Effective approach and decoupling in QFT

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Based on collaborations with

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What is QFT? – September, 2011

Contents:

- Why the decoupling theorem is important?
- Quantum corrections to vacuum action for massive fields.
- Electromagnetic field and Multiplicative Anomaly.
- Gravitational form factors.
- Application to conformal anomaly.

Why the decoupling theorem is important?

It is supposed that the QFT is describing matter at the most fundamental level, e.g., at the scale of particle physics.

However what happens with this fundamental physics when we go to the lower energy scale? For example, to Nuclear Physics, or to Quantum Mechanics?

The answer is that some d.o.f. become "frozen" if the energy of the process is insufficient to make it active.

This idea is in the heart of the Effective approach to QFT. The recent historical review:

Steven Weinberg, Effective Field Theory, Past and Future. PoS CD09:001, 2009; arXiv:0908.1964.

One can find an example of decoupling even in Classical Mechanics.

Consider a closed system of particles with masses m_i. The kinetic energy is given by

$$K = K_C + \frac{M V^2}{2}, \qquad (1)$$

where the first term is the kinetic energy in the center of mass reference frame and the second is the kinetic energy of the system as a whole.

Formula (1) explains why we can describe the motion of macroscopic bodies without thinking about the motion of molecules and atoms inside it.

Field theory: Decoupling at the classical level.

Consider propagator of massive field at very low energy

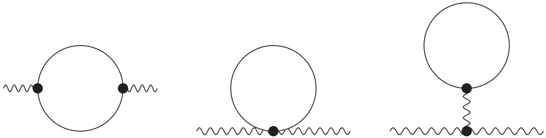
$$\frac{1}{k^2+m^2} = \frac{1}{m^2}\left(1 - \frac{k^2}{m^2} + \frac{k^4}{m^4} + ...\right).$$

In case of $k^2 \ll m^2$ there is no propagation of particle.

What about quantum theory, loop corrections?

Formally, in loops integration goes over all values of momenta.

Is it true that the effects of heavy fields always become irrelevant at low energies?



The QED example (flat space). 1-loop vacuum polarization

$$-\frac{e^2\,\theta_{\mu\nu}}{2\pi^2}\int_0^1 dx\,x(1-x)\,\ln\frac{m_e^2+p^2x(1-x)}{4\pi\mu^2},$$

where $\theta_{\mu\nu}=(p_\mu p_\nu-p^2g_{\mu\nu})$, **and** μ **is the dimensional parameter.**

$\beta^{\overline{\rm MS}}$ **is** $\frac{e}{2}\mu\frac{d}{d\mu}$ **acting on the formfactor of** $\theta_{\mu\nu}$,

$$\beta_e^{\overline{\rm MS}} = \frac{e^3}{12\pi^2}.$$

In the physical (mass-dependent) scheme one has to subtract at $p^2=M^2$ **and take the limit** $\frac{e}{2}M\frac{d}{dM}$.

$$\beta_e = \frac{e^3}{2\pi^2}\int_0^1 dx\,x(1-x)\,\frac{M^2x(1-x)}{m_e^2+M^2x(1-x)}.$$

The UV limit $M\gg m_e$ **and** $\beta_e = \beta_e^{\overline{\rm MS}}$. **The IR limit** $M\ll m_e$

$$\beta_e = \frac{e^3}{60\pi^2}\cdot\frac{M^2}{m_e^2} + \mathcal{O}\left(\frac{M^4}{m^4}\right).$$

Appelquist & Carazzone, “Decoupling Theorem” (PRD, 1975)

General observation on vacuum quantum corrections for massive fields.

Quantum effects of massive fields can be quite different from the ones of massless fields.

In order to see this difference one has to use a physical renormalization scheme, e.g., the one based on momentum subtraction.

The Appelquist and Carazzone theorem in QED states that when the typical energy of the scattering process ε tends to zero, the finite contribution of the loop tends to zero as $(\varepsilon/m)^2$.

Do we have a well defined IR limit?

