A gradient expansion

Describing non-linear structures in a perturbative expansion

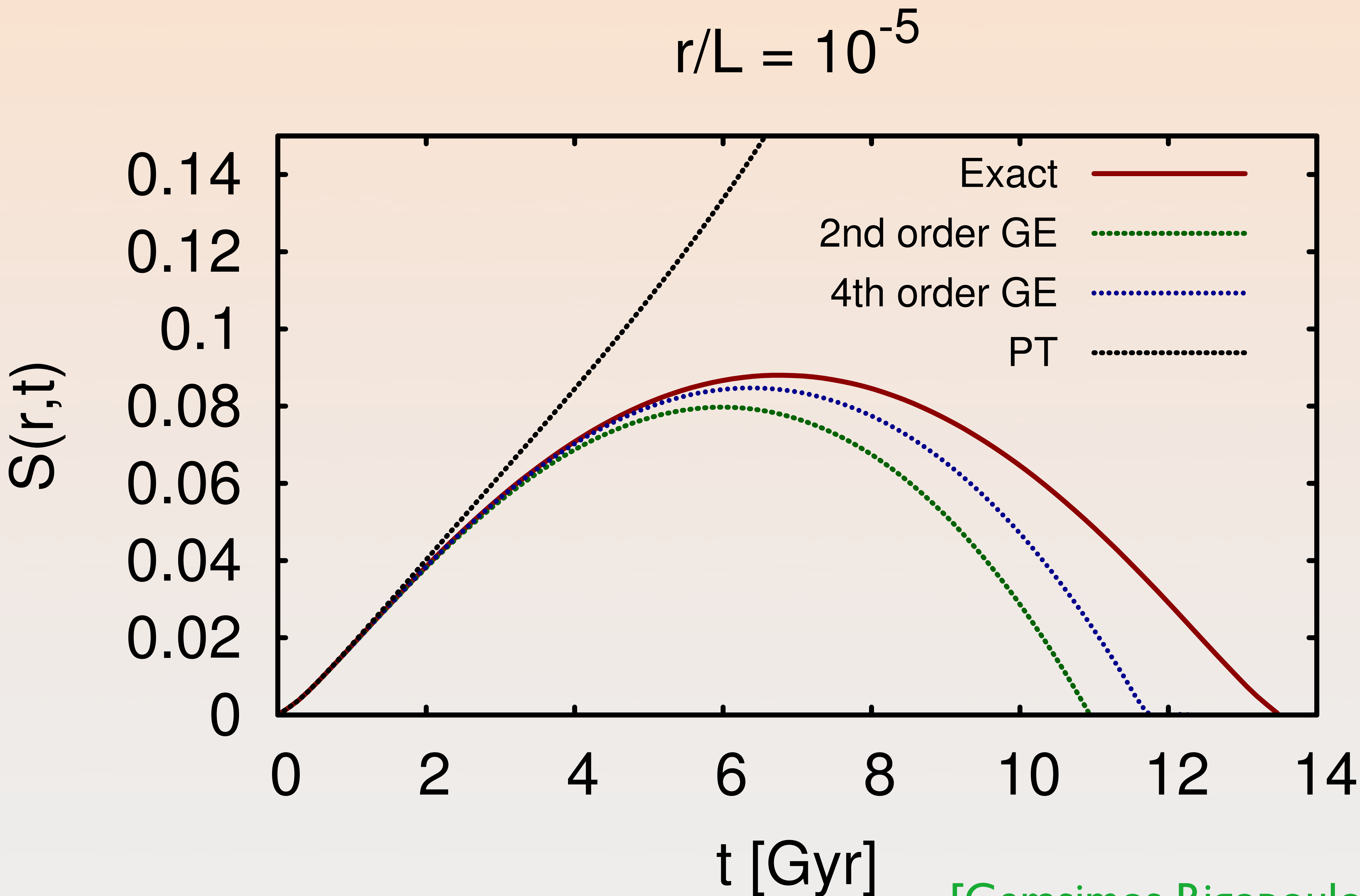
Wessel Valkenburg (Leiden University / Heidelberg University)
at Benasque Cosmology, 22 August 2012

[Gerasimos Rigopoulos, WV, PRD 2012]

