Quaternion - Octonion QCD and SU(3) Flavor Symmetry

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Outline of the Talk

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1. Quaternions
2. Octonions
3. Quaternionic Lagrangian Formalism
4. Quaternionic SU(2) Global Gauge Symmetry
5. Quaternionic SU(2) Local Gauge Symmetry
6. Quaternionic Representation of Isospin $SU(2)$ Group
7. Gellmann λ matrices
8. Relation Between Octonion and Gellmann Matrices
9. Octonionic Reformulation of QCD
10. Octonion Formulation of SU(3) Flavor Group
11. Commutation Relations for Octonion Valued Shift Operetors
12. Conclusion

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Quaternions

- Quaternion are the natural extension of complex numbers and form an algebra under addition and multiplication.

- Quaternion were first described by Irish mathematician Sir William Rowan Hamilton in 1843.
- A striking feature of quaternions is that the product of two quaternions is non commutative.

Quaternions

The algebra $\mathbb{H}$ of quaternion is a four - dimensional algebra over the field of real numbers $\mathbb{R}$ and a quaternion ϕ is expressed in terms of its four base elements as

$$\phi = \phi_\mu e_\mu = \phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 \quad (\mu = 0, 1, 2, 3), \tag{1}$$

where ϕ_0 ,ϕ_1 ,ϕ_2,ϕ_3 are the real quarterate of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$e_0 e_A = e_A e_0 = e_A; \quad e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C. \, (\forall\, A, B, C = 1, 2, 3) \tag{2}$$

where δ_{AB} is the Kronecker delta symbol and f_{ABC} is the Levi Civita three index symbol.

Quaternions

As such we may write the following relations among quaternion basis elements

$$
\begin{aligned}
[e_A,\ e_B] &= 2\, f_{ABC}\, e_C; \\
\{e_A,\ e_B\} &= -2\, \delta_{AB} e_0; \\
e_A(\, e_B\, e_C) &= (e_A\, e_B\,)\, e_C
\end{aligned}
\tag{3}
$$

where brackets [,] and { , } are used respectively for commutation and the anti commutation relations.

- $\mathbb{H}$ is an associative but non commutative algebra.
- Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$.
- Quaternion elements are non-Abelian in nature and thus represent a non commutative division ring.

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Octonions

- The octonions form the widest normed algebra after the algebra of real numbers, complex numbers and quaternions.
- The octonions are an 8 - dimensional algebra with basis 1, e_1, e_2, e_3, e_4, e_6, e_7.
- Set of octets $(e_0,\ e_1,\ e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and satisfy the following multiplication rules

$$e_0 = 1;\ e_0 e_A = e_A e_0 = e_A$$
$$e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C.\ (A, B, C = 1, 2, ..., 7). \tag{4}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for following combinations,

$$f_{ABC} = +1; \forall (ABC) = (123), (471), (257), (165), (624), (543), (736). \tag{5}$$

Quaternionic Lagrangian Formalism

Let us consider that we have two spin 1/2 fields, ψ_a and ψ_b. The Lagrangian without any interaction is thus defined as

$$L = [i\overline{\psi}_a\gamma^\mu\partial_\mu\psi_a - m\overline{\psi}_a\psi_a] + [\quad i\overline{\psi}_b\gamma^\mu\partial_\mu\psi_b - m\overline{\psi}_b\psi_b \quad] \tag{6}$$

where m is the mass of particle, $\overline{\psi}_a$ and $\overline{\psi}_b$ are respectively used for the adjoint representations of ψ_a and ψ_b and the γ matrices are defined as

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} (\forall j = 1, 2, 3). \tag{7}$$

Here σ_j are the well known 2×2 Pauli spin matrices. Lagrangian density (6) is thus the sum of two Lagrangian for particles a and b.

Quaternionic Lagrangian Formalism

We can write above equation more compactly by combining ψ_a and ψ_b into two component column vector;

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{8}$$

where $\psi_a = (\psi_0 + e_1\psi_1)$ and $\psi_b = (\psi_2 - e_1\psi_3)$ described in terms of the field of real number representations. So, we may write the quaternionic form of the Lagrangian in terms of ψ as

$$L = [i\overline{\psi}\gamma^\mu\partial_\mu\psi \quad -m\,\overline{\psi}\psi] \tag{9}$$

Solving the Lagrangian, the Dirac equation expressed as

$$i\gamma^\mu(\partial_\mu\psi) - m\psi = 0. \tag{10}$$

which provide the four current as

$$j^\mu = \overline{\psi}\,\gamma^\mu\psi \tag{11}$$

Here we developed interrelationship between $SU(2)$ non - Abelian gauge theory with quaternion algebra.