• Quantum correction to photon sector

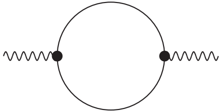
Consider one-loop QED on curved background. The one-loop effective action (EA) in the case of $g_{\mu\nu}$ and A_μ background can be defined via the path integral

$$e^{i\Gamma^{(1)}[g_{\mu\nu},A_\mu]} = \int D\psi D\bar{\psi}\; e^{iS_{QED}}\,,$$

or (here the Det does not take into account Grassmann parity)

$$\bar{\Gamma}^{(1)} = -i\,\mathrm{Log\,Det}\,\hat{H}\,, \quad \textbf{where} \quad \hat{H} = i\big(\gamma^\mu\nabla_\mu - im - ie\gamma^\mu A_\mu\big)\,.$$

The Feynman diagram of our interest is (up to gravitational external lines)



Instead of diagrams, at one loop we can use the heat-kernel method and the Schwinger-DeWitt technique.

- **Reducing the problem to the derivation of** Log Det $\hat{\mathcal{O}}$,

$$\hat{\mathcal{O}} \,=\, \widehat{\Box} + 2\hat{h}^{\mu}\nabla_{\mu} + \hat{\Pi}\,.$$

- • **Multiply** $\hat{H}$ **by an appropriate conjugate** $\hat{H}^{*}$,

$$\hat{\mathcal{O}} \,=\, \hat{H} \cdot \hat{H}^{*}$$

and use the relation

$$\text{Log Det } \hat{H} \,=\, \text{Log Det } \hat{\mathcal{O}} \,-\, \text{Log Det } \hat{H}^{*}\,.$$

- **The simplest choice,**

$$\hat{H}_1^* = -\,i\left(\gamma^\mu \nabla_\mu + im - ie\gamma^\mu A_\mu\right).$$

According to G. De Berredo-Peixoto, M.Ph.L.A 16 (2001),

$$\text{Log Det } \hat{H} = \text{ Log Det } \hat{H}_1^*\,,$$

$$\textbf{then} \qquad \text{Log Det } \hat{H} = \frac{1}{2}\,\text{Log Det }\left(\hat{H}\hat{H}_1^*\right).$$

- **An alternative choice:**

$$\hat{H}_2^* = -\,i\left(\gamma^\mu \nabla_\mu + im\right).$$

This operator does not depend on A_μ, hence

$$\text{Log Det } \hat{H}\Big|_{FF} = \text{ Log Det }\left(\hat{H}\hat{H}_2^*\right)\Big|_{FF}.$$

If the relation

$$\text{Det}\,(\hat{A}\cdot\hat{B}) = \text{Det}\,\hat{A}\cdot\,\text{Det}\,\hat{B}$$

holds in this case, we are going to meet equal expressions,

$$\frac{1}{2}\,\text{Log Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} \,=\, \text{Log Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}\,.$$

It is so?

In reality, we meet

$$\frac{1}{2}\,\text{Log Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} \,=\, \text{Log Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}$$

for divergencies, but

$$\frac{1}{2}\,\text{Log Det}\,\big(\hat{H}\hat{H}_1^*\big)\Big|_{FF} \,\neq\, \text{Log Det}\,\big(\hat{H}\hat{H}_2^*\big)\Big|_{FF}$$

for the nonlocal finite parts of the two effective actions.

This is nothing else, but Multiplicative Anomaly (MA)

M. Kontsevich & S. Vishik, hep-th/9406140/9404046.
E. Elizalde, L. Vanzo & S. Zerbini, Com.Math.Phys. 194 (1998);
G. Cognola, E. Elizalde & S. Zerbini, Com.Math.Phys. 237(2003);
NPB 532(1998); ...

The MA in ζ- regularization may be just due to μ - ambiguity.

T.S. Evans, PLB 457 (1999);
J.S. Dowker, hep-th/9803200;
J.J. McKenzie-Smith and D.J. Toms, PRD 58 (1998).

