From small scales to the horizon: a nonlinear post-Friedmann framework for structure formation in ΛCDM

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Credits

- work with Irene Milillo (Rome) and Daniele Bertacca (Cape Town)
- current developments with Daniel B. Thomas and David Wands (ICG)

Outline

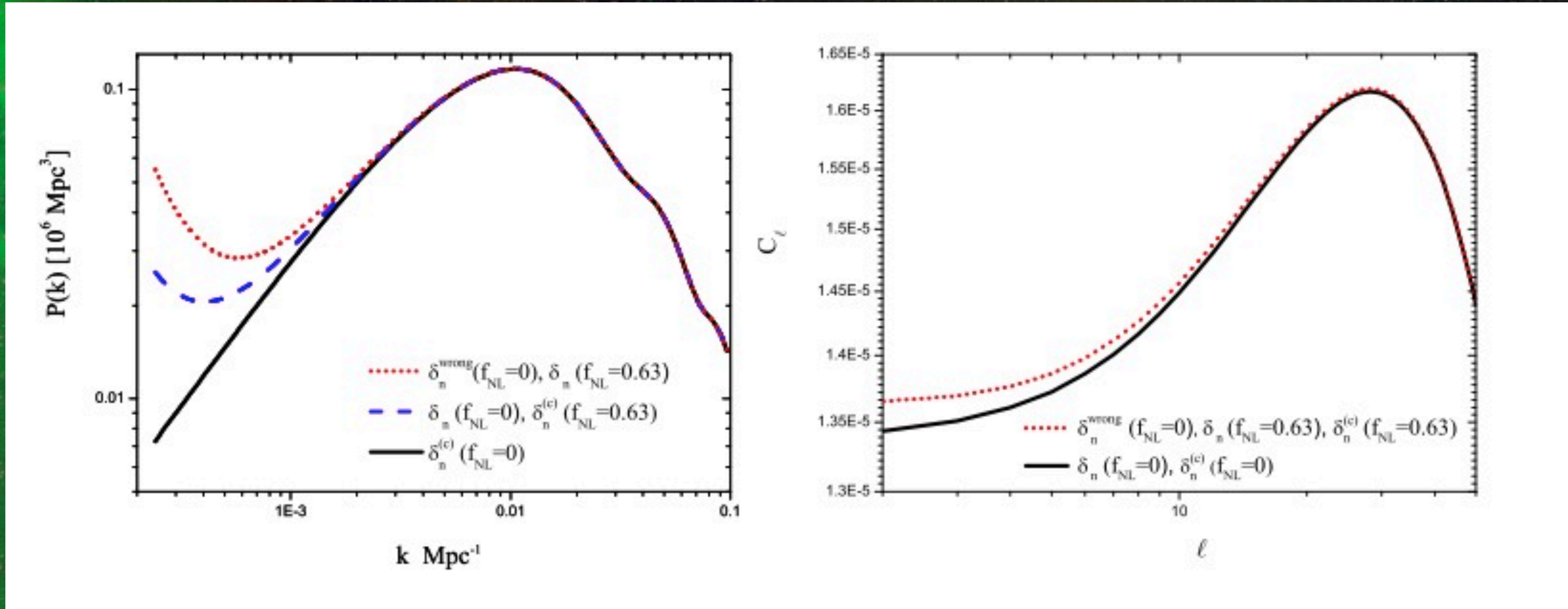
- the 3 ingredients of standard cosmology and the standard model, ΛCDM
- aims of Relativistic Cosmology
- non-linear Post-Friedmann ΛCDM: a new post-Newtonian type approximation scheme for cosmology
- Outlook and work in progress

"take home message"

- it is important to consider relativistic effects in structure formations
  - e.g. the matter power spectrum on large scales

MB, Crittenden, Koyama, Maartens, Pitrou & Wands, *Disentangling non-Gaussianity, bias and GR effects in the galaxy distribution*, arXiv:1106.3999, PRD 85 (2012)

see Bonvin & Durrer PRD 84 (2011) and Challinor & Lewis PRD 84 (2011)

