoretic setting.

At the end of this journey, we believe it to be rewarding to know that many sophisticated methods of Classical Physics may find their way into the formalism of Quantum Physics.

A Nijenhuis torsions and Nijenhuis Tensors

Nijenhuis Torsions and Tensors on Smooth Manifolds

Let us consider, to begin with, the set $\mathfrak{X}(\mathcal{M})$ of vector fields over some (smooth) manifold $\mathcal{M}$. $\mathfrak{X}(\mathcal{M})$ has, as is well known, the structure of a (actually an infinite-dimensional) Lie algebra defined by the Lie bracket:

$$[.,.] : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})\,; \ \ (X,Y) \mapsto [X,Y] =: \mathcal{L}_X Y = -\mathcal{L}_Y X;\ X,Y \in \mathfrak{X}(\mathcal{M}) \tag{A.1}$$

with $\mathcal{L}.$ the Lie derivative. Let then T be a $(1-1)$ tensor viewed as a map: $T : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$. One can associate[124] with T an antiderivation d_T of degree one whose actions on zero- and one-forms is given by:

$$d_T f\,(X) = df\,(TX) \tag{A.2}$$

on functions, and:

$$d_T\theta\,(X,Y) = (\mathcal{L}_{TX}\theta)\,(Y) - (\mathcal{L}_{TY}\theta)\,(X) + \theta\,(T\,[X,Y]) \tag{A.3}$$

on one-forms (recall that a (anti)derivation is entirely defined [41] by its action on zero- and one-forms). One proves that d_T^2 is a derivation (of degree two) commuting with d: $d \circ d_T^2 = d_T^2 \circ d$. As such, its action is entirely defined [41] by that on zero-forms (functions), and one finds:

$$\left(d_T^2 f\right)(X,Y) = -df\,(N_T\,(X,Y)) \tag{A.4}$$

where [78, 152, 186, 194] the*Nijenhuis torsion* N_T of T is the $(1-2)$-type tensor defined by[125]:

$$N_T\,(X,Y) = \{(T \circ \mathcal{L}_X\,(T)) - (\mathcal{L}_{TX}\,(T))\}\,(Y) \tag{A.5}$$

or, more explicitly:

$$N_T\,(X,Y) = T\,[TX,Y] + T\,[X,TY] - T^2\,[X,Y] - [TX,TY] \tag{A.6}$$

T will be said to be a *Nijenhuis tensor* if its Nijenhuis torsion vanishes, i.e. if:

$$N_T = 0 \tag{A.7}$$

Remark 66 *In local coordinates x^i, if:*

$$T = T^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j \tag{A.8}$$

[124] See Ref. [186] for more details

[125] Note that what we call here, following the literature, the "Nijenhuis torsion" was called the "Nijenhuis tensor" in Ref. [186].

then:

$$N_T = \frac{1}{2} \left(N_T \right)^i {}_{km} \frac{\partial}{\partial x^i} \otimes dx^k \wedge dx^m \tag{A.9}$$

where:

$$\left(N_T \right)^i {}_{km} = \frac{\partial T^i {}_k}{\partial x^j} T^j {}_m + T^i {}_j \frac{\partial T^j {}_m}{\partial x k} - (k \longleftrightarrow m) \tag{A.10}$$

and, obviously: $N_T = 0$ *whenever the representative matrix of* T *is a matrix with constant entries.*

Nijenhuis Torsions and Tensors on Associative Algebras

Eqn.(A.5) defines the Nijenhuis torsion on a Lie algebra. Nijenhuis-type tensors and torsions can be given however a more general setting [33, 34]in the framework of associative algebras. We recall[126] that an associative algebra $(\mathcal{A}, *)$ becomes also a Lie algebra under commutation, i.e. with a bracket defined as:

$$[A, B] =: A * B - B * A; \quad A, B \in \mathcal{A} \tag{A.11}$$

and associativity of the algebra guarantees that the bracket does satisfy the Jacobi identity, so it is indeed a Lie bracket.

Let then $(\mathcal{A}, *)$ be an associative algebra over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ for our purposes), an let: $T : \mathcal{A} \to \mathcal{A}$ be a linear map. T will be a *derivation* of the algebra $(\mathcal{A}, *)$ if (and only if):

$$T (A * B) = T (A) * B + A * T (B) \;\; \forall A, B \in \mathcal{A} \tag{A.12}$$

Be it as it may, given T one can define in general the bilinear map:

$$*_T : (A, B) \to A *_T B = T (A) * B + A * T (B) - T (A * B) \tag{A.13}$$

and $*_T$ will be trivial if (and only if) T is a derivation. In general (with T not a derivation), $*_T$ will define a (non-trivial) new algebra structure $(\mathcal{A}, *_T)$.