• **Schwinger-DeWitt technique (Sch-DW) is perhaps the most useful method for many 1-loop calculations.**

Consider the typical form of the operator $\hat{H}$

$$\hat{\mathbf{H}} \,=\, \hat{1}\Box + \hat{\Pi} + \hat{1}m^2\,.$$

The one-loop EA is given by the expression

$$\bar{\Gamma}^{(1)} \,=\, \mathrm{sTr}\,\lim_{x'\to x}\int\limits_0^\infty \frac{ds}{s}\,\hat{U}(x,x'\,;s)\,,$$

where the evolution operator satisfies the equation

$$i\,\frac{\partial\hat{U}(x,x'\,;s)}{\partial s} \,=\, -\,\hat{\mathbf{H}}\hat{U}(x,x'\,;s)\,,$$

$$U(x,x'\,;0) = \delta(x,x')\,.$$

A useful representation for the evolution operator $\hat{U}(x,x'\,;s)$ is

$$\hat{U}(x,x'\,;s) \,=\, \hat{U}_0(x,x'\,;s)\,\sum_{k=0}^{\infty}(is)^k\,\hat{a}_k(x,x')\,,$$

$\hat{a}_k(x,x')$ **are Schwinger-DeWitt coefficients.**

$$\hat{U}_0(x,x'\,;s) = \frac{\mathcal{D}^{1/2}(x,x')}{(4\pi i\,s)^{n/2}}\,\exp\left\{\frac{i\sigma(x,x')}{2s} - m^2 s\right\}.$$

$\sigma(x,x')$ **- geodesic distance between** x **and** x'**.**

$\mathcal{D}$ **is the Van Vleck-Morett determinant**

$$\mathcal{D}(x,x') \,=\, \det\left[-\,\frac{\partial^2\sigma(x,x')}{\partial x^\mu\,\partial x'^\nu}\right]\,,$$

It is sufficient to know the coincidence limits

$$\lim_{x\to x'}\hat{a}_k(x,x') \,=\, \hat{a}_k\Big|\,.$$

If we consider more general operator

$$S_2 \,=\, \hat{H} \,=\, \hat{1}\Box + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi}\,,$$

the linear term can be absorbed into $\nabla_\mu \to \mathcal{D}_\mu = \nabla_\mu + \hat{h}_\mu$.

The commutator of the new covariant derivatives will be

$$\hat{S}_{\mu\nu} \,=\, \hat{R}_{\mu\nu} - (\nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu) - (\hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu)$$

and we arrive at

$$\hat{a}_1\Big| \,=\, \hat{a}_1(x,x) \,=\, \hat{P} \,=\, \hat{\Pi} + \frac{\hat{1}}{6}\,R - \nabla_\mu \hat{h}^\mu - \hat{h}_\mu \hat{h}^\mu$$

and

$$\hat{a}_2\Big| \,=\, \hat{a}_2(x,x) \,=\, \frac{\hat{1}}{180}\,(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R) + \frac{1}{2}\,\hat{P}^2 + \frac{1}{6}\,(\Box\hat{P}) + \frac{1}{12}\,\hat{S}^2_{\mu\nu}\,.$$

The great advantage of the general expressions for $\hat{a}_k\Big|$ is their universality.

These coefficients enable one to analyze the effective action in a given space-time dimension for numerous field theory models.

In 2-dimensional space-time

- $\hat{a}_1\Big|$ **defines logarithmic divergences**

In 4-dimensional space-time

- $\hat{a}_2\Big|$ **defines logarithmic divergences**

while $\hat{a}_1\Big|$ defines quadratic divergences.

In 6-dimensional space-time

- $\hat{a}_3\Big|$ **defines logarithmic divergences**

while $\hat{a}_2\Big|$ defines quadratic divergences.

and $\hat{a}_1\Big|$ defines quartic divergences. **etc.**

The general expressions for the Sch-DW coefficients do not depend on the space-time dimension. However, the traces do have such dependence!

It is well established fact that the logarithmic divergences are always universal and hence scheme-independent.

Then, as far as the coincidence limits $\hat{a}_k\big|$ are universal in the "right" dimensions (this means the dimension where they define logarithmic divergences), they should be scheme-dependent in other dimensions!

One can check it for $\hat{a}_1\big|$, $\hat{a}_2\big|$, $\hat{a}_3\big|$, and then go to the sum.