Conclusion

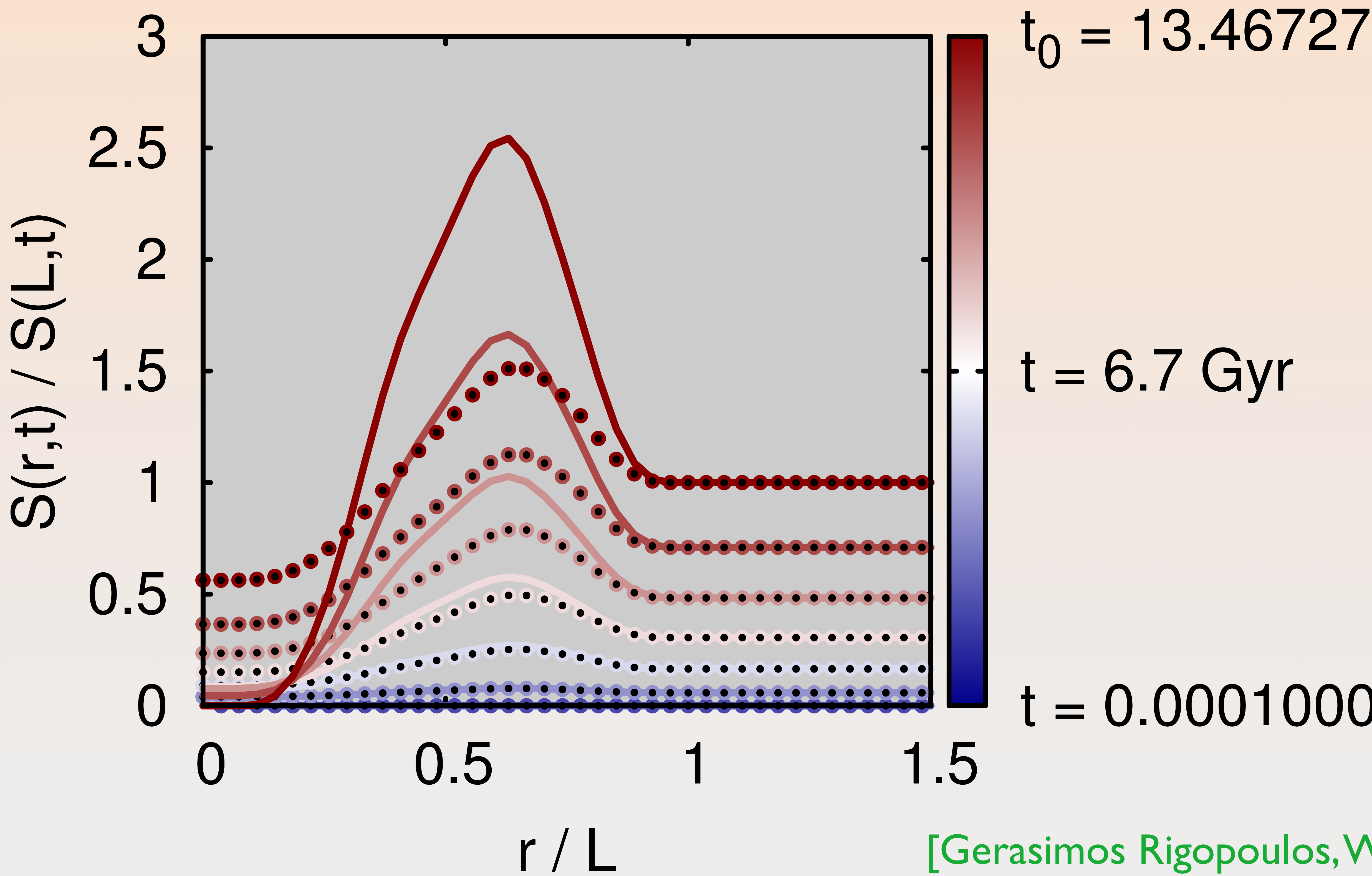
- An expansion in gradients rather then density perturbations, allows for non-linear densities
- The solution to the metric is given in terms of initial conditions and a number of *time*-dependent functions
- The gradient expansion follows exact gravitational collapse remarkably well

Results



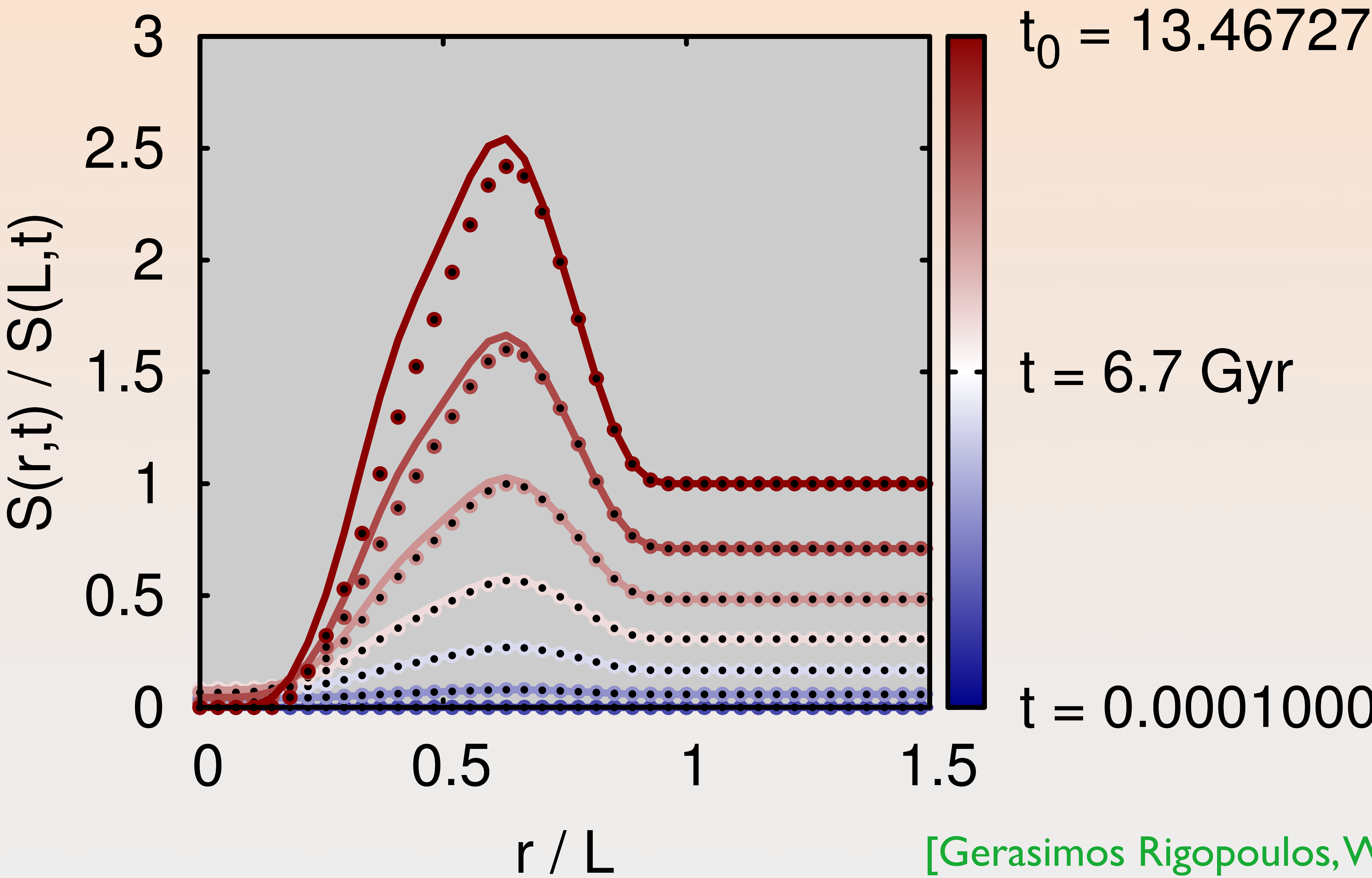
[Gerasimos Rigopoulos, WV, PRD 2012]