As a simple example, let's take $T \in \mathcal{A}$, an hence: $T (A) = T * A$. Then, a simple calculation shows that:

$$A *_T B = A * T * B \tag{A.14}$$

Products of this sort will be employed in the text in the discussion of alternative commutation relations in Quantum Mechanics.

[126]It goes without saying that the "star-product" $*$ we are talking about here has nothing to do with the Moyal product.

A Digression on: Hochschild Cohomologies

Given an associative algebra $(\mathcal{A}, *)$ and an $\mathcal{A}$-bimodule V (what we will have in mind will be the case in which V is the additive group of $\mathcal{A}$ and the bimodule structure is given by left and right multiplication), an *n-cochain* will be an n-linear mapping:

$$\alpha : \underbrace{\mathcal{A} \times \mathcal{A} \times ... \mathcal{A}}_{n \ \ times} \rightarrow V \tag{A.15}$$

The space $C^n(\mathcal{A}, V)$ of n-cochains has a group structure under addition. Then, for every n, the *Hochschild coboundary operator*[127]: $\delta_* : C^n(\mathcal{A}, V) \rightarrow C^{n+1}(\mathcal{A}, V)$ is defined $(\alpha \in C^n(\mathcal{A}, V), \ a_1, ..., a_{n+1} \in \mathcal{A})$ via [101]:

$$\begin{aligned}(\delta_* \alpha)(a_1, ..., a_{n+1}) &= a_1 \alpha(a_2, ..., a_{n+1}) + \\ +\sum_{n=1}^{n} &(-1)^i \alpha(a_1, .., a_i * a_{i+1}, .., a_{n+1}) + \\ &+ (-1)^{n+1} \alpha(a_1, ..., a_n) a_{n+1}\end{aligned} \tag{A.16}$$

where $a\alpha(..)$ and $\alpha(..)a$ denote the left and right actions of $\mathcal{A}$ on V respectively. One can check directly that:

$$\delta_* \circ \delta_* = 0 \tag{A.17}$$

As an example, for $n = 1$:

$$(\delta_* \alpha)(a_1, a_2) = a_1 \alpha(a_2) + \alpha(a_1) a_2 - \alpha(a_1 * a_2) \tag{A.18}$$

An n-cochain α is called an $n-$*cocycle* if $\delta_* \alpha = 0$, an $n-$*coboundary* if $\alpha = \delta_* \beta$ for some $(n-1)-$cochain β. $n-$cocycles form an additive group usually denoted as $\mathbb{Z}^n(\mathcal{A}, V)$, and (in view of (A.17)) $n-$coboundaries form an subgroup $\mathbb{B}^n(\mathcal{A}, V)$ of $\mathbb{Z}^n(\mathcal{A}, V)$. The *$n-$(Hochschild) cohomology group* $\mathbb{H}^n(\mathcal{A}, V)$ is defined then as the quotient:

$$\mathbb{H}^n(\mathcal{A}, V) = \mathbb{Z}^n(\mathcal{A}, V) / \mathbb{B}^n(\mathcal{A}, V) \tag{A.19}$$

The linear mapping T can be considered as a one-cochain and, looking then at Eqn.(A.13)we can conclude that:

$$A *_T B = \delta_* T(A, B) \tag{A.20}$$

and hence we can rephrase what has been said previously by saying that T will be a derivation if and only if it is a one-cocycle in the Hochschild cohomology associated with the "star-product".

The $*-$*Nijenhuis torsion* of T is defined as:

$$N_T(A, B) = T(A *_T B) - T(A) * T(B) \tag{A.21}$$

[127]The suffix serves here to stress that the operators and the ensuing properties are all relative to the binary product ("star-product") in the algebra.

or, more explicitly:

$$N_T(A,B) = T(T(A)*B) + T(A*T(B)) - T^2(A*B) - T(A)*T(B) \quad \text{(A.22)}$$

It is clear from Eqn.(A.21) that the Nijenhuis torsion of T measures the obstruction for the linear map T to be a homomorphism of the two products.

Here too it will be said that T is a $*-$*Nijenhuis tensor* if its Nijenhuis torsion vanishes. For example, it is easy to see that $N_T = 0$ if $T \in \mathcal{A}$ and the associated product is given by Eqn.(A.14). Hence, T is a Nijenhuis tensor.