The EA is a sum of the coefficients $\hat{a}_k\big|$, this is the main point!

Let us start from the simplest nontrivial coefficient, $a_1\Big|$ **.**

For the general d-dimensional case we obtain the expressions

$$a_{1,\,(1)}\Big| = -\Big(4m^2 - \frac{1}{3}\,R\Big)\,. \qquad (1)$$

and

$$a_{1,\,(2)}\Big| = -\Big\{4m^2 - \frac{1}{3}\,R \;+\; 2ie\,(\nabla^\mu A_\mu) - (d-2)\,e^2\,A^\mu A_\mu\Big\}\,. \qquad (2)$$

The expression (1) is independent on A_μ and is gauge invariant.

However, (2) is obviously different in both respects. In particular it becomes gauge invariant only in $d=2$ case.

In $d=4$ there is an explicit scheme dependence!

The second Sch-DW coefficients are

$$a_{2,1}\Big| = \frac{d}{288}\left(48e^2F_{\mu\nu}F^{\mu\nu} + R^2 - 24Rm^2 - 3R^2_{\mu\nu\alpha\beta} + 144m^4\right). \quad (1)$$

for the first scheme based on $\hat{H}_1^*$ **and**

$$a_{2,2}\Big| = a_{2,1}\Big| + \frac{d}{288}(d-4)\Big\{24\,e^2(\nabla^\mu A^\nu)(\nabla_\mu A_\nu)$$

$$+ 6e^2A^\mu A_\mu\Big[2(R-12m^2) + (d-2)e^2A_\nu A^\nu\Big]\Big\} \qquad (2)$$

for an alternative scheme of calculation based on $\hat{H}_2^*$**.**

The two expressions do coincide in $d=4$ **and ONLY in** $d=4$**.**

Moreover, only in $d=4$ **the** $a_{2,2}\Big|$ **is gauge invariant.**

We have a direct evidence of that the UV divergent part of Effective Action is scheme invariant, while the finite part is not.

Most important: One of these features follows from another !!!

What all this means, after all?

We assumed the relation

$$\mathrm{Log\ Det}\,(\hat{H}\cdot\hat{H}_2^*) \,=\, \mathrm{Log\ Det}\,\hat{H} + \mathrm{Log\ Det}\,\hat{H}_2^*\,. \qquad (1)$$

The two terms in the *r.h.s.* **are gauge invariant, because the first term is a gauge-invariant functional integration and the second term simply does not depend neither on A_μ nor on fermion.**

Hence if we find the violation of gauge symmetry in the *l.h.s.* **(as we actually did), this indicates the violation of the "identity"** (1).

The relation similar to (1),

$$\det(A\cdot B) = \det A\cdot\det B \qquad (2)$$

can be easily proved for the finite-size square matrices A and B. **However, this proof can not be generalized for the differential operators, which have an infinite-size matrix representations.**

What about the sum of the Sch-DW series?

Since in such series, in any particular dimension, there is only one universal term, the sum of the series can not be universal, of course.

The practical calculations can be performed by using Feynman diagrams or by the heat-kernel method, which is technically much simpler.

$$\bar{\Gamma}^{(1)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}\,K(s)\,,$$

where the heat kernel is

$$\mathrm{Tr}\,K(s) \,=\, \frac{\mu^{4-2\omega}}{(4\pi s)^\omega}\,\int d^4x\,\sqrt{g}\;e^{-sm^2}\,\mathrm{tr}\,\Big\{\,\hat{1} + s\hat{P} \,+\, s^2\,\big[\,R_{\mu\nu}f_1(-s\Box)R^{\mu\nu}$$

$$+Rf_2(-s\Box)R + \hat{P}f_3(-s\Box)R + \hat{P}f_4(-s\Box)\hat{P} + \hat{S}_{\mu\nu}f_5(-s\Box)\hat{S}^{\mu\nu}\big]\,\Big\}\,.$$