Results



[Gerasimos Rigopoulos, WV, PRD 2012]

Results



[Gerasimos Rigopoulos, WV, PRD 2012]

Metric up to 4 gradients

$$
\begin{aligned}
\gamma_{ij}^{(4)} \;=\;& A^2(t)k_{ij} + \lambda(t)(\hat{R}k_{ij} - 4\hat{R}_{ij}) \\
& + A^2(t)\int^t \frac{C_1}{A^2}\,\hat{R}^2 k_{ij} + A^2(t)\int^t \frac{C_2}{A^2}\,\hat{R}^{kl}\hat{R}_{kl}k_{ij} + A^2(t)\int^t \frac{C_3}{A^2}\,\hat{R}\hat{R}_{ij} + A^2(t)\int^t \frac{C_4}{A^2}\,\hat{R}_{ik}\hat{R}^k{}_j \\
& + A^2(t)\int^t \frac{D_1}{A^2}\,\hat{R}^{|k}{}_{|k}k_{ij} + A^2(t)\int^t \frac{D_2}{A^2}\,\hat{R}_{|ij} + A^2(t)\int^t \frac{D_3}{A^2}\,\hat{R}_{ij}{}^{|k}{}_{|k}\,.
\end{aligned}
$$

$$
\begin{aligned}
k_{ij} &= \text{initial seed spatial metric (synch. comov. gauge)} \\
\hat{} &= \text{quantities evaluated on } k_{ij}
\end{aligned}
$$

[Gerasimos Rigopoulos, WV, PRD 2012]

Action for a dusty scalar field in GR

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}\left({}^{(4)}R - 2\Lambda\right) - \frac{1}{2}\rho\left(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + 1\right)\right]$$

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$$\pi^{ij} \equiv \frac{\delta\mathcal{S}}{\delta\dot{\gamma}_{ij}}$$

$$\pi^{\chi} \equiv \frac{\delta\mathcal{S}}{\delta\dot{\chi}} :$$

$$g_{00} = -N^2 + h_{ij}N^iN^j\,, \qquad g_{0i} = \gamma_{ij}N^j\,, \qquad g_{ij} = \gamma_{ij}$$

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$$\mathcal{U} = \frac{2\kappa}{\sqrt{\gamma}} \left(\pi_{ij} \pi^{ij} - \frac{\pi^2}{2} \right) - \frac{\sqrt{\gamma}}{2\kappa} (R - 2\Lambda) + \pi^{\chi} \sqrt{1 + \gamma^{ij} \partial_i \chi \partial_j \chi}$$

$$\mathcal{U}_i = -2\nabla_k \pi^k_i + \pi^{\chi} \partial_i \chi$$

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[Salopek, Stewart, CQG 1993]
[Croudace, Parry, Salopek, Stewart, ApJ 1993]

Hamilton-Jacobi equation

Not treating action as functional of all possible paths, but as function integration bounds.

$$\mathcal{S} = \int d^4x \left(\pi^\chi \frac{\partial \chi}{\partial t} + \pi^{ij} \frac{\partial \gamma_{ij}}{\partial t} - N\mathcal{U} - N_i \mathcal{U}^i \right)$$

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Synchronous co-moving gauge

[Salopek, Stewart, CQG 1993]
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$$\frac{\delta S}{\delta N_i} = 0 \quad \longrightarrow \quad \mathcal{U}_i = -2\nabla_k \pi^k_i + \pi^\chi \partial_i \chi = -2\nabla_k \pi^k_i = 0$$

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$$\frac{\partial \mathcal{S}[\pi_{ij}, \gamma_{ij}, t]}{\partial t} + \mathcal{H}[\pi_{ij}, \gamma_{ij}, t] =$$

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Canonical transformation

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$$\mathcal{H}[\pi_{ij}, \gamma_{ij}, t] = \int d^3x \left\{ \frac{2\kappa}{\sqrt{\gamma}} \left(\pi_{ij}\pi^{ij} - \frac{(\pi^i_i)^2}{2} \right) - \frac{\sqrt{\gamma}}{2\kappa}(R - 2\Lambda) \right\}$$

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$$\hat{\pi}^{ij} \equiv \frac{\delta F[\pi_{ij}, \gamma_{ij}, t]}{\delta \gamma_{ij}}$$

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$$\hat{\pi}^{ij} \equiv \frac{\delta F[\pi_{ij}, \gamma_{ij}, t]}{\delta \gamma_{ij}}$$

$$\mathcal{H}'[\hat{\pi}_{ij}, \gamma_{ij}, t] = \mathcal{H}[\hat{\pi}_{ij}, \gamma_{ij}, t] + \frac{\partial F[\pi_{ij}, \gamma_{ij}, t]}{\partial t} = 0$$