Making Contacts

To make contact with the initial definition of the Nijenhuis torsion, we recall what has already been said, i.e. that an associative algebra can be made into a Lie algebra using the commutator (A.11). If we substitute the "star-product" with the commutator, then Eqn.(A.22) becomes:

$$N_T(A,B) = T[T(A),B] + T[A,T(B)] - T^2[A,B] - [T(A),T(B)] \quad \text{(A.23)}$$

which coincides with Eqn.(A.6) if we substitute for $A, B, ..$ vector fields on a manifold and the commutator with the Lie bracket. This establishes the link between the two definitions of the Nijenhuis torsion that have been given here. The Nijenhuis torsion defined on an associative algebra will play a rôle in the discussion, in the text, of alternative associative products on the algebra of (bounded) operators on a Hilbert space. Completeness would require discussing also how the (Lie) algebra of vector fields can be embedded into a larger associative algebra (the enveloping algebra), but we will not insist on this point not too lengthen too much the discussion.

B Recursion Operators

Some Preliminaries

Let T be a $(1,1)$-type tensor field: $T \in \mathcal{F}^1_1(\mathcal{M})$. As is already known, the action of T on vector fields (denoted with the same symbol) and one-forms (defined as $\widetilde{T}$) is defined uniquely by:

$$\langle TX|\alpha\rangle =: \left\langle X|\widetilde{T}\alpha\right\rangle, \; X \in \mathcal{X}(\mathcal{M}), \alpha \in \mathcal{X}^*(\mathcal{M}) \tag{B.1}$$

where $\langle .|.\rangle$ denotes the usual pairing. In coordinates, if:

$$T = T^i_j dx^j \otimes \frac{\partial}{\partial x^i} \tag{B.2}$$

is represented by the matrix[128]: $T = \left\|T^i{}_j\right\|$ then $\widetilde{T}$ will be represented by the matrix: $\widetilde{T} =: \left\|\widetilde{T}_j{}^i\right\|$ and Eqn.(B.1) implies:

$$\widetilde{T}_j{}^i = T^i{}_j \tag{B.3}$$

i.e. that $\widetilde{T}$ be the *transpose* of T:

$$\widetilde{T} = T^t \tag{B.4}$$

All this is well known and is repeated here only for completeness.

One can consider extending the action of the $\widetilde{T}$ on forms oh higher rank, as well as that of T on multivectors. We will concentrate here only on the former, recollecting some results that can be found in the literature ([186]).

The extension under consideration is not unique. Let, e.g., ω be a two-form. In particular, ω will be considered as the map:

$$\begin{gathered} \omega : \mathcal{X}(\mathcal{M}) \to \mathcal{X}^*(\mathcal{M}); \; \omega \; : Y \to \omega(.,Y) = -i_Y\omega \\ \langle \omega(.,Y)|X\rangle = -i_X i_Y \omega = \omega(X,Y) \end{gathered} \tag{B.5}$$

$((\omega(.,Y)) = \omega_{ij}Y^j dx^i)$. Hence we can compose $\widetilde{T}$ with ω to obtain the $(0,2)$ tensor:

$$\widetilde{T}\circ\omega : \;\; (X,Y) \to \left\langle \widetilde{T}\circ\omega(.,Y)|X\right\rangle = \langle\omega(.,Y)|TX\rangle \tag{B.6}$$

i.e.:

$$\left(\widetilde{T}\circ\omega\right)(X,Y) = \omega(TX,Y) \tag{B.7}$$

This is a linear extension. In terms of representative matrices $\widetilde{T}\circ\omega$ is represented by the matrix $T^t\omega$, i.e. (cfr. Eqn.(B.3)):

$$\widetilde{T}\circ\omega = \left(T^t\omega\right)_{ij} dx^i \otimes dx^j = T^k{}_i \omega_{kj} dx^i \otimes dx^j \tag{B.8}$$

[128] With some abuse of notation, we will denote here with the same symbol $(1,1)$ tensors and their representative matrices.

Another possible and more symmetric linear extension is provided by:

$$\left(\widetilde{T}\circ\omega\right)(X,Y)=\omega\left(TX,Y\right)+\omega\left(X,TY\right) \tag{B.9}$$

Also, a nonlinear extension such as:

$$\left(\widetilde{T}\circ\omega\right)(X,Y)=\omega\left(TX,TY\right) \tag{B.10}$$

may be envisaged, with even more possibilities for forms of higher rank.

Notice that, while the extensions (B.9) and (B.10) map two-forms into two-forms, this is not true in general for the extension (B.7) which will yield in general a $(0,2)$-type tensor but not a two-form.