I. Avramidi, Sov.J.Nucl.Phys. 49 (1989);
A. Barvinsky & G.A. Vilkovisky, Nucl Phys. B333 (1990) 471.

The elements of the heat-kernel solution are as follows:

$$f_1(\tau) = \frac{f(\tau) - 1 + \tau/6}{\tau^2}\,, \qquad f_2(\tau) = \frac{f(\tau)}{288} + \frac{f(\tau) - 1}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2}\,,$$

$$f_3 = \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau}\,, \qquad f_4 = \frac{f(\tau)}{2}\,, \qquad f_5 = \frac{1 - f(\tau)}{2\tau}\,,$$

where

$$f(\tau) = \int_0^1 d\alpha\, e^{\alpha(1-\alpha)\tau}\,, \qquad \tau = -s\Box\,.$$

It is remarkable that one can integrate this out for massive theory and that the result fits perfectly with the one from the Feynman diagram approach.

In this way one can arrive at the most complete form of decoupling theorem.

Ed. Gorbar & I.Sh., JHEP 02 (2003).

The result for the one-loop Euclidean effective action is

$$\bar{\Gamma}^{(1)}_{\sim F^2} = \frac{e^2}{2(4\pi)^2}\int d^4x\sqrt{g}\;F_{\mu\nu}\,\left[\frac{2}{3\epsilon} + k_1^{FF}\right]\,F^{\mu\nu},$$

where $$k_1^{FF} \,=\, k_1^{FF}(a) \,=\, Y\left(2 - \frac{8}{3a^2}\right) - \frac{2}{9}\,.$$

$$Y = 1 - \frac{1}{a}\log\left(\frac{2+a}{2-a}\right)\,, \qquad a^2 = \frac{4\Box}{\Box - 4m^2}\,.$$

ϵ **is the parameter of dimensional regularization**

$$\frac{1}{\epsilon} = \frac{2}{4-d} + \log\left(\frac{4\pi\mu^2}{m^2}\right).$$

This expression represents a complete one-loop contribution to the photon propagator.

B. Gonçalves, G. de Berredo-Peixoto & I.Sh., PRD-80 (2009) 104013.

Is this result universal and unambiguous?

Calculating within the second scheme we get a different result,

$$\bar{\Gamma}^{(1)}_{\sim AA}\Big| = -\frac{e^2}{2(4\pi)^2}\int d^4x\sqrt{g}\,\Big\{F_{\mu\nu}\,\Big[\frac{2}{3\,\epsilon} + k_2^{FF}(a)\Big]F^{\mu\nu}$$

$$+\,2\nabla_\mu A^\mu\Big[Y\big(\frac{8}{3a^2}-2\big)+\frac{2}{9}\Big]\nabla_\nu A^\nu + \nabla_\mu A^\nu\Big[\frac{16\,Y}{3a^2}+\frac{4}{9}\Big]\nabla_\nu A^\mu + \mathcal{O}(R{\cdot}A{\cdot}A)\Big\}\,,$$

where

$$k_2^{FF}(a) = Y\Big(1+\frac{4}{3a^2}\Big)+\frac{1}{9}\,,$$

and $\mathcal{O}(R\cdot A\cdot A)$ **are terms proportional to scalar curvature.**

The difference between the effective actions derived within the two schemes, do differ by the non-local terms, therefore it can not be reduced to the renormalization μ **- ambiguity.**

High-energy limit of quantum correction.

The mass of the quantum field is negligible, this corresponds to the limit $a \to 2$ in both of the formfactors, and we arrive at the universal result

$$\beta^{MS} \cdot F^{\mu\nu} \ln\left(\frac{\Box}{\mu^2}\right) F_{\mu\nu} \,.$$

The same asymptotic behavior can be seen in the MS scheme of renormalization.

In fact, the difference between the two schemes, with

$$\hat{H}_1^* \quad \textbf{and} \quad \hat{H}_2^*$$

is still possible, but only for surface terms. This is intimately related to the ambiguity in the conformal anomaly which we will discuss next.

UV and IR limits for quantum correction.

The Appelquist and Carazzone decoupling theorem can be directly obtained from our formfactors.