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$F[\pi_{ij}, \gamma_{ij}, t]$ $\Longleftrightarrow$ satisfies HJ-equation $\Longleftrightarrow$ $S[\pi_{ij}, \gamma_{ij}, t]$

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Gradient expansion

- Choose $F[\pi_{ij}, \gamma_{ij}, t]$ such that it preserves covariance of Hamiltionian and action
- Expand in a hierarchy of number of gradients in the metric

Ansatz: $$S[\pi_{ij}, \gamma_{ij}, t] = F[\pi_{ij}, \gamma_{ij}, t] = \int d^3x \sqrt{\gamma}\mathcal{F}[\pi_{ij}, \gamma_{ij}, t]$$

$$\mathcal{F} = -2H(t) + J(t)R + L_1(t)R^2 + L_2(t)R^{ij}R_{ij} + \ldots$$

[Salopek, Stewart, CQG 1993]
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$$\mathcal{F} = -2H(t) + J(t)R + L_1(t)R^2 + L_2(t)R^{ij}R_{ij} + \ldots$$

$$\frac{\partial \mathcal{F}[\pi_{ij}, \gamma_{ij}, t]}{\partial t} + \int d^3x \left\{ \frac{2\kappa}{\sqrt{\gamma}} \frac{\delta \mathcal{F}}{\delta \gamma_{ij}} \frac{\delta \mathcal{F}}{\delta \gamma_{kl}} \left(\gamma_{ik}\gamma_{jl} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\right) - \frac{\sqrt{\gamma}}{2\kappa}(R - 2\Lambda) \right\} = 0$$

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$$\mathcal{F} = -2H(t) + J(t)R + L_1(t)R^2 + L_2(t)R^{ij}R_{ij} + \ldots$$

$$\begin{aligned} \frac{dH}{dt} + \frac{3}{2}H^2 - \frac{\Lambda}{2} &= 0 \\ \frac{dJ}{dt} + JH - \frac{1}{2} &= 0 \\ \frac{dL_1}{dt} - L_1 H - \frac{3}{4}J^2 &= 0 \\ \frac{dL_2}{dt} - L_2 H + 2J^2 &= 0 \end{aligned}$$

Solving the Hamilton-Jacobi equation for each number of gradients separately.

$$\frac{\partial \mathcal{F}[\pi_{ij}, \gamma_{ij}, t]}{\partial t} + \int d^3x \left\{ \frac{2\kappa}{\sqrt{\gamma}} \frac{\delta \mathcal{F}}{\delta \gamma_{ij}} \frac{\delta \mathcal{F}}{\delta \gamma_{kl}} \left(\gamma_{ik}\gamma_{jl} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\right) - \frac{\sqrt{\gamma}}{2\kappa}(R - 2\Lambda) \right\} = 0$$

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$$\frac{\partial \gamma_{ij}}{\partial t} = \frac{\partial \mathcal{H}'[\hat{\pi}_{ij}, \gamma_{ij}, t]}{\partial \hat{\pi}_{ij}}$$

$$= \frac{2}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} (2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl})$$

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$$\mathcal{F} = -2H(t) + J(t)R + L_1(t)R^2 + L_2(t)R^{ij}R_{ij} + \dots$$

$$\begin{aligned}
\gamma_{ij}^{(4)} \;=\;& A^2(t)k_{ij} + \lambda(t)(\hat{R}k_{ij} - 4\hat{R}_{ij}) \\
& + A^2(t)\int^t \frac{C_1}{A^2}\hat{R}^2 k_{ij} + A^2(t)\int^t \frac{C_2}{A^2}\hat{R}^{kl}\hat{R}_{kl}k_{ij} + A^2(t)\int^t \frac{C_3}{A^2}\hat{R}\hat{R}_{ij} + A^2(t)\int^t \frac{C_4}{A^2}\hat{R}_{ik}\hat{R}^k{}_j \\
& + A^2(t)\int^t \frac{D_1}{A^2}\hat{R}^{|k}{}_{|k}k_{ij} + A^2(t)\int^t \frac{D_2}{A^2}\hat{R}_{|ij} + A^2(t)\int^t \frac{D_3}{A^2}\hat{R}_{ij}{}^{|k}{}_{|k} \,.
\end{aligned}$$


Anybody still listening?

Let's compare to something real

- Exact solutions for a spherically symmetric collapsing object embedded in FLRW, using the LTB metric [WV, GERG 2012]

$$ds^2 = -dt^2 + \frac{a(r,t) + ra'(r,t)}{1+2E(r)} dr^2 + r^2 a^2(r,t) d\Omega^2$$

$$H_0 \left[t(A) - t_{BB}(r) \right] = \frac{1}{\sqrt{\Omega_\Lambda}} \frac{(-1)^{-\frac{3n}{2}}}{\sqrt{\prod_{m=1}^{n} y_m}} \int_0^\infty \frac{d\,b}{(b+\frac{1}{A})\sqrt{\prod_{m=1}^{n}(b+\frac{1}{A}-\frac{1}{y_m})}}$$

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$$ds^2 = -dt^2 + \frac{a(r,t) + ra'(r,t)}{1+2E(r)} dr^2 + r^2 a^2(r,t) d\Omega^2$$

$$H_0\left[t(A) - t_{BB}(r)\right] = \frac{1}{\sqrt{\Omega_\Lambda}} \frac{(-1)^{-\frac{3n}{2}}}{\sqrt{\prod_{m=1}^{n} y_m}} \int_0^\infty \frac{d\,b}{(b+\frac{1}{A})\sqrt{\prod_{m=1}^{n}(b+\frac{1}{A}-\frac{1}{y_m})}}$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \frac{(-1)^{-\frac{9}{2}}}{\sqrt{\prod_{m=1}^{3} y_m}} R_J\left(\frac{1}{A}-\frac{1}{y_1}, \frac{1}{A}-\frac{1}{y_2}, \frac{1}{A}-\frac{1}{y_3}, \frac{1}{A}\right)$$