Quaternionic SU(2) Global gauge symmetry

In global gauge symmetry, the unitary transformations are independent of space and time. The Lagrangian density is invariant under SU(2) global gauge transformations i.e. $\delta L = 0$. The Lagrangian density thus yields the continuity equation after taking the variations and the definitions of Euler Lagrange equations as

$$\partial_{\mu}\left\{\frac{\partial L}{\partial(\partial_{\mu}\psi)}e_{k}\psi\right\} = \partial_{\mu}\left\{i\overline{\psi}\gamma^{\mu}e_{k}\psi\right\} = \partial_{\mu}(j^{\mu})^{k} = 0 \;(\forall\, k = 1,2,3) \quad (12)$$

where the $SU(2)$ gauge current is defined as

$$(j^{\mu})^{k} = \left\{i\overline{\psi}\gamma^{\mu}e_{k}\psi\right\}. \quad (13)$$

which is the global current of the fermion field.

Quaternionic SU(2) Local Gauge Symmetry

- $SU(2)$ local gauge transformation we may replace the unitary gauge transformation as space - time dependent.
- Replacing partial derivative of global gauge symmetry to covariant derivative of local gauge symmetry, we may write the invariant Lagrangian density for the quaternion $SU(2)$ gauge fields in the following form

$$L = i\overline{\psi}\gamma_{\mu}(D_{\mu}\psi) - m\overline{\psi}\psi, \tag{14}$$

which yields the following current densities of electric and magnetic charges of dyons i.e

$$J_{\mu} = (j_{\mu})_{electric} + (j_{\mu})_{magnetic} = ie\overline{\psi}\,\gamma_{\mu}\psi + ig\overline{\psi}\,\gamma_{\mu}\psi. \tag{15}$$

Quaternionic representation of isospin SU(2) group

Using the appropriate properties of quaternions and its relation with Pauli matrices we may now describe the $SU(2)$ isospin in terms of quaternions as

$$I_a = \frac{ie_a}{2} \;\; (\forall\, a = 1, 2, 3) \;\; and \;\; I_{\pm} = \frac{i}{2}(e_1 \pm ie_2) \,. \tag{16}$$

Thus, we may write the quaternion basis elements in terms of $SU(2)$ isospin as

$$e_1 = \frac{1}{i}(I_+ + I_-) \,; \quad e_2 = \frac{1}{i}(I_+ - I_-) \,; \quad e_3 = \frac{1}{i}(I_3) \,; \tag{17}$$

which satisfy the following commutation relation

$$[I_+, I_-] = ie_3; \qquad [I_3, I_{\pm}] = \pm\frac{i}{2}(e_1 \pm ie_2) \,. \tag{18}$$

Quaternionic representation of isospin SU(2) group

Here $SU(2)$ group acts upon the fundamental representation of $SU(2)$ doublets of up (u) and down(d) quark spinors

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{19}$$

Up (u) and down(d) quark spinors spans the self representation space of the flavor $SU(2)$ group. We get for up quarks

$$\begin{aligned} I_{+}\left|u\right\rangle &= \frac{i\left(e_{1}+ie_{2}\right)}{2}\left|u\right\rangle = 0; \\ I_{-}\left|u\right\rangle &= \frac{i\left(e_{1}-ie_{2}\right)}{2}\left|u\right\rangle = \frac{1}{2}\left|d\right\rangle; \\ I_{3}\left|u\right\rangle &= \frac{ie_{3}}{2}\left|u\right\rangle = \frac{1}{2}\left|u\right\rangle; \end{aligned} \tag{20}$$

Quaternionic representation of isospin SU(2) group

For down quarks we have

$$
\begin{aligned}
I_{+}\left|d\right\rangle &= \frac{i\left(e_{1}+ie_{2}\right)}{2}\left|d\right\rangle = \frac{1}{2}\left|u\right\rangle ; \\
I_{-}\left|d\right\rangle &= \frac{i\left(e_{1}-ie_{2}\right)}{2}\left|d\right\rangle = 0; \\
I_{3}\left|d\right\rangle &= \frac{ie_{3}}{2}\left|d\right\rangle = -\frac{1}{2}\left|d\right\rangle
\end{aligned} \tag{21}
$$

Conjugates of above equations are now be described as

$$
\begin{aligned}
\left\langle d\right| I_{+} &= \left\langle d\right| \frac{i\left(e_{1}+ie_{2}\right)}{2} = 0; \\
\left\langle d\right| I_{-} &= \left\langle d\right| \frac{i\left(e_{1}-ie_{2}\right)}{2} = \frac{1}{2}\left\langle u\right| ; \\
\left\langle d\right| I &= \left\langle d\right| \frac{ie_{3}}{2} = \frac{1}{2}\left\langle d\right| ;
\end{aligned} \tag{22}
$$

Quaternionic representation of isospin SU(2) group

$$\langle u| I_+ = \langle u| \frac{i\left(e_1 + ie_2\right)}{2} = \frac{1}{2}\langle d|\,;$$
$$\langle u| I_- = \langle u| \frac{i\left(e_1 - ie_2\right)}{2} = 0;$$
$$\langle u| I_3 = \langle u| \frac{ie_3}{2} = \frac{1}{2}\langle u|\,; \tag{23}$$

The effect of quaternion operator on up $|u\rangle$ and down $|d\rangle$ quarks states leads to

$$ie_1\,|u\rangle = |d\rangle\,; \quad ie_1\,|d\rangle = |u\rangle\,;$$
$$e_2\,|u\rangle = |d\rangle\,; \quad e_2\,|d\rangle = -\,|u\rangle$$
$$ie_3\,|u\rangle = |u\rangle\,; \quad ie_3\,|d\rangle = -\,|d\rangle\,. \tag{24}$$

So, we may write

$$e_1 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ d \end{pmatrix}; \tag{25}$$

$$e_2 \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} d \\ -u \end{pmatrix}; \tag{26}$$

$$e_3 \begin{pmatrix} u \\ d \end{pmatrix} = i \begin{pmatrix} u \\ -d \end{pmatrix}; \tag{27}$$

transform a neutron (down quark) state into a proton (up quark) state or vice verse. Only e_2 gives real doublets of up and down quarks.