From k_1^{FF} we arrive at the complete momentum-subtraction β-function

$$\beta_e^1 = \frac{e^3}{6a^3\,(4\pi)^2}\,\left[20a^3 - 48a + 3(a^2-4)^2\,\ln\left(\frac{2+a}{2-a}\right)\right]\,,$$

In one special case we meet the UV limit $p^2 \gg m^2$,

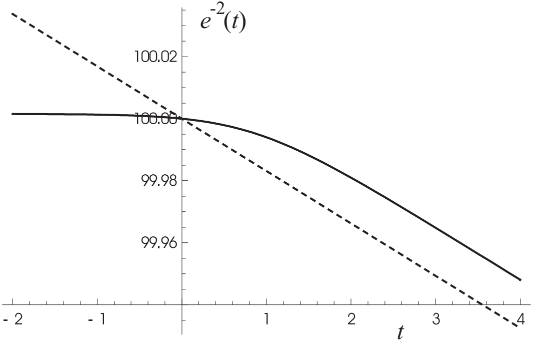
$$\beta_e^{1\;UV} = \frac{4\,e^3}{3\,(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right)\,.$$

In the IR regime $p^2 \ll m^2$ and the result is quite different

$$\beta_e^{1\;IR} = \frac{e^3}{(4\pi)^2}\cdot\,\frac{4\,M^2}{15\,m^2} + \mathcal{O}\left(\frac{M^4}{m^4}\right)\,.$$

This is exactly the standard form of the Appelquist and Carazzone decoupling theorem (PRD, 1977).

As we see, the general expression interpolates between the UV and IR limits.



These plots show the effective electron charge as a function of $\log(\mu/\mu_0)$ **in the case of the MS-scheme and for the momentum-subtraction scheme, with** $\ln(p/\mu_0)$.

An interesting high-energy effect is a small apparent shift of the initial value of the effective charge.

Similar calculations starting from the formfactor $k_2^{FF}(a)$ give

$$\beta_e^2 \,=\, \frac{e^3}{12a^3\,(4\pi)^2}\,\,\left\{4a(12+a^2)\,-\,3(a^4-16)\,\ln\left(\frac{2-a}{2+a}\right)\right\}\,.$$

In the UV limit $p^2 \gg m^2$ the above β-function is in agreement with the standard MS result.

As a consequence of the Multiplicative Anomaly, in the IR limit $p^2 \ll m^2$ we obtain slightly different result,

$$\beta_e^{2\ IR} \,=\, \frac{e^3}{5\,(4\pi)^2}\,\cdot\,\frac{M^2}{m^2}\,+\,\mathcal{O}\Big(\frac{M^4}{m^4}\Big)\,.$$

The slight difference in just in details of how the two β-functions go to zero in the IR limit.

Similar form factors were derived for gravity.

E.g., for a massive scalar field *(Gorbar & I.Sh., JHEP, 2003).*

$$L = \frac{1}{2}\left\{ (\nabla\varphi)^2 + m^2\varphi^2 + \left(\tilde{\xi} + \frac{1}{6}\right)R\varphi^2 \right\}.$$

The vacuum action is $S = \int d^4\sqrt{g}\,(L_{HE} + L_{HD})$,

$$L_{HE} = -\frac{1}{16\pi G}(R + 2\Lambda) \quad \textbf{and} \quad L_{HD} = a_1 C^2 + a_2 E + a_3\Box R + a_4 R^2 .$$

For the formfactors, we find $k_\Lambda = \dfrac{3m^4}{8(4\pi)^2}, \qquad k_R = \dfrac{m^2}{2(4\pi)^2}\,\tilde{\xi},$

$$k_1(a) = \frac{8Y}{15a^4} + \frac{2}{45a^2} + \frac{1}{150},$$

$$k_4(a) = Y\tilde{\xi}^2 + \tilde{\xi}\left(\frac{2Y}{3a^2} - \frac{Y}{6} + \frac{1}{18}\right) + \frac{8Y(2-a^2) + Ya^4}{144a^4} + \frac{20 - 7a^2}{2160a^2}.$$

Equivalent expressions were obtained earlier

Yu. Gusev and A. Zelnikov, PRD (1998) ???