The linear extension (B.9) allows for the association with T of an antiderivation of degree one usually denote as d_T that acts on zero- and one-forms as:

$$d_T f=\widetilde{T}df;\ d_T f\left(X\right)=:df\left(TX\right) \tag{B.11}$$

and:

$$\left(d_T\theta\right)(X,Y)=\left(\mathcal{L}_{TX}\theta\right)(Y)-\left(\mathcal{L}_{TY}\theta\right)(X)+\theta\left(T\left[X,Y\right]\right) \tag{B.12}$$

d_T can be shown to be nilpotent $\left(d_T\circ d_T=:d_T^2=0\right)$ like the ordinary exterior differential d if and only if T has a vanishing Nijenhuis torsion, but we will not insist on that.

Returning instead to the extension (B.7), one can prove the following:

The extension of the action of T on two-forms defined by:

$$\left(\widetilde{T}\circ\omega\right)(X,Y)=:\omega\left(TX,Y\right) \tag{B.13}$$

will be a two-form (i.e. it will be skew-symmetric) if and only if:

$$\omega\left(TX,Y\right)=\omega\left(X,TY\right)\quad\forall X,Y \tag{B.14}$$

Indeed, if the condition (B.14) holds, then:

$$\begin{aligned}\left(\widetilde{T}\circ\omega\right)(X,Y)\ &=:\ \omega\left(TX,Y\right)=-\omega\left(Y,TX\right)=\\ &=\ -\omega\left(TY,X\right)=-\left(\widetilde{T}\circ\omega\right)(Y,X)\end{aligned} \tag{B.15}$$

and $\widetilde{T}\circ\omega$ is skew-symmetric. Viceversa, if $\omega_1=:\widetilde{T}\circ\omega$ is skew-symmetric, then:

$$\begin{aligned}\omega\left(X,TY\right)\ &=\ -\omega\left(TY,X\right)=-\omega_1\left(Y,X\right)=\\ &=\ \omega_1\left(X,Y\right)=\omega\left(TX,Y\right)\end{aligned} \tag{B.16}$$

and (B.14) holds.■

Notice that, in this case:

$$\omega\left(TX,Y\right)=\frac{1}{2}\left\{\omega\left(TX,Y\right)+\omega\left(X,TY\right)\right\} \tag{B.17}$$

and there is no real difference between the two linear extensions.

$\mathcal{H}-$weak and ω-weak Recursion Operators. Strong Recursion Operators

Let Γ be a Hamiltonian vector field with Hamiltonian $\mathcal{H}$ w.r.t. a given symplectic form ω, i.e.:

$$i_\Gamma \omega = d\mathcal{H} \tag{B.18}$$

Then [53, 120, 179, 239], a $(1,1)$-type tensor field T compatible with the dynamics, i.e. such that:

$$\mathcal{L}_\Gamma T = 0 \tag{B.19}$$

is called:

- A *$\mathcal{H}$-weak recursion operator* if it "generates new Hamiltonians" in the sense that:

  $$d\left(\widetilde{T}^k d\mathcal{H}\right) = 0, \;\; k = 1, 2, 3, ... \tag{B.20}$$

  i.e., locally at least:

  $$\widetilde{T}^k d\mathcal{H} = d\mathcal{H}_k, \;\; k \geq 1 \tag{B.21}$$

  for some $\mathcal{H}_k \in \mathcal{F}(\mathcal{M})$. It is called instead:

- A *ω-weak recursion operator* if it "generates new symplectic forms" in the sense that:

  $$\omega_k =: \underbrace{\widetilde{T} \circ \widetilde{T} \circ .... \circ \widetilde{T}}_{k\ times}\omega =: \widetilde{T}^k \circ \omega, \;\; k = 1, 2, 3... \tag{B.22}$$

  is closed and skew-symmetric (and hence a symplectic form if T is invertible). Finally, T is called:

- A *strong recursion operator* if it is both $\mathcal{H}$-weak and ω-weak.

Before discussing the conditions under which a $(1,1)$ tensor is $\mathcal{H}$-weak and/or ω-weak, let us examine some consequences of these definitions.

First of all, if T is $\mathcal{H}$-weak, it may well happen that: $d\mathcal{H}_k \wedge d\mathcal{H} = 0$ for some k (even for $k = 1$[129]) , and the process of generating new Hamiltonian functions will stop at this stage. Barring this case, one can generate then a set of ω-Hamiltonian vector fields Γ_k via:

$$i_{\Gamma_k}\omega = d\mathcal{H}_k, \quad k \geq 1 \tag{B.23}$$

Taking the Lie derivative w.r.t. Γ of Eqn.(B.21) and taking into account the invariance of T one finds at once:

$$d\left(\mathcal{L}_\Gamma \mathcal{H}_k\right) = 0 \tag{B.24}$$

[129] This seems to be the case for the Kepler problem [179]

This implies only: $\mathcal{L}_\Gamma \mathcal{H}_k = const.$ and not that $\mathcal{H}_k$ is a constant of the motion for Γ. This will require some additional assumptions that will be discussed shortly below.