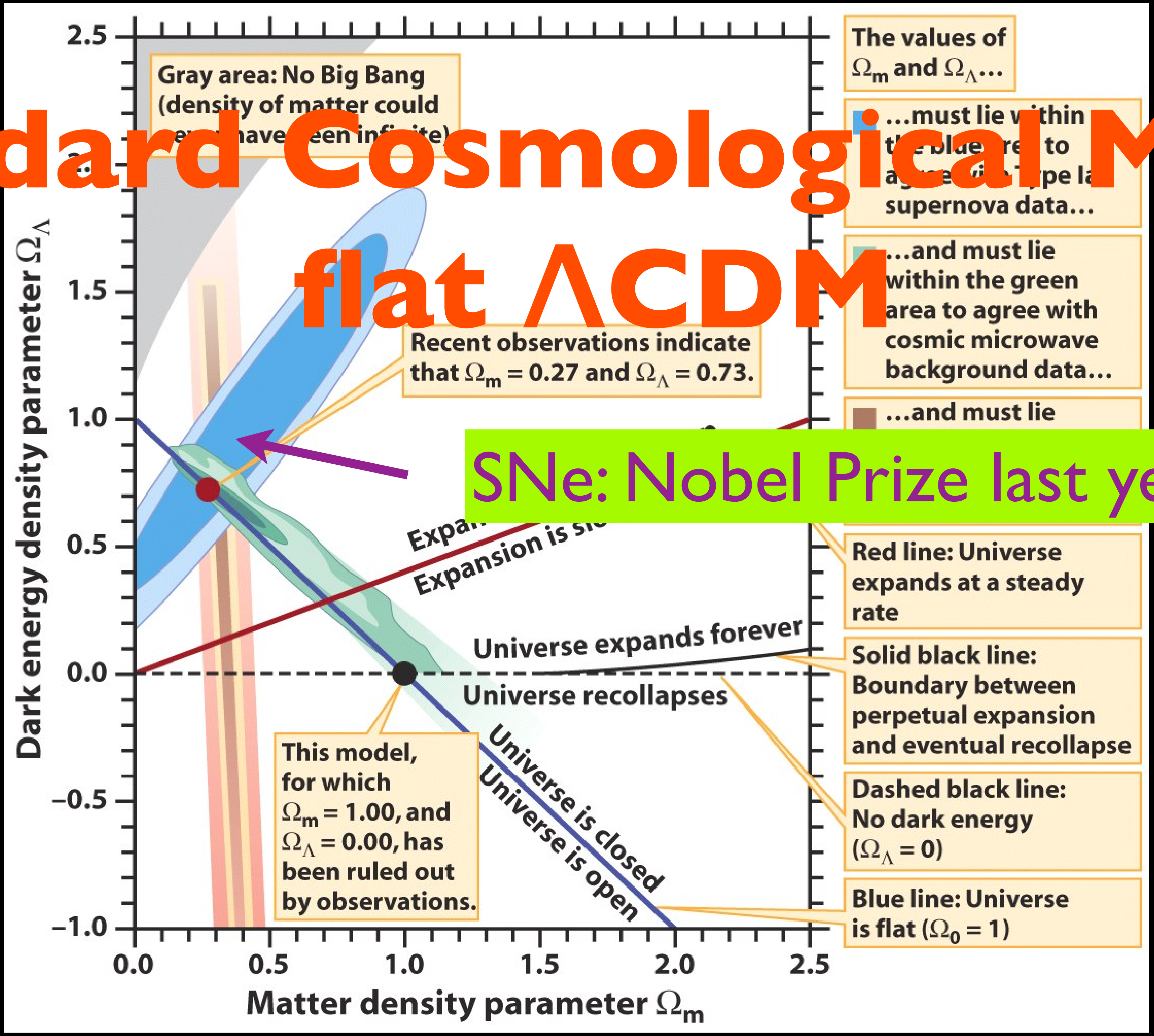


Standard Cosmology

- Recipe for modelling based on 3 main ingredients:
  1. Homogeneous isotropic background, FLRW models
  2. Relativistic Perturbations (e.g. CMB)
  3. Newtonian study of non-linear structure formation (numerical simulations or approx. techniques)
- on this basis, well supported by observations, the flat ΛCDM model has emerged as the Standard "Concordance" Model of cosmology.

Standard Cosmology

Standard Cosmological Model: flat ΛCDM



SNe: Nobel Prize last year!

Questions on ΛCDM

- Recipe for modelling based on 3 main ingredients:
  1. Homogeneous isotropic background, FRW models
  2. Relativistic Perturbations (e.g. CMB)
  3. Newtonian study of non-linear structure formation (numerical simulations or approx. techniques)
- do we really need $\Omega_\Lambda \approx 0.7$? (or some other form of Dark Energy)
- Is 3 good enough? (more data, precision cosmology, observations and simulations covering large fraction of H^{-1}, etc...)

Alternatives to ΛCDM

ΛCDM is the simplest and very successful model supporting the observations that, assuming the Cosmological Principle, are interpreted as acceleration of the Universe expansion

Going beyond ΛCDM, two main alternatives:

1. Maintain the Cosmological Principle (FLRW background), then either

   a) maintain GR + dark components (CDM+DE or UDM)

   b) modified gravity (f(R), branes, etc...)

Alternatives to ΛCDM

Going beyond ΛCDM, two main alternatives:

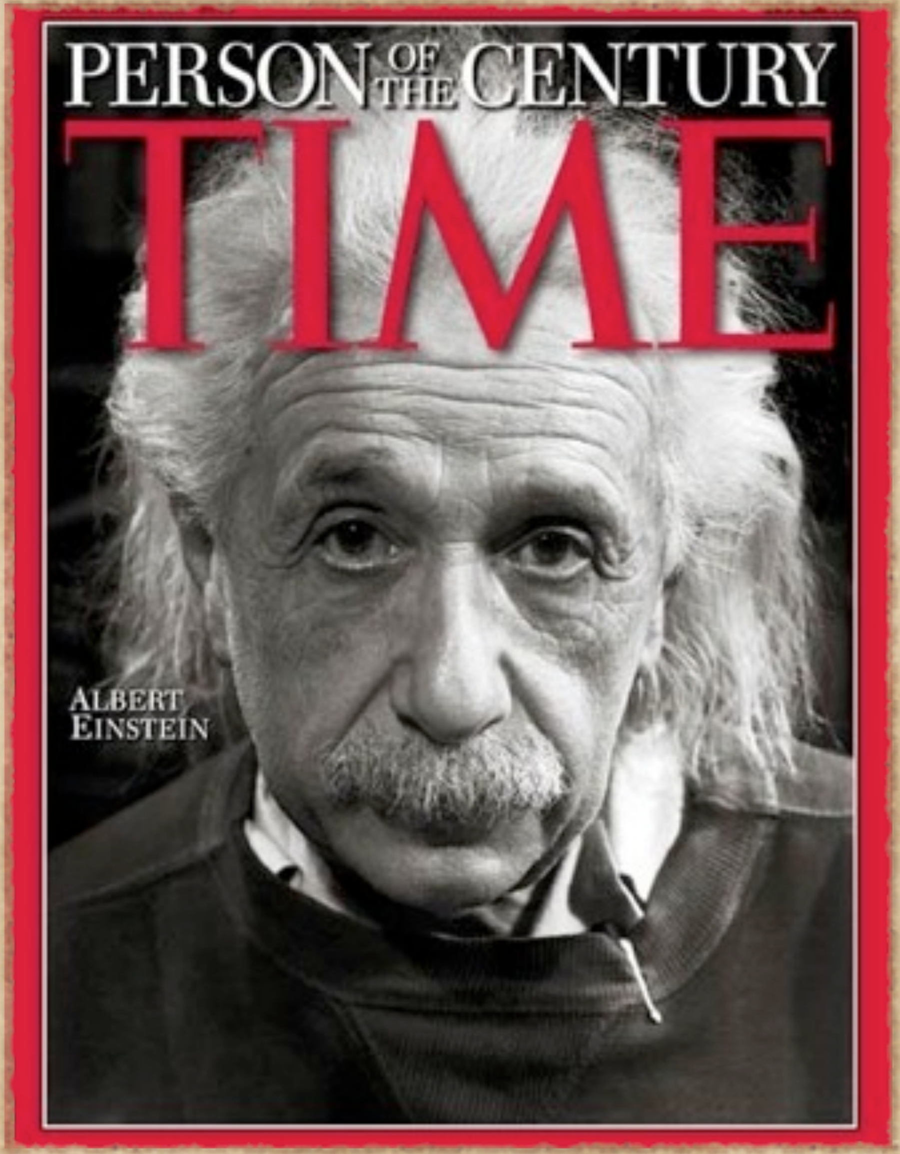
2. Maintain GR, then either

   a) consider inhomogeneous models, e.g. LTB (violating the CP) or Szekeres (not necessarily violating the CP): back-reaction on observations

   b) try to construct an homogeneous isotropic model from averaging, possibly giving acceleration: dynamical back-reaction

Aims of Relativistic Cosmology

- in view of future data, is Newtonian non-linear structure formation good enough?
- GR itself highly successful theory of gravitational interaction between bodies, but we don't know how to average E.E.s
- back-reaction may be relevant: if not dynamically, on light propagation through inhomogeneities (e.g. effects on distances)
- relativistic effects relevant on large scales (e.g. Power Spectrum)



TIME cover, January 2000

back-reaction

- in essence, back-reaction is typical of non-linear systems, a manifestation of non-linearity
- in cosmology, we may speak of two types of BR(*):
  - Strong BR: proper dynamical BR, i.e. the growth of structure really changes the expansion
    - in perturbation theory BR neglected by construction
    - In essence, in a a Newtonian N-body simulations a big volume is conformally expanded, neglecting back-reaction
  - Weak BR: optical BR, i.e. effects of inhomogeneities on observations (neglected in SNa, but the essence of lensing and ISW)