Application to Conformal anomaly

Consider theory with $g_{\mu\nu}$ and fields Φ (conformal weights k_Φ).
The Noether identity for the local conformal symmetry is

$$\left[-2\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right]\,S(g_{\mu\nu},\,\Phi)\;=\;0$$

On shell we have

$$-\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}}\;=\;T^\mu_\mu\;=\;0\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ has to be replaced by the effective action $\Gamma_{vac}(g_{\mu\nu})$. For free field only 1-loop order is relevant.

$$\Gamma_{div}\;=\;\frac{1}{\varepsilon}\int d^4x\sqrt{g}\left\{\beta_1 C^2+\beta_2 E+\beta_3\Box R\right\}\;.$$

For global conformal symmetry the renormalization group tells

$$<T^\mu_\mu>=\;\left\{\beta_1 C^2+\beta_2 E+a'\Box R\right\}\;,$$

where $a'=\beta_3$. But in the local symmetry case a' is ambiguous.

The conformal anomaly represents the most efficient instrument to evaluate the effective action. Hence it is important to know the real origin of this ambiguity.

The β-functions depend on the number of the fields of different spin, $N_0,\, N_{1/2},\, N_1$,

$$\left(\begin{array}{c} \beta_1 \\ -\beta_2 \\ \beta_3 \end{array} \right) = \frac{1}{360(4\pi)^2} \left(\begin{array}{c} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{array} \right)$$

We can see that the ambiguity in a' is typical for many local terms, which come from total derivatives in divergences. This ambiguity has essentially classical oriigin.

Anomaly-Induced Effective Action (EA)

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = T\,.$$

An exact solution for the conformal factor *Davies, (1978)*;

Covariant solution: *Riegert, Fradkin & Tseytlin, (1984).*

$$\bar{\Gamma}_{ind} = S_c[g_{\mu\nu}] + \frac{3\beta_3 - 2\beta_2}{6}\int_x R^2,$$

$$+\frac{\beta_1}{4}\int_x\int_y \left(E - \frac{2}{3}\Box R\right)_x G(x,y)\,\left(C^2\right)_y$$

$$-\frac{\beta_2}{8}\int_x\int_y \left(E - \frac{2}{3}\Box R\right)_x G(x,y)\,\left(E - \frac{2}{3}\Box R\right)_y\,.$$

Here $\int_x = \int d^4x\sqrt{\bar{g}}$ **and** $\Delta_x G(x,x') = \delta(x,x')\,.$

$$\Delta = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu\,.$$

$S_c[g_{\mu\nu}]$ **is an arbitrary conformal functional.**

Consider some part of our result for massive scalar, in the second order in curvatures

$$\bar{\Gamma}^{(1)}_{RR} = \frac{1}{2(4\pi)^2}\int_x R\Big\{\frac{\tilde{\xi}^2}{2\epsilon}+Y\tilde{\xi}^2+\frac{\tilde{\xi}\,Y(4-a^2)}{6a^2}+\frac{\tilde{\xi}}{18}+\frac{Y\,\big(16-8a^2+a^4\big)}{144a^4}+\frac{20-7a^2}{2160\,a^2}$$

where $\tilde{\xi} = \xi - \dfrac{1}{6}\,, \quad Y = 1+\dfrac{1}{a}\ln\left|\dfrac{2-a}{2+a}\right|$ **and** $a^2 = \dfrac{4\Box}{4m^2-\Box}\,.$

In the $m=0,\ \ \xi = 1/6$ **limit we obtain**

$$-\frac{1}{12\cdot 180(4\pi)^2}\int d^4x\, g^{1/2}\, R^2\,,$$

fitting perfectly with the conformal anomaly obtained by point-splitting *Christensen, (1978),* ζ**-reg.** *Cristley & Dowker, (1976); Hawking, (1977)*, **and other methods**

$$< T^\mu_\mu >= \frac{1}{180(4\pi)^2}\,\Box R + \,...\,.$$

The ambiguity of local anomalous term $\int\sqrt{-g}R^2$ in the EA and, correspondingly, $\Box R$ in the anomaly can be seen either in dimensional or covariant Pauli-Villars regularizations.