If instead T is ω-weak, taking again the Lie derivative w.r.t. Γ of Eqn.(B.22), invariance of T leads at once to:

$$\mathcal{L}_\Gamma \omega_k = 0, \quad k \geq 1 \tag{B.25}$$

In other words, Γ will be also locally ω_k-Hamiltonian. Then, locally at least:

$$i_\Gamma \omega_k = d\widetilde{\mathcal{H}}_k \tag{B.26}$$

for some $\widetilde{\mathcal{H}}_k \in \mathcal{F}(\mathcal{M})$, and this will provide *alternative Hamiltonian descriptions* for the same dynamics. Notice that the $\widetilde{\mathcal{H}}_k$'s are not related (at least not in a simple way) to the $\mathcal{H}_k$'s of Eqn.(B.21). Alternatively, one can define a new set of vector fields $\widetilde{\Gamma}_k$ via:

$$i_{\widetilde{\Gamma}_k} \omega_k = d\mathcal{H} \tag{B.27}$$

and these will be all Hamiltonian vector fields associated with different symplectic structure but with *the same* Hamiltonian function.

Some relevant results concerning $\mathcal{H}$-weak and/or ω-weak recursion operators have been proved in the literature. The main results that we will summarize here (referring to the literature for details of the proof) are:

1. If T satisfies the condition (B.20) for $k = 1$, i.e.:

   $$d\left(\widetilde{T} d\mathcal{H}\right) = 0 \tag{B.28}$$

   and has vanishing Nijenhuis torsion:

   $$N_T = 0 \tag{B.29}$$

   then it is a $\mathcal{H}$-weak recursion operator (i.e. Eqn.(B.20) will hold for every k).$\blacksquare$

2. If, moreover, $\widetilde{T} \circ \omega$ is skew-symmetric, which means, in terms of the representative matrices, ω being already skew-symmetric:

   $$T^t \omega = \omega T \tag{B.30}$$

   then the $\mathcal{H}_k$'s defined by Eqn.(B.21) are all constants of the motion for Γ pairwise in involution:

   $$\{\mathcal{H}_k, \mathcal{H}_l\} =: \omega(\Gamma_l, \Gamma_k) = 0 \ \forall k, l \geq 0 \tag{B.31}$$

   where $\{.,.\}$ denotes the Poisson bracket associated with the symplectic form ω.$\blacksquare$

Remark 67 *This last result has the following implications:*

- *As ω is non-degenerate, there can be at most a set of $k \leq n = (1/2)\dim(\mathcal{M})$ (functionally) independent constants of the motion pairwise in involution, and:*

- *If the set is maximal (i.e. $k = n$), the dynamics is completely integrable in the Liouville sense.*

Concerning ω-weak recursion operators, it has also been proved in the literature that, if T has a vanishing Nijenhuis torsion and, moreover, $\widetilde{T} \circ \omega$ is closed:

$$d\left(\widetilde{T} \circ \omega\right) = 0 \tag{B.32}$$

and is skew-symmetric (Eqn.(B.30)), then T is a ω-weak recursion operator.■

All this has the consequence that:

- *If T has a vanishing Nijenhuis torsion:*

$$N_T = 0 \tag{B.33}$$

If :

- *$\widetilde{T} \circ \omega$ is skew-symmetric, i.e., in terms of the representative matrices:*

$$T^t \omega = \omega T \tag{B.34}$$

and if:

- *both $\widetilde{T} \circ \omega$* and *$\widetilde{T} d\mathcal{H}$ are closed:*

$$d\left(\widetilde{T} \circ \omega\right) = d\left(\widetilde{T} d\mathcal{H}\right) = 0 \tag{B.35}$$

then T is a strong recursion operator.■

In the next Section we shall discuss a relevant class of recursion operators that happen to satisfy almost all of the above conditions.

Factorizable Recursion Operators

We will consider here dynamical systems that are *bi-Hamiltonian*[130]. A dynamical vector field Γ is bi-Hamiltonian if there exist two pairs $(\omega_1, \mathcal{H}_1)$ and $(\omega_2, \mathcal{H}_2)$ such that[131]:

$$i_\Gamma \omega_1 = d\mathcal{H}_1 \tag{B.36}$$

[130] Or, for that matter, *bi-Lagrangian*.