(*) Kolb, E.W., Marra, V. & Matarrese, S., 2010, GRG 42(6), pp.1399–1412.

the strong BR challenge

- standard flat ΛCDM: $\Omega_M + \Omega_\Lambda = 1, \qquad \Omega_\Lambda^{'} = 3\Omega_\Lambda(1-\Omega_\Lambda)$

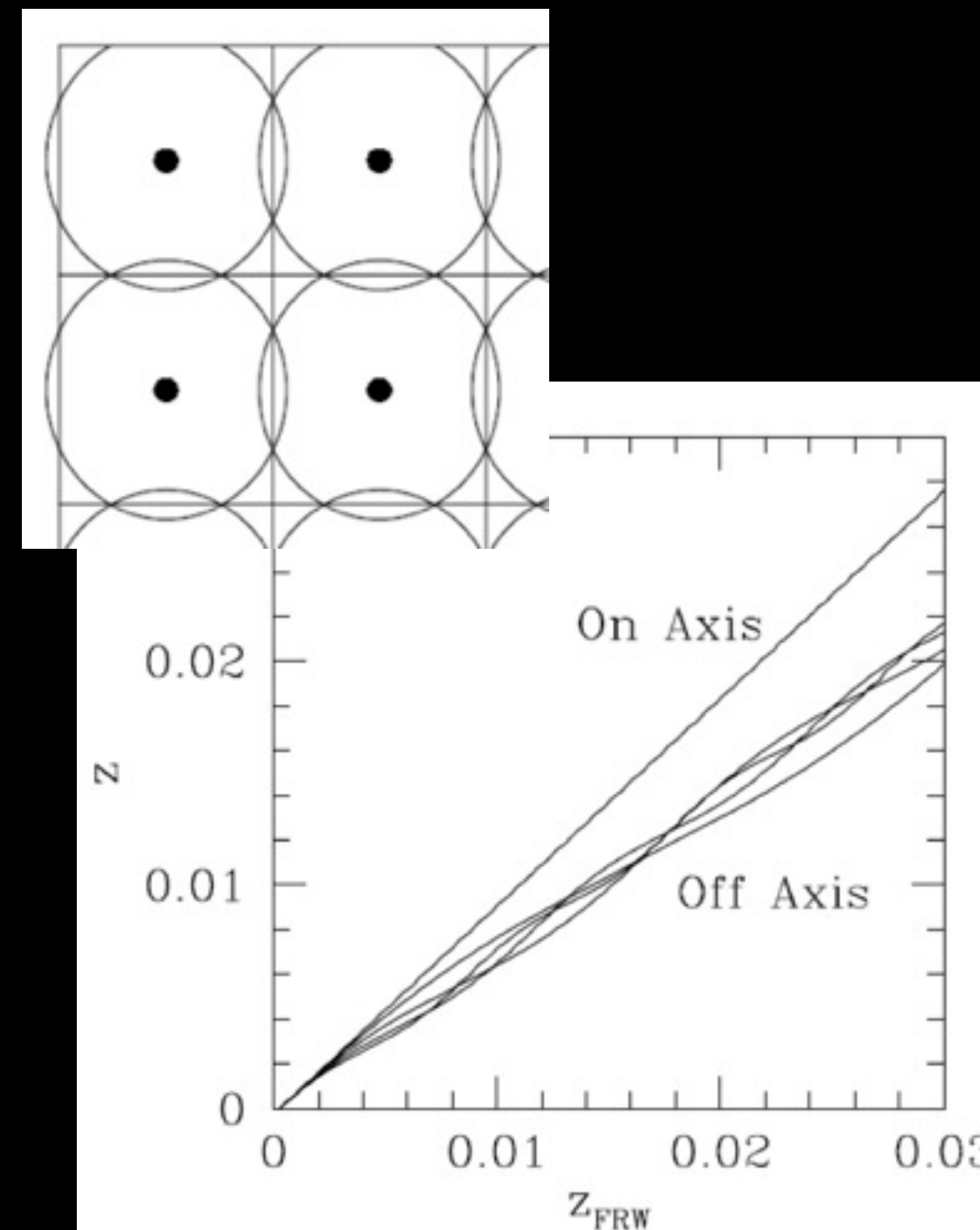
$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

- BR cosmology: from EdS to an accelerated attractor, an effective de Sitter model or something else

Motivations for weak BR

- dynamical (strong) BR may be irrelevant, the overall cosmological dynamics is FLRW, yet effects of inhomogeneities on light propagation may affect redshifts and distances. e.g. Clifton & Ferreira, PRD 80, 10 (2009) [arXiv:0907.4109], based on Lindquist and Wheeler, Rev. Mod. Phys. 29, 432 (1957)

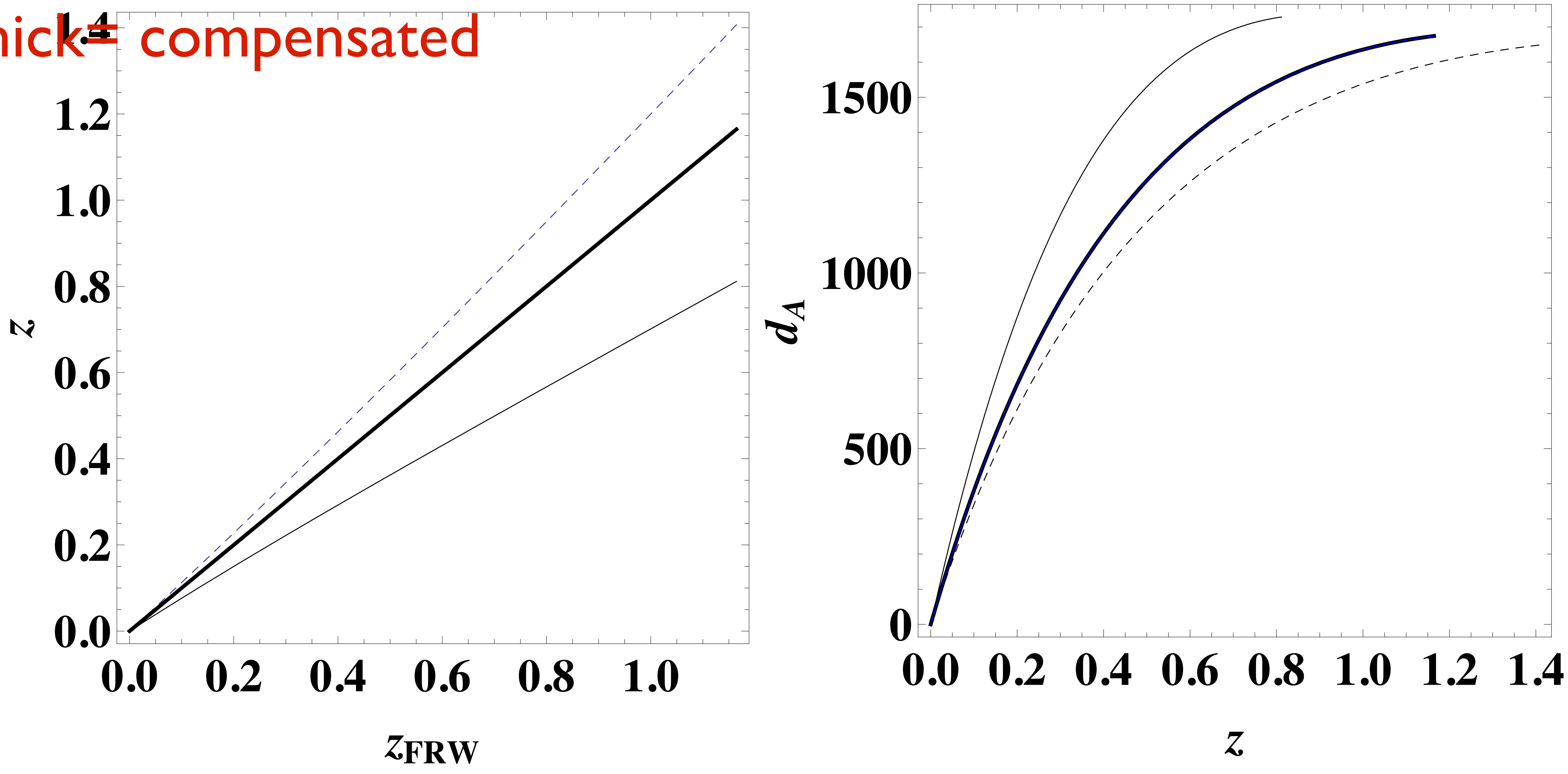
- less radical scenario, based on inhomogeneous Szekeres models (matter continuously distributed and evolving from standard growing mode in ΛCDM) seems to indicate that effects are small (but depends crucially on the "right background").
Meures, N. & MB, PRD, 8 (2011) arXiv:1103.0501
Meures, N. & MB, MN 419 (2012) arXiv:1107.4433