Let's compare to something real

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$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{s(t)(t+p)}$$

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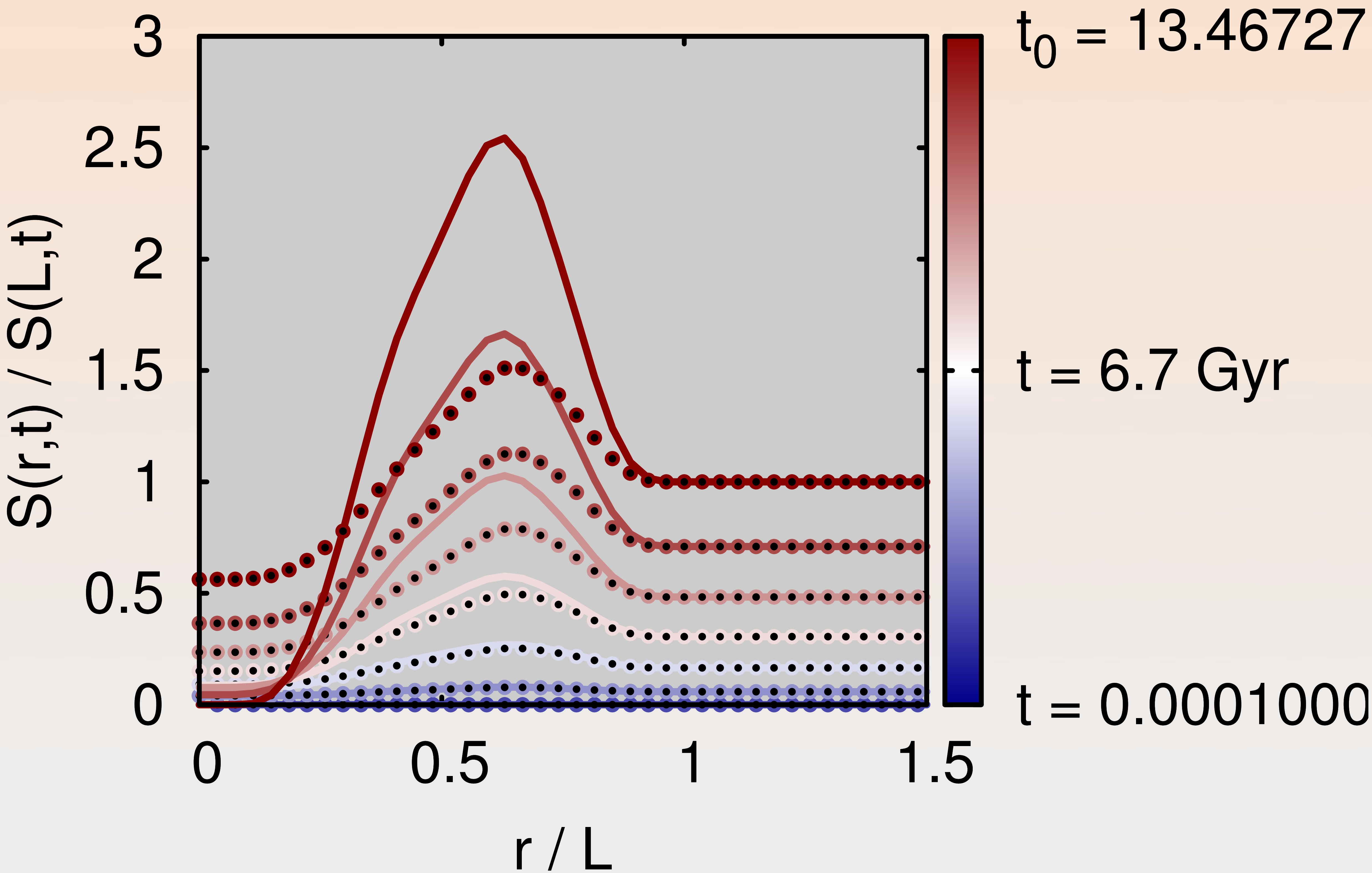
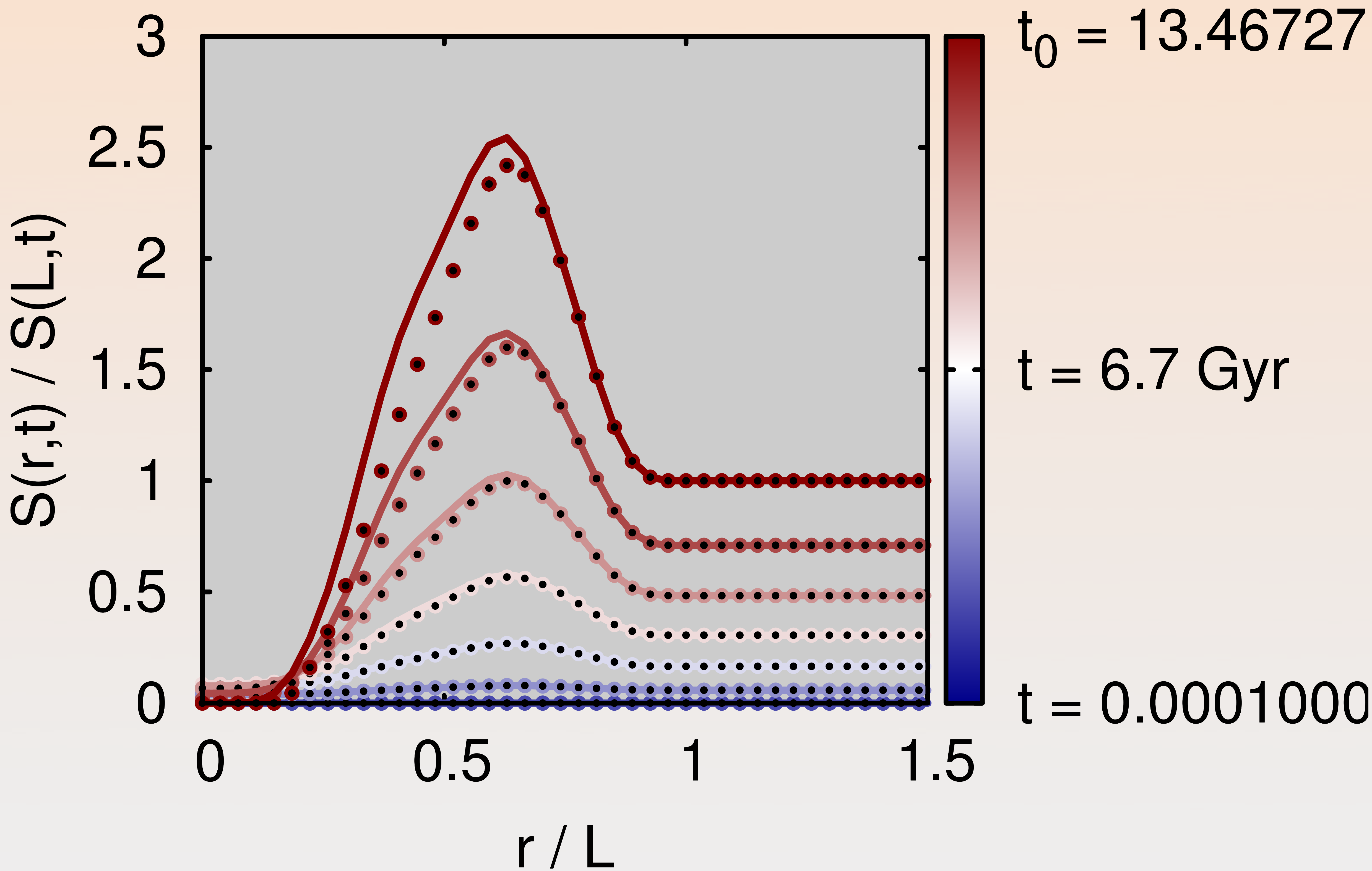
$$\begin{aligned}
\tilde{M}a'(r,t) = &-\frac{Y'}{Z}\frac{A}{(z_1-z_2)(z_1-z_3)} - \frac{\sqrt{Z}\sqrt{(A-z_1)(A-z_2)(A-z_3)}}{\sqrt{A}} \times \\
&\left[\frac{Y'}{3Z^{\frac{3}{2}}}\frac{i}{(z_1 z_2 z_3)^{\frac{1}{2}}}\left(\frac{1}{(z_2-z_1)(z_2-z_3)} - \frac{1}{(z_1-z_2)(z_1-z_3)}\right) R_D\left(\frac{1}{A}-\frac{1}{z_3}, \frac{1}{A}-\frac{1}{z_1}, \frac{1}{A}-\frac{1}{z_2}\right)\right. \\
&+\frac{Y'}{3Z^{\frac{3}{2}}}\frac{i}{(z_1 z_2 z_3)^{\frac{1}{2}}}\left(\frac{1}{(z_3-z_1)(z_3-z_2)} - \frac{1}{(z_1-z_2)(z_1-z_3)}\right) R_D\left(\frac{1}{A}-\frac{1}{z_1}, \frac{1}{A}-\frac{1}{z_2}, \frac{1}{A}-\frac{1}{z_3}\right) \\
&\left.+\tilde{M}\partial_r t_{BB}(r) \vphantom{\frac{1}{1}}\right],
\end{aligned}$$