[131] In the Lagrangian case the same rôle will be played by the Lagrangian two-forms and the associated energy functions.

as well as:

$$i_\Gamma \omega_2 = d\mathcal{H}_2 \tag{B.37}$$

At least one of the two closed two- forms, say ω_1, will be assumed to be non-degenerate, hence a symplectic form. As such, it will have an inverse ω_1^{-1} which will be the bivector (actually a $(2,0\ )$ tensor, a Poisson tensor):

$$\omega_1^{-1} = \frac{1}{2}(\omega_1)^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}; \quad (\omega_1)^{ik}(\omega_1)_{kj} = \delta^i{}_j \tag{B.38}$$

Out of the two symplectic forms we can then build up the $(1,1)$ tensor T defined via:

$$\left(\widetilde{T} \circ \omega_1\right)(X,Y) =: \omega_1 (TX, Y) = \omega_2 (X,Y) \tag{B.39}$$

or, for short:

$$T = \omega_1^{-1} \circ \omega_2 \tag{B.40}$$

Explicitly:

$$T = T^i{}_j dx^j \otimes \frac{\partial}{\partial x^i};\ T^i{}_j = (\omega_1)^{ik}(\omega_2)_{kj} \tag{B.41}$$

$(1,1)$ tensors that can constructed via the composition of a $(2,0)$ and of a $(0,2)$ tensor will be called *factorizable*.

From now on, ω_1 and $\mathcal{H}_1$ will play the rôle of the $\omega, \mathcal{H}$ of the previous Section.

Remark 68 *It is pretty obvious from the definition (B.39) that:*

$$Ker\,(T) \equiv Ker\,(\omega_2) \tag{B.42}$$

As the kernel of a closed two-form is is a Lie subalgebra of $\mathcal{X}(\mathcal{M})$, i.e. it is involutive, if: $\dim Ker\,(T)$ *has constant dimension, it is also a distribution. Moreover, T will be invertible* $\left(\det \left\|T^i{}_j\right\| \neq 0\right)$ *iff, besides ω_1, ω_2 is also non-degenerate, and hence symplectic as well.*

The $(1,1)$ tensor T is a natural candidate for a recursion operator. Indeed, let us prove first that the closure condition for $\widetilde{T}d\mathcal{H}_1$is satisfied. We have:

$$Td\mathcal{H}_1 = \frac{\partial \mathcal{H}_1}{\partial x^i} T^i{}_j dx^j \equiv \frac{\partial \mathcal{H}_1}{\partial x^i} (\omega_1)^{ik} (\omega_2)_{kj}\, dx^j \tag{B.43}$$

But: $i_\Gamma \omega_1 = d\mathcal{H}_1$ implies:

$$\frac{\partial \mathcal{H}_1}{\partial x^i} (\omega_1)^{ik} = \Gamma^k \tag{B.44}$$

and hence:

$$Td\mathcal{H}_1 = d\mathcal{H}_2 \tag{B.45}$$

which proves that $Td\mathcal{H}_1$ is not only closed, but also exact.■

Moreover:

$$\begin{aligned}\omega_1(TX,Y) &= \omega_2(X,Y) = -\omega_2(Y,X) \\ &= -\omega_1(TY,X) = \omega_1(X,TY)\end{aligned} \tag{B.46}$$

which proves (cfr. Eqn. (B.14)) that $\widetilde{T} \circ \omega_1$ is skew-symmetric.

This result could have been inferred more directly from Eqn.(B.39) which states that:

$$\widetilde{T} \circ \omega_1 = \omega_2 \tag{B.47}$$

which allows us also to conclude that $\widetilde{T} \circ \omega_1$ is a closed two-form.

Therefore we obtain the following result:

If the $(1,1)$ *tensor field (*B.40*) satisfies the Nijenhuis condition, i.e. if:*

$$N_T = 0 \tag{B.48}$$

then T *is a strong recursion operator.*■

C Symplectic Fourier Transform

Introduction

Let us consider, for simplicity [77, 240], $\mathbb{R}^2 \approx \mathbb{T}^*\mathbb{R}$ with coordinates (q, p). The standard Fourier transform (e.g. in $L_2\left(\mathbb{R}^2\right)$) of a function $f = f\,(q, p)$ is defined as:

$$\mathcal{F}(f)\,(\eta, \xi) = \iint \frac{dqdp}{2\pi} \exp\{-i(q\eta + p\xi)\} f\,(q, p) \tag{C.1}$$

with the known inversion formula (again in the sense of $L_2\left(\mathbb{R}^2\right)$) :

$$f\,(q, p) = \iint \frac{d\eta d\xi}{2\pi} \exp\{i(q\eta + p\xi)\} \mathcal{F}(f)\,(\eta, \xi) \tag{C.2}$$