cf. Clarkson et al. *Interpreting supernovae observations in a lumpy universe* arXiv1109.2484

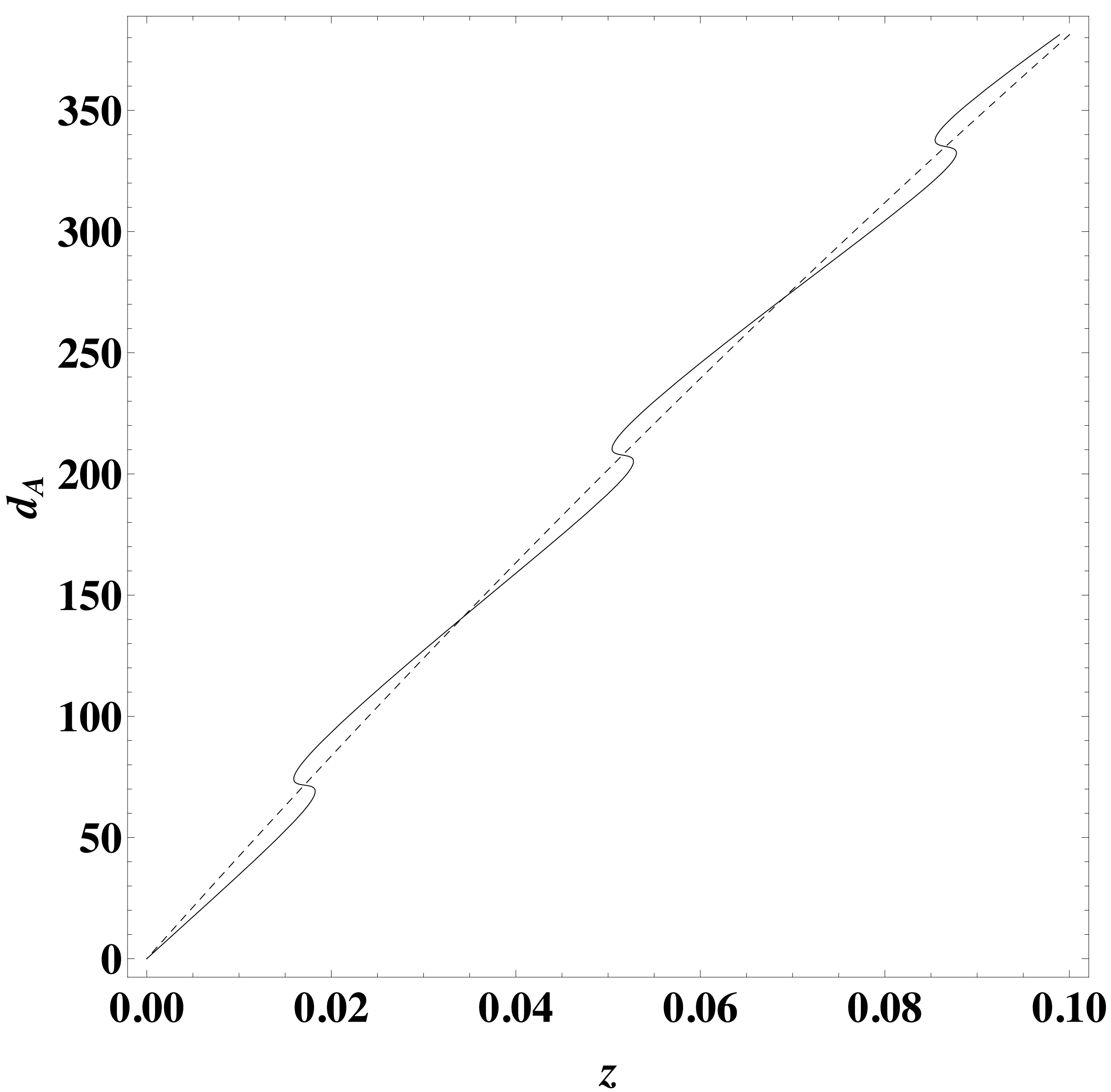
Light tracing

dashed = underdensities
thin = overdensities
thick = compensated

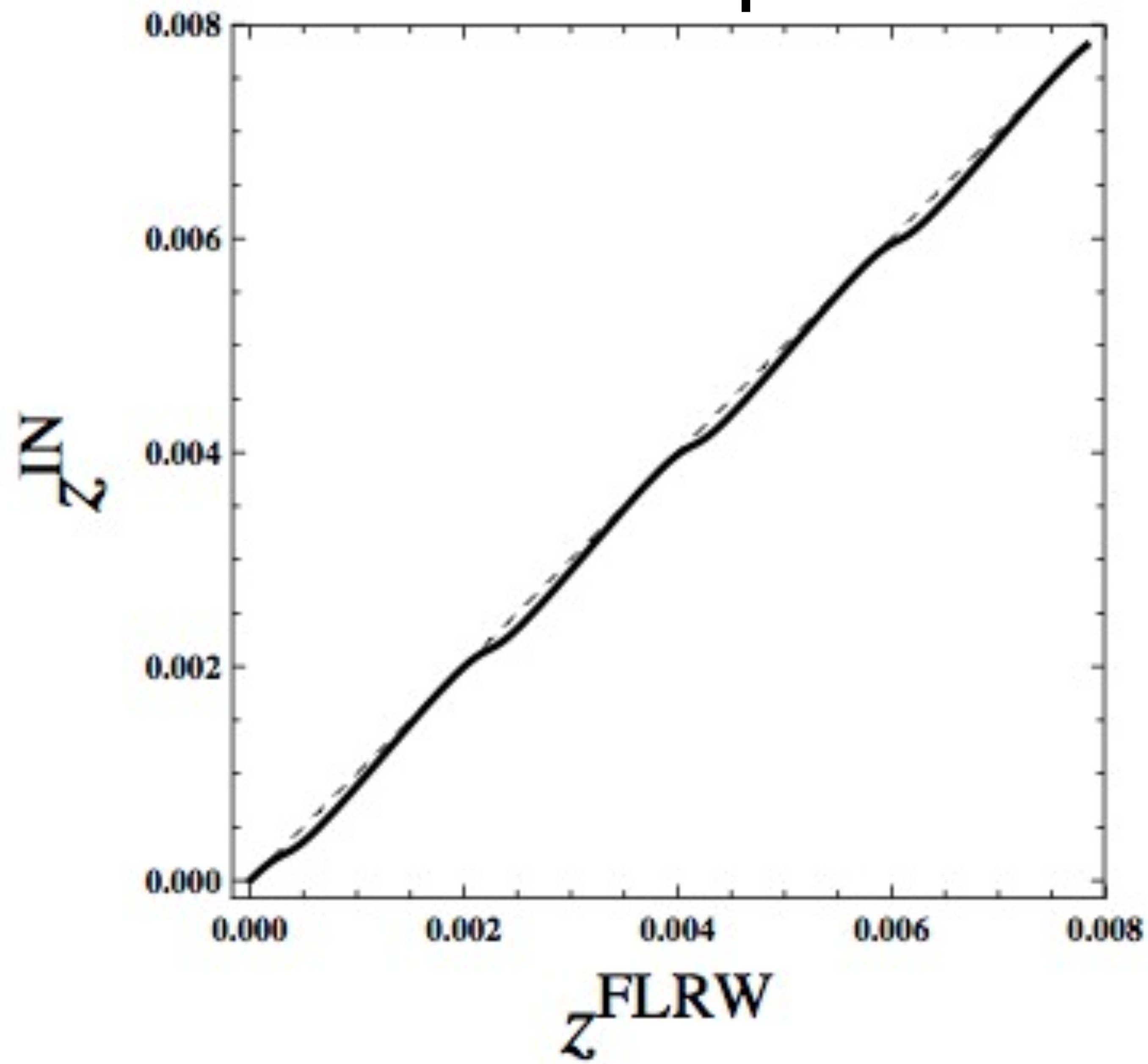


blow up

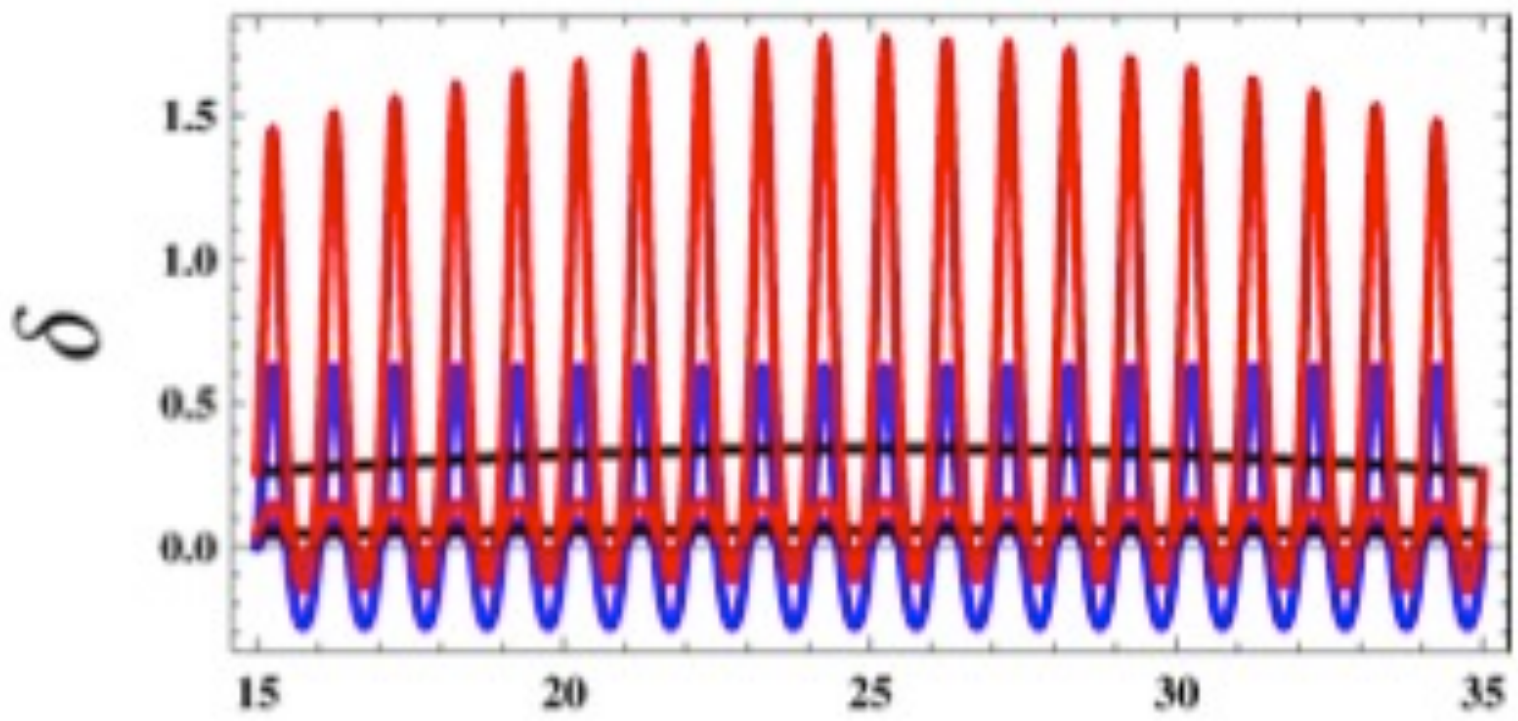
δ=20 today,
λ=100 Mpc



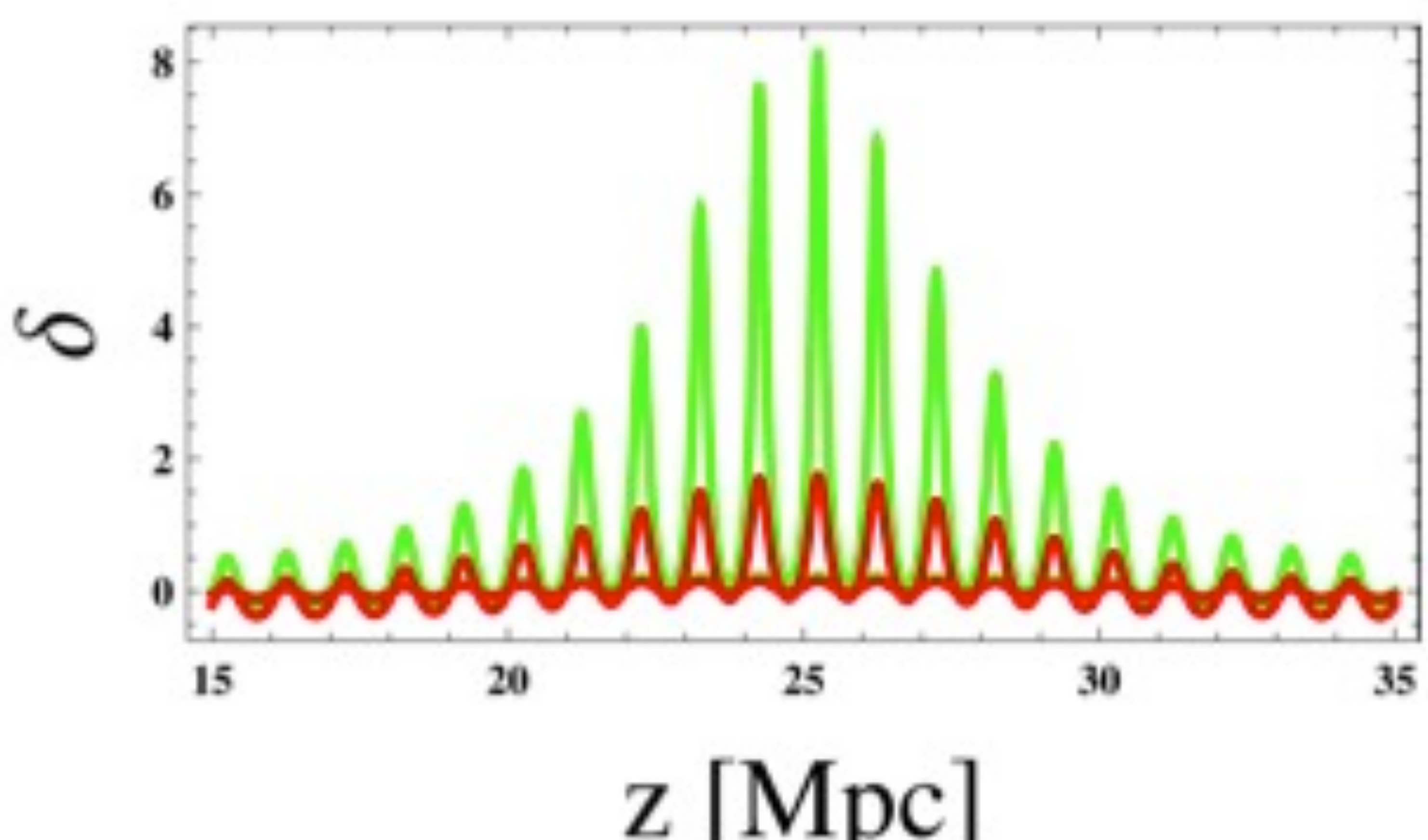
$\delta \approx 1$ today,
λ=8 Mpc



“bias” in action



black=100 Mpc
blue=1 Mpc
red=non-linear interaction



initial superposition of:
red=1+20 Mpc
green=1+20+100 Mpc

Menu of the Day

- Maintain standard ΛCDM, i.e. Cold Dark Matter and Λ on a flat Robertson-Walker background in GR
- develop a non-linear post-Friedmann(*) formalism, unifying small and large scales

(*) a post-Newtonian type approximation to cosmology

motivations for a non-linear Post-Friedmann ΛCDM Cosmology

- assume simplest standard cosmology, flat ΛCDM, trying to bridge the gap between relativistic perturbation theory and non-linear Newtonian structure formation