M. Asorey, E. Gorbar & I.Sh., CQG **21** *(2003).*

In the dimensional regularization, the counterterm $\int\sqrt{-g}\Box R$ doesn't contribute to anomalous violation of local conformal symmetry.

According to *Duff (1977)*, **the anomaly comes from the C^2-type counterterm.**

$$-\frac{2}{\sqrt{-g}}\; g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\;\int d^n x\sqrt{-g}\,C^2(d)\Big|_{d=4,\,n\to 4} \;=\; C^2-\frac{2}{3}\,\Box R\,.$$

However, the finiteness of renormalized EA and locality leave us the freedom to choose the parameter d. If we take

$$d = n + \gamma \cdot [n - 4]\,,$$

where γ is an arbitrary parameter, we meet $a' \sim \gamma$.

Changing α' is $\equiv$ to adding a $\int R^2$-term to the classical action.

Why we are allowed to add a $\int R^2$-term?

Because it belongs to the action of external field, $g_{\mu\nu}$ which doesn't break the conformal symmetry of quantum fields.

In order to fix the arbitrariness, one has to do the following:

- Introduce $\int R^2$-term into the classical action.
- Calculate quantum correction.
- Fix overall $\int R^2$ -term by the renormalization condition.

In the covariant Pauli-Villars regularization one has to introduce a set of "regulator" fields. E.g., in case of a massless conformal scalar φ we have to start from the action

$$S_{\rm reg} = \sum_{i=0}^{N} \int d^4x\sqrt{g}\,\big\{(\nabla\varphi_i)^2 + (\xi_i R + m_i^2)\varphi_i^2\big\}\,.$$

The physical scalar field $\varphi \equiv \varphi_0$ is conformal $\xi = 1/6$, $m_0 = 0$ and bosonic $s_0 = 1$, while PV regulators φ_i are massive $m_i = \mu_i M$ and can be bosonic $s_i = 1$ or fermionic $s_i = -2$.

The UV limit $M \to \infty$ produces the vacuum EA. The Pauli-Villars regulators may have conformal $\xi_i = 1/6$ or non-conformal couplings $\xi_i \neq 1/6$.

$\int R^2$**-term depends on the choice of ξ_i and hence is arbitrary.**

Even stronger arbitrariness

M. Asorey, G. de Berredo-Peixoto, I.Sh, PRD-2006.

Consider interacting conformal scalar theory

$$S = \int d^4x\sqrt{g}\,\Big\{\,\frac{1}{2}\,\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{12}\,R\phi^2 - \frac{\lambda}{24}\phi^4\,\Big\}\,.$$

The Noether identity $\quad \mathcal{T} \,=\, \frac{1}{\sqrt{g}}\Big(\phi\,\frac{\delta S}{\delta\phi} - 2\,g_{\mu\nu}\,\frac{\delta S}{\delta g_{\mu\nu}}\Big) \,=\, 0\,.$

At quantum level it is indeed violated $<\mathcal{T}>\neq 0$.

The ambigous part of anomaly $<\mathcal{T}> = \alpha_2'\Box\phi^2 + ...$ **because the corresponding term in the EA is local**

$$\Gamma_{ind} \,=\, \frac{\alpha'}{6}\,\int d^4x\sqrt{g}\;R\phi^2 \,+\, ...$$

Changing $R\varphi^2$-term in the classical action implies an essential change in the dynamics of quantum fields !!

Conclusions

- An exact formfactors by diagrams or within the heat-kernel method provide a complete form of the decopupling AC theorem.

- At least in the fermionic case one can observe a qualitatively new kind of ambiguity, called Multiplicaive Anomaly. It is due to the physical renormalization scheme in the massive fields case. It looks like for massless fields we have nothing like that.

- In case of conformal massless fields there is another ambiguity related to the local part of quantum EA.

- It might happen that the massless limit of Multiplicaive Anomaly is (sometimes, at least) related to the ambiguity of local anomalous terms, since both cases are related to the surface counterterms.

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Parabens, Manolo!

Feliz Aniversario!!