Notice that, with the standard Euclidean metric in $\mathbb{R}^2$, $g = diag\,(1, 1)$, $q\eta + p\xi = g\,((q, p), (\eta, \xi)$. Introducing the canonical symplectic form $\omega_D = dq \wedge dp$, with representative matrix:

$$\Omega_D = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{C.3}$$

the *symplectic* Fourier transform $\mathcal{F}_s\,(f)$ is defined as:

$$\mathcal{F}_s\,(f)\,(\eta, \xi) =: \iint \frac{dqdp}{2\pi} \exp\left\{-i\omega_D\left((q, p), (\xi, \eta)\right\} f\,(q, p)\right. \tag{C.4}$$

where, explicitly:

$$\omega_D\left((q, p), \xi, \eta\right) = \begin{vmatrix} q & p \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} \xi \\ \eta \end{vmatrix} = q\eta - p\xi \tag{C.5}$$

Therefore:

$$\mathcal{F}_s\,(f)\,(\eta, \xi) = \mathcal{F}\,(f)\,(\eta, -\xi) \tag{C.6}$$

and the transform can be inverted into:

$$f\,(q, p) = \iint \frac{d\eta d\xi}{2\pi} \exp\left\{i\omega_D\left((q.p), (\xi, \eta)\right)\right\} \mathcal{F}_s\,(f)\,(\eta, \xi) \tag{C.7}$$

or:

$$f\,(q, p) = \iint \frac{d\eta d\xi}{2\pi} \exp\left\{-i\omega_D\left((\xi, \eta), (q, p)\right)\right\} \mathcal{F}_s\,(f)\,(\eta, \xi) \tag{C.8}$$

where, explicitly: $\omega_D\left((q, p), (\xi, \eta)\right) = q\eta - p\xi$.

A generic constant symplectic structure ω in $\mathbb{R}^2$ is of course associated with a (real) skew-symmetric matrix of the form:

$$\Omega = \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}, \quad a \neq 0 \tag{C.9}$$

and there exists a nonsingular matrix $T \in \mathcal{A}ut\left(\mathbb{R}^2\right) = GL\left(2,\mathbb{R}\right)$ (a $(1,1)$ tensor) such that:

$$\Omega = \widetilde{T}\omega_D T \tag{C.10}$$

i.e. (always remember that, by definition: $\left(\widetilde{T}\right)_i{}^j = T^j{}_i$):

$$\omega(x,y) = \omega_D\left(Tx, Ty\right),\ x, y \in \mathbb{R}^2 \tag{C.11}$$

Indeed, if:

$$T = \begin{vmatrix} \lambda & \mu \\ \nu & \rho \end{vmatrix} \tag{C.12}$$

then the previous condition only requires:

$$\det T = \lambda\rho - \mu\nu = a \tag{C.13}$$

and T will be actually defined "modulo" left multiplication by any matrix U with $\det U = 1$, i.e.: $U \in Sp\left(2,\mathbb{R}\right) \approx SL\left(2,\mathbb{R}\right)$: $\widetilde{U}\omega_D U = \omega_D$. In this slightly more general setting, the symplectic Fourier transform is defined as:

$$\mathcal{F}_{sT}\left(f\right)\left(\eta,\xi\right) = \frac{J}{2\pi}\iint dqdp \exp\left\{-i\omega\left(\left(q,p\right),\left(\xi,\eta\right)\right\} f\left(q,p\right) \tag{C.14}$$

where: $J =: \det T$. Now, if: $T(q,p) =: (x,k)$, then:

$$\frac{\partial\left(q,p\right)}{\partial\left(x,k\right)} = J^{-1} \tag{C.15}$$

Moreover, with: $X =: \left(q,p\right), Y =: (x,k)$, $TX = Y$ and: $Z = (\xi,\eta)$, we have: $\omega\left(\left(q,p\right),\left(\xi,\eta\right) = \omega\left(T^{-1}Y, Z\right) = \omega_D\left(Y, TZ\right)$. Hence, changing variables:

$$\mathcal{F}_{sT}\left(f\right)\left(\eta,\xi\right) = \iint \frac{dxdk}{2\pi}\left(f \circ T^{-1}\right)\left(x,k\right) \exp\left\{-i\omega_D\left(\left(x,k\right), T\left(\xi,\eta\right)\right)\right\} \tag{C.16}$$

i.e., setting: $\left(\xi_T, \eta_T\right) =: T\left(\xi,\eta\right)$:

$$\mathcal{F}_{sT}\left(f\right)\left(\eta,\xi\right) = \mathcal{F}_s\left(f \circ T^{-1}\right)\left(\eta_T, \xi_T\right) \tag{C.17}$$