- an attempt to include leading order relativistic effects in non-linear structure formation
- related question: how we interpret Newtonian simulations from a relativistic point of view (cf. Chisari & Zaldarriaga, PRD 83 (2011), Green & Wald, PRD 83 (2011) and arXiv1111.2997)

Post-Newtonian cosmology

- post-Newtonian: expansion in 1/c powers (more later)
- various attempts and studies:
  - Tomita Prog. Theor. Phys. 79 (1988) and 85 (1991)
  - Matarrese & Terranova, MN 283 (1996)
  - Takada & Futamase, MN 306 (1999)
  - Carbone & Matarrese, PRD 71 (2005)
  - Hwang, Noh & Puetzfeld, JCAP 03 (2008)
- even in perturbation theory it is important to distinguish post-Newtonian effects, e.g. in non-Gaussianity, cf. Bartolo et al. CQG 27 (2010)

post-N vs. post-F

- possible assumptions on the 1/c expansion:
  - Newton: field is weak, appears only in g_{00}; small velocities
  - post-Newtonian: next order, in 1/c, add corrections to g_{00} and g_{ij}
  - post-Minkowski (weak field): velocities can be large, time derivatives ~ space derivative
  - post-Friedmann: something in between, using a FLRW background, Hubble flow is not slow but peculiar velocities are small

$$\boxed{\dot{\vec{r}} = H\vec{r} + a\vec{v}}$$

  - post-Friedmann: we don't follow an iterative approach

metric and matter

starting point: the 1-PN cosmological metric (Chandrasekhar)

$$g_{00} = -\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi)\right] + O\left(\frac{1}{c^6}\right),$$

$$g_{0i} = -\frac{a}{c^3}P_i - \frac{a}{c^5}\tilde{P}_i + O\left(\frac{1}{c^7}\right),$$

$$g_{ij} = a^2\left[\left(1 + \frac{2V}{c^2} + \frac{1}{c^4}(2V^2 + 4\Psi)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + O\left(\frac{1}{c^6}\right)$$

we assume a Newtonian-Poisson gauge: P_i is solenoidal and h_{ij} is TT, at each order 2 scalar DoF in g_{00} and g_{ij}, 2 vector DoF in frame dragging potential P_i and 2 TT DoF in h_{ij} (not GW!)