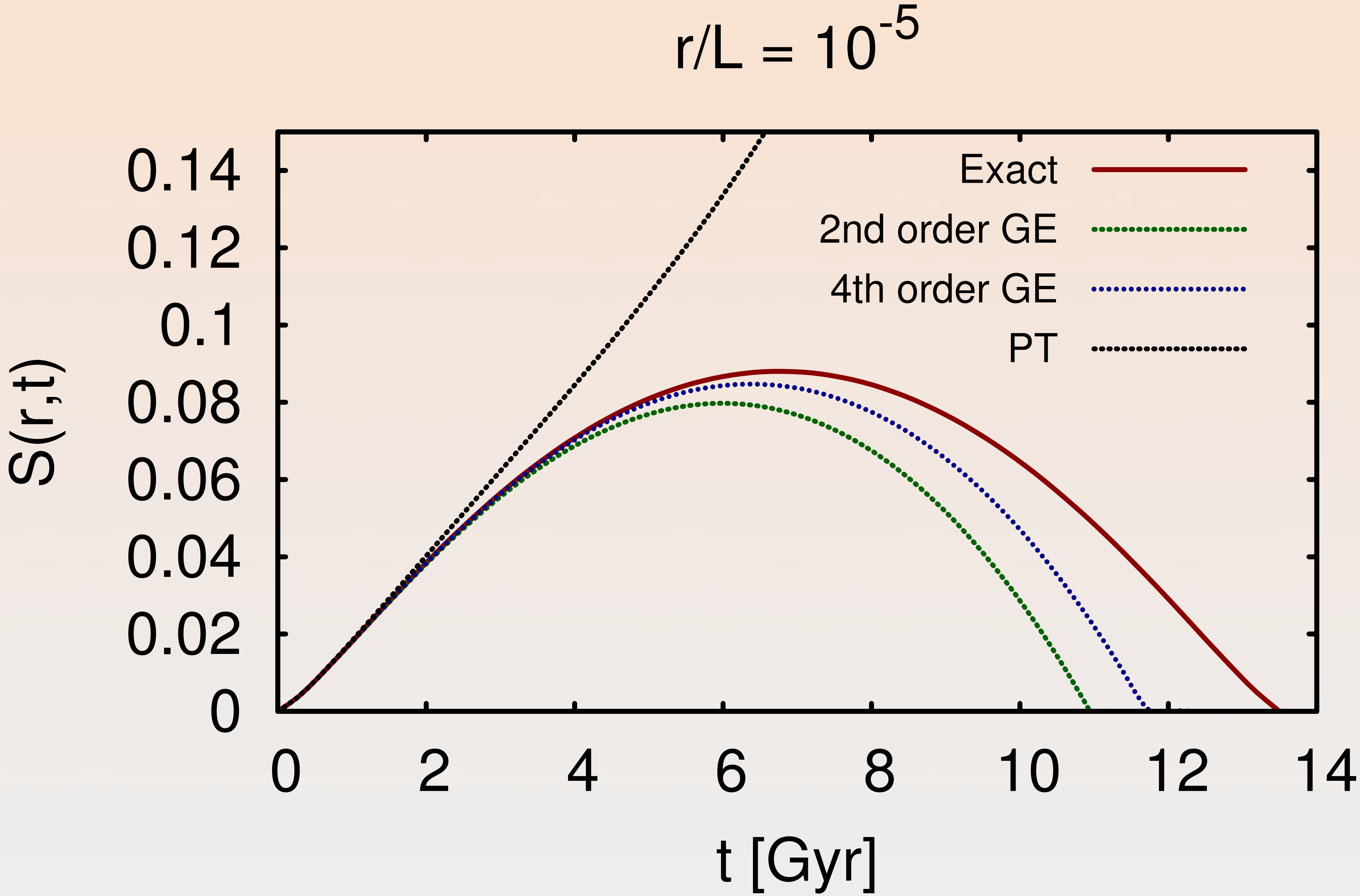
Results



Results



Results



r/L = 10^-5
Exact
2nd order GE
4th order GE
PT
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
S(r,t)
0
2
4
6
8
10
12
14
t [Gyr]

Range of applicability

$$k_{ij} = \left(\frac{t_{\rm i}}{t_0}\right)^{4/3} \delta_{ij} \left(1 + \frac{10}{3}\Phi(\mathbf{x})\right)$$

$$\gamma_{ij} \simeq \left(\frac{t}{t_0}\right)^{4/3} \left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right] + \mathcal{O}(6)$$

$$\frac{t_{\rm con}}{t_0} \simeq 3.4 \times \frac{H_0^3}{\left(\nabla^2\Phi\right)^{3/2}} \longrightarrow \text{Poisson} \longrightarrow \delta\rho_0 \simeq 2.5 \times \Omega_m .$$

Zel'dovich approximation

$$\gamma_{ij} \simeq \left(\frac{t}{t_0}\right)^{4/3} \left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right] + \mathcal{O}(6)$$

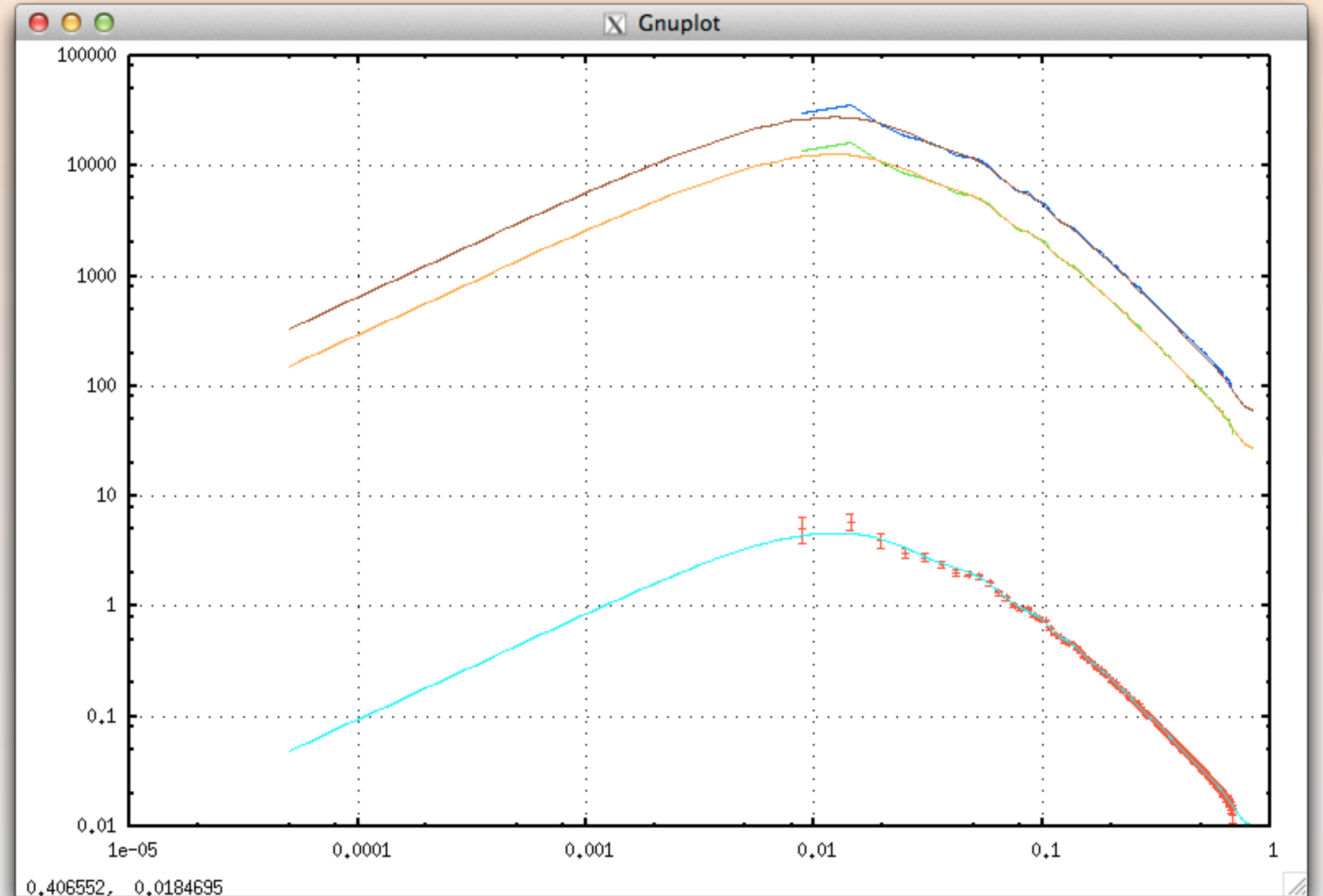
$$\rho \propto \frac{1}{\sqrt{\det \gamma_{ij}}}$$