Noticing that:

$$f\left(q,p\right) \equiv \left(f \circ T^{-1}\right)\left(T\left(q,p\right)\right) \tag{C.18}$$

we can write, using the inversion formula for the "canonical" symplectic transform:

$$f\left(q,p\right) = \iint \frac{d\xi_T d\eta_T}{2\pi}\mathcal{F}_s\left(f \circ T^{-1}\right)\left(\eta_T, \xi_T\right) \exp\left\{-i\omega_D\left(\left(\xi_T, \eta_T\right), T\left(q,p\right)\right)\right\} \tag{C.19}$$

or:

$$f\left(q,p\right) = \iint \frac{d\xi_T d\eta_T}{2\pi}\mathcal{F}_{sT}\left(f\right)\left(\eta,\xi\right) \exp\left\{-i\omega_D\left(T\left(\xi,\eta\right), T\left(q,p\right)\right)\right\} \tag{C.20}$$

and eventually $\left(\partial\left(\xi_T, \eta_T\right)/\partial\left(\xi,\eta\right) = J\right)$ we obtain the inversion formula:

$$f\left(q,p\right) = \frac{J}{2\pi}\iint d\xi d\eta \mathcal{F}_{sT}\left(f\right)\left(\eta,\xi\right) \exp\left\{-i\omega\left(\left(\xi,\eta\right),\left(q,p\right)\right)\right\} \tag{C.21}$$

Equivariance

What remains to be discussed is the role of the ambiguity in the definition of T (T and UT, $U \in Sp\,(2,\mathbb{R})$ playing the same role). The question is whether or not $\mathcal{F}_{sT}\,(f)\,(\eta,\xi)$ and $\mathcal{F}_{sUT}\,(f)\,(\eta,\xi)$, i.e. $\mathcal{F}_s\left(f\circ T^{-1}\right)(\eta_T,\xi_T)$ and $\mathcal{F}_s\left(f\circ (UT)^{-1}\right)(\eta_{UT},\xi_{UT})$ define the same symplectic Fourier transform. From the definition:

$$\begin{gathered}\mathcal{F}_s\left(f\circ (UT)^{-1}\right)(\eta_{UT},\xi_{UT}) = \\ = \iint \tfrac{dqdp}{2\pi}\left(f\circ T^{-1}\circ U^{-1}\right)(q,p)\exp\left\{-i\omega_D\left((q,p)\,,U\circ T\,(\xi,\eta)\right)\right\}\end{gathered} \tag{C.22}$$

Setting: $U^{-1}\,(q,p) = (x,k)$ ($\det U = 1$):

$$\begin{gathered}\mathcal{F}_s\left(f\circ (UT)^{-1}\right)(\eta_{UT},\xi_{UT}) = \\ = \iint \tfrac{dqdp}{2\pi}\left(f\circ T^{-1}\right)(x,k)\exp\left\{-i\omega_D\left(U\,(x,k)\,,U\circ T\,(\xi,\eta)\right)\right\}\end{gathered} \tag{C.23}$$

But: $\omega_D\,(U(.),U(.)) = \omega_D\,((.),(.))$, and hence:

$$\begin{gathered}\mathcal{F}_s\left(f\circ (UT)^{-1}\right)(\eta_{UT},\xi_{UT}) = \\ = \iint \tfrac{dqdp}{2\pi}\left(f\circ T^{-1}\right)(x,k)\exp\left\{-i\omega_D\left((x,k)\,,T\,(\xi,\eta)\right)\right\}\end{gathered} \tag{C.24}$$

i.e.:

$$\mathcal{F}_s\left(f\circ (UT)^{-1}\right)(\eta_{UT},\xi_{UT}) = \mathcal{F}_s\left(f\circ T^{-1}\right)(\eta_T,\xi_T) \tag{C.25}$$

$\mathcal{F}_{sT}$ depends then only on the right coset of T in $GL\,(2,\mathbb{R})$ relative to the subgroup $Sp\,(2,\mathbb{R})$ of the symplectic linear maps. This result can be summarized by writing (for $T = \mathbb{I}$, otherwise we substitute f with $f\circ T^{-1}$):

$$\mathcal{F}_s\left(f\circ U^{-1}\right)\circ U = \mathcal{F}_s\,(f) \tag{C.26}$$

or, according to the standard definition of "pull-back" of a map:

$$\phi^*\mathcal{F}_s\,(f) = \mathcal{F}_s\,(\phi^* f) \tag{C.27}$$

where: $\phi = U^{-1} \in Sp(2,\mathbb{R})$, which can then be rephrased by saying that the symplectic Fourier transform is equivariant, or that it is "natural", w.r.t. the symplectic group.

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