metric and matter

velocities, matter and the energy momentum tensor

Having in mind the Newtonian cosmology it is natural to define the peculiar velocity as $v^i = adx^i/dt$, obtain

$$u^i = \frac{dx^i}{cd\tau} = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0 \,.$$

$$u^i = \frac{1}{c}\frac{v^i}{a}u^0 \,,$$

$$u^0 = 1 + \frac{1}{c^2}\left(U + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[\frac{1}{2}U^2 + 2\Phi + v^2V + \frac{3}{2}v^2U + \frac{3}{8}v^4 - P_i v^i\right] \,,$$

$$u_i = \frac{av_i}{c} + \frac{a}{c^3}\left[-P_i + v_iU + 2v_iV + \frac{1}{2}v_iv^2\right] \,,$$

$$u_0 = -1 + \frac{1}{c^2}\left(U - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[2\Phi - \frac{1}{2}U^2 - \frac{1}{2}v^2U - v^2V - \frac{3}{8}v^4\right] \,.$$

$$T^\mu{}_\nu = c^2\rho u^\mu u_\nu \,,$$

$$T^0{}_0 = -c^2\rho - \rho v^2 - \frac{1}{c^2}\rho\left[2(U+V)v^2 - P^iv_i + v^4\right] \,,$$

$$T^0{}_i = c\rho a v_i + \frac{1}{c}\rho a\left\{v_i[v^2 + 2(U+V)] - P_i\right\} \,,$$

$$T^i{}_j = \rho v^i v_j + \frac{1}{c^2}\rho\left\{v^iv_j[v^2 + 2(U+V)] + v^iP_j\right\} \,,$$

$$T^\mu{}_\mu = T = -\rho c^2 .$$

note:

ρ is a non-perturbative quantity

Quiz Time!

Which metric would you say is right in the Newtonian regime?
Which terms would you retain?

$$g_{00} = -\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi)\right] + O\left(\frac{1}{c^6}\right) ,$$

$$g_{0i} = -\frac{a}{c^3}P_i - \frac{a}{c^5}\tilde{P}_i + O\left(\frac{1}{c^7}\right) ,$$

$$g_{ij} = a^2\left[\left(1 + \frac{2V}{c^2} + \frac{1}{c^4}(2V^2 + 4\Psi)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + O\left(\frac{1}{c^6}\right)$$

Answer

- The question is not well posed: the answer depends on what you are interest in!
- passive approach, gravitational field is given (geodesics):
  - particle or fluid motion: just U is relevant;
  - photons: U and V
- active approach: matter tells space how to curve, curvature tells matter how to move:
  - self-consistent derivation of Newtonian equations from Einstein equations requires U, V and P_i (i.e. all leading order terms)

$$g_{00} = -\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi)\right] + O\left(\frac{1}{c^6}\right) ,$$

$$g_{0i} = -\frac{a}{c^3}P_i - \frac{a}{c^5}\tilde{P}_i + O\left(\frac{1}{c^7}\right) ,$$

$$g_{ij} = a^2\left[\left(1 + \frac{2V}{c^2} + \frac{1}{c^4}(2V^2 + 4\Psi)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + O\left(\frac{1}{c^6}\right)$$

Newtonian ΛCDM, with a bonus

- insert leading order terms in E.M. conservation and Einstein equations
- subtract the background, getting usual Friedmann equations
- introduce usual density contrast by $\rho = \rho_b(1+\delta)$

from E.M. conservation:
Continuity & Euler equations

$$\frac{d\delta}{dt} + \frac{v^i_{\ ,i}}{a}(\delta + 1) = 0\,,$$
$$\frac{dv_i}{dt} + \frac{\dot{a}}{a}v_i = \frac{1}{a}U_{,i}\,.$$

Poisson

$$G^0_{\ 0} + \Lambda = \frac{8\pi G}{c^4}T^0_{\ 0} \;\rightarrow\; \frac{1}{c^2}\frac{1}{a^2}\nabla^2 V = -\frac{4\pi G}{c^2}\rho_b\delta\,,$$

Newtonian ΛCDM, with a bonus

what do we get from the ij and 0i Einstein equations?

$$\text{trace of } G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\rightarrow\; \frac{1}{c^2}\frac{2}{a^2}\nabla^2(V-U) = 0\,,$$

zero "Slip"

$$\text{traceless part of } G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\rightarrow\; \frac{1}{c^2}\frac{1}{a^2}\left[(V-U)_{,i}{}^{,j} - \frac{1}{3}\nabla^2(V-U)\delta_i^j\right] = 0\,.$$

bonus

$$G^0{}_i = \frac{8\pi G}{c^4}T^0{}_i \;\rightarrow\; \frac{1}{c^3}\left[-\frac{1}{2a^2}\nabla^2 P_i + 2\frac{\dot a}{a^2}U_{,i} + \frac{2}{a}\dot V_{,i}\right] = \frac{8\pi G}{c^3}\rho_b(1+\delta)v_i$$

- Newtonian dynamics at leading order, with a bonus: the frame dragging potential P_i is not dynamical at this order, but cannot be set to zero: doing so would force a constraint on Newtonian dynamics
- result entirely consistent with vector relativistic perturbation theory
- in a relativistic framework, gravitomagnetic effects cannot be set to zero even in the Newtonian regime, cf. Kofman & Pogosyan (1995), ApJ 442:

magnetic Weyl tensor at leading order

$$H_{ij} = \frac{1}{2c^3}\left[P_{\mu,\nu(i}\varepsilon_{j)}{}^{\mu\nu} + 2v_\mu(U+V)_{,\nu(i}\varepsilon_{j)}{}^{\mu\nu}\right]$$

Post-Friedmannian ΛCDM

next to leading order: the 1-PF variables

- resummed scalar potentials
$$\phi_P = -(U + \frac{2}{c^2}\Phi),$$
$$\psi_P = -(V + \frac{2}{c^2}\Psi),$$

- resummed gravitational potential
$$\phi_G = \frac{1}{2}(\psi_P + \phi_P),$$

- resummed “Slip” potential
$$\frac{D_P}{c^2} = \frac{1}{2}(\psi_P - \phi_P);$$

- resummed vector “frame dragging” potential
$$P_i^* = P_i + \frac{1}{c^2}\tilde{P}_i.$$

- Chandrasekhar velocity:
$$v_i^* = v_i - \frac{1}{c^2}P_i\ ,$$

Post-Friedmannian ΛCDM

The 1-PF equations: scalar sector

Continuity & Euler

$$\frac{d\delta}{dt} + \frac{v^{*i}{}_{,i}}{a}(\delta+1) - \frac{1}{c^2}\left[(\delta+1)\left(3\frac{d\phi_G}{dt} + \frac{v^*_k\phi_{G,k}}{a} + \frac{\dot{a}}{a}v^{*2}\right)\right] = 0\,.$$

$$\frac{dv^*_i}{dt} + \frac{\dot{a}}{a}v^*_i + \frac{1}{a}\phi_{G,i} + \frac{1}{c^2}\left[\frac{\phi_{G,i}}{a}(4\phi_G + v^{*2}) - 3v^*_i\frac{d\phi_G}{dt} - \frac{D_{P,i}}{a} - \frac{v^*_i}{a}v^*_j\phi_{G,j} - \frac{\dot{a}}{a}v^{*2}v^*_i + \frac{P_{j,i}v^{*j}}{a}\right] = 0\,.$$