$$\rho(t, \mathbf{x}) = \frac{1}{6\pi G\, t^2} \frac{1}{\sqrt{\mathrm{Det}\left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right]}}$$

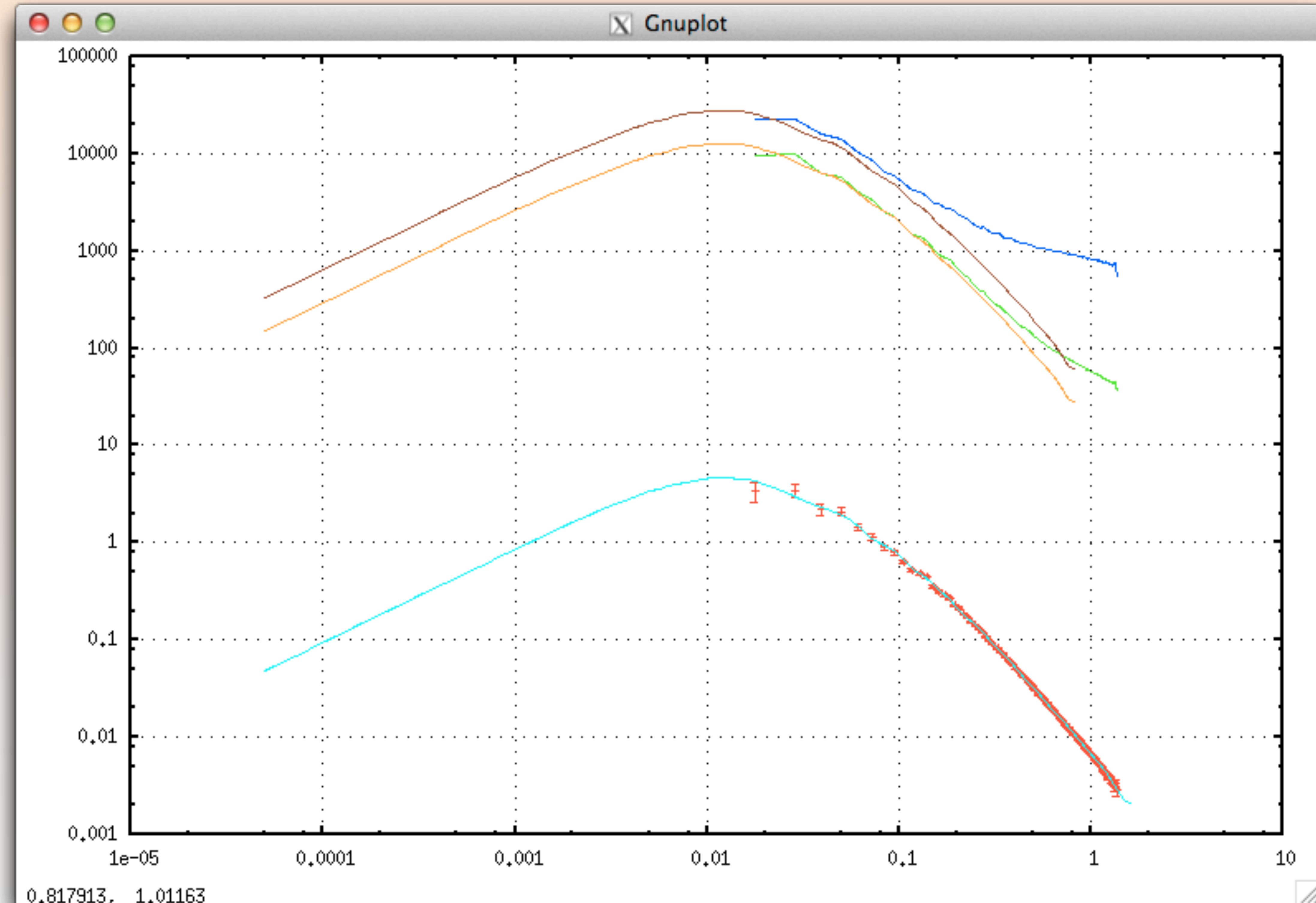
Outlook

- Get P(k) for standard cosmology and compare to other non-linear extensions

Preprepreliminary



Preprepreliminary



Conclusion

- An expansion in gradients rather then density perturbations, allows for non-linear densities
- The solution to the metric is given in terms of initial conditions and a number of *time*-dependent functions
- The gradient expansion follows exact gravitational collapse remarkably well