generalized Poisson: a non-linear wave eq. for φ_g

$$\frac{1}{c^2}\frac{2}{3a^2}\nabla^2\phi_G + \frac{1}{c^4}\left[\ddot{\phi}_G + 2\frac{\dot{a}}{a}\dot{\phi}_G + 2\frac{\ddot{a}}{a}\phi_G - \left(\frac{\dot{a}}{a}\right)^2\phi_G + \frac{2}{3a^2}\nabla^2\phi_G^2 - \frac{3}{2a^2}\phi_{G,i}\phi_G{}^{,i}\right] = \frac{4\pi G}{3}\rho_b\left[\frac{1}{c^2}\delta + \frac{1}{c^4}\rho_b(1+\delta)v^{*2}\right]$$

non-dynamical "Slip"

$$\frac{1}{c^4}\frac{1}{3a^2}\nabla^2\nabla^2 D_{PN} = -\frac{1}{c^2}\frac{1}{3a^2}\nabla^2\nabla^2\phi_G - \frac{1}{c^4}\left[\frac{1}{3a^2}\nabla^2\nabla^2\phi_G^2 - \frac{5}{6a^2}\nabla^2(\phi_{G,i}\phi_{G,i})\right]$$
$$+\frac{4\pi G}{3}\rho_b\left\{\frac{1}{c^2}\nabla^2\delta - \frac{1}{c^4}\left[\nabla^2((1+\delta)v^{*2})) + \dot{a}((1+\delta)v^*_k)^{,k}\right]\right\}\,,$$

Post-Friedmannian ΛCDM

The 1-PF equations: vector and tensor sectors

- the frame dragging vector potential becomes dynamical at this order

- the TT metric tensor h_{ij} is not dynamical at this order, but it is instead determined by a non-linear constraint in terms of the scalar and vector potentials

Post-Friedmannian ΛCDM

The 1-PF equations: simplifying variables and simpler equations

new density and velocity variables

$$\bar{\rho} = -a^{-3}\rho\left(\frac{(-g)^{1/2}}{u^0}\right)^{-1} = \rho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^{*2} - 3\phi_G\right)\right]$$

$$\bar{v}_i = v_i^*\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^{*2} - 3\phi_G\right)\right]$$

Continuity & Euler:

$$\frac{d\bar{\delta}}{dt} + \frac{v^{*i}{}_{,i}}{a}(\bar{\delta}+1) = 0$$

$$\frac{d\bar{v}_i}{dt} + \frac{\dot{a}}{a}\bar{v}_i + \frac{1}{a}\phi_{G,i} + \frac{1}{c^2}\left[\frac{\phi_{G,i}}{a}\left(\phi_G + \frac{3}{2}\bar{v}^2\right) - \frac{D_{P,i}}{a} + \frac{P_{j,i}\bar{v}^j}{a}\right] = 0$$

Post-Friedmannian ΛCDM

The 1-PF equations: simplifying variables and simpler equations

$$\frac{1}{c^2}\frac{2}{3a^2}\nabla^2\phi_G + \frac{1}{c^4}\left[\ddot{\phi}_G + 2\frac{\dot{a}}{a}\dot{\phi}_G + 2\frac{\ddot{a}}{a}\phi_G - \left(\frac{\dot{a}}{a}\right)^2\phi_G + \frac{2}{3a^2}\nabla^2\phi_G^2 - \frac{3}{2a^2}\phi_{G,i}\phi_G^{,i}\right] = \frac{4\pi G}{3}\rho_b\left[\frac{1}{c^2}\bar{\delta} + \frac{1}{c^4}\rho_b(1+\bar{\delta})\left(3\phi_G + \frac{1}{2}v^{*2}\right)\right].$$

wave eq. for φ_g

$$\frac{1}{c^4}\frac{1}{3a^2}\nabla^2\nabla^2 D_{PN} = -\frac{1}{c^2}\frac{1}{3a^2}\nabla^2\nabla^2\phi_G - \frac{1}{c^4}\left[\frac{1}{3a^2}\nabla^2\nabla^2\phi_G^2 - \frac{5}{6a^2}\nabla^2(\phi_{G,i}\phi_{G,i})\right] + \frac{4\pi G}{3}\rho_b\left\{\frac{1}{c^2}\nabla^2\bar{\delta} - \frac{1}{c^4}\left[\nabla^2\left((1+\bar{\delta})\left(\frac{1}{2}v^{*2} - 3\phi_G\right)\right) + \dot{a}((1+\bar{\delta})v^*_k)^{,k}\right]\right\}$$

eq. for Slip D_p

linearized equations

linearized equations:
scalar and vector perturbation equations
in the Poisson gauge

$$\nabla^2\psi_P - \frac{3}{c^2}a^2\left[\frac{\dot a}{a}\dot\psi_P + \left(\frac{\dot a}{a}\right)^2\phi_P\right] = 4\pi G\rho_b a^2\delta \,,$$

$$-\nabla^2(\psi_P-\phi_P) + \frac{3}{c^2}a^2\left[\frac{\dot a}{a}(\dot\phi_P+3\dot\psi_P) + 2\frac{\ddot a}{a}\phi_P + \left(\frac{\dot a}{a}\right)^2\phi_P + \ddot\psi_P\right] = 0$$

$$\nabla^2\left(\frac{\dot a}{a}\phi_P+\dot\psi_P\right) = -4\pi G a\rho_b\theta \,,$$

$$\frac{1}{c^2a^2}\frac{2}{3}\nabla^2\nabla^2(\phi_P-\psi_P) = 0 \,,$$

$$\dot\delta + \frac{\theta}{a} - \frac{3}{c^2}\dot\psi_P = 0 \,,$$

$$\dot\theta + \frac{\dot a}{a}\theta + \frac{\nabla^2\phi_P}{a} = 0 \,.$$

Summary

- "Resummed" equations include Newtonian and 1-PF non-linear terms together
- at leading Newtonian order, consistency of Einstein equations requires a non-zero gravito-magnetic vector potential
- framework provides a straightforward relativistic interpretation of Newtonian simulations: quantities are those of Newton-Poisson gauge
- 2 scalar potentials, become 1 in the Newtonian limit and in the linear regime, valid at horizon scales: slip non-zero in relativistic mildly non-linear (intermediate scales?) regime
- non-trivial important result: linearised equations coincide with 1-order relativistic perturbation theory in Poisson gauge
- formalism therefore provides a unified framework valid from Newtonian non-linear small scales to H^{-1} scales

Outlook and work in progress

- more work needed to really quantify effects of inhomogeneities on light tracing
- back-reaction of structure formation on observations and dynamics still poorly understood
- applications of Post Friedmannian formalism in many directions: linear/non-linear power spectrum, lensing, modified N-bodies, etc...
- extension to parametrised non-linear post-F to complement existing linear post-F work
- current work in progress: with Dan Thomas and David Wands, we are working on extracting the vector potential from N-body simulations, see Dan talk on friday