Gellmann λ matrices

In order to extend the symmetry from $SU(2)$ to $SU(3)$ we replace three Pauli spin matrices by eight Gellmann λ matrices. $\lambda_j \ \ (j = 1, 2, ...8)$ be the 3×3 traceless Hermitian matrices introduced by Gell-Mann. Their explicit forms are;

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Gellmann λ matrices

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{28}$$

which satisfy the following property as

$$[\lambda_j, \lambda_k] = 2F_{jkl}\lambda_l \ (\forall \ j, k, l = 1, 2, 3, 4, 5, 6, 7, 8) \tag{29}$$

where F_{jkl} are the structure constants of $SU(3)$ group defined as

$$F_{123} = 1; \ F_{147} = F_{257} = F_{435} = F_{651} = F_{637} = \frac{1}{2};$$
$$F_{458} = F_{678} = \sqrt{\frac{3}{2}}. \tag{30}$$

Relation between Octonion and Gellmann Matrices

Here we establish the relationship between octonion basis elements e_A and Gellmann λ matrices. Comparing the structure constants of octonion with structure coefficients of Gell Mann λ matrices, we get,

$$\frac{[e_A, e_B]}{[\lambda_A, \lambda_B]} = \frac{e_C}{i\lambda_C} \;(\forall\, A, B, C = 1, 2, 3)$$
$$\Rightarrow [e_A, e_B] = [\lambda_A, \lambda_B] \;\; (\forall\, e_C = i\lambda_C) \tag{31}$$

On the other hand we get,

$$\frac{[e_A, e_B]}{[\lambda_A, \lambda_B]} = \frac{e_C}{2i\lambda_C} \;(\forall ABC = 516, 624, 471, 435, 673, 572)$$
$$\Rightarrow [e_A, e_B] = [\lambda_A, \lambda_B] \;\; (\forall\, e_C = i\frac{\lambda_C}{2}). \tag{32}$$

Relation between Octonion and Gellmann Matrices

Now we describe λ_8 in terms of octonion units as

$$\lambda_8 = -\frac{2}{i\sqrt{3}}\{[e_4, e_5] + [e_6, e_7]\} = \frac{8e_3}{i\sqrt{3}}. \tag{33}$$

where $\mathtt{k} = \frac{i\sqrt{3}}{8}$. In general these commutation relations are given as

$$\frac{[e_{a+3}, e_7]}{[\lambda_{a+3}, \lambda_7]} = \frac{e_a}{2i\lambda_a} \tag{34}$$

$$\frac{[e_7, e_a]}{[\lambda_7, \lambda_a]} = \frac{e_{a+3}}{2i\lambda_{a+3}} \tag{35}$$

Relation between Octonion and Gellmann Matrices

$$\frac{[e_a, e_{a+3}]}{[\lambda_a, \lambda_{a+3}]} = \frac{e_7}{2i\lambda_7} \tag{36}$$

where a=1,2,3. Now we find the following relationship between Gell Mann λ matrices and octonion units:

$$\begin{aligned} \lambda_1 = -ie_1 k_1; \lambda_2 = -ie_2 k_2; \lambda_3 = -ie_3 k_3; \lambda_4 = -ie_4 k_4; \\ \lambda_5 = -ie_5 k_5; \lambda_6 = -ie_6 k_6; \lambda_7 = -\ ie_7 k_7 \end{aligned} \tag{37}$$

where $k_a = -1 \ \ (\forall a = 1, 2, 3, ..., 7)$ are proportionality constants. And also λ_8 are related with e_3 as

$$\lambda_8 = -ik_8 e_3 \ \ where \ \ k_8 = \frac{8}{\sqrt{3}} \tag{38}$$

Octonionic Reformulation of QCD

- Quantum chromodynamics (QCD) is a non - Abelian local gauge theory based on a symmetry implying a quantum number called color.
- The theory of strong interaction might be built by considering the color symmetry as a local gauge symmetry, suggests that quarks appear in three colors.
- It describes the interaction between point like colored quarks and gluon's.
- The local gauge theory of color $SU(3)$ group gives the theory of QCD. The QCD (quantum chromodynamics) is just a Yang Mills theory with $SU(3)$ gauge group.

Octonionic Reformulation of QCD

The theory of strong interactions, quantum chromodynamics (QCD) is based on $SU(3)_C$ group. This is a group which acts on the colour indices of quark flavors described in the form of a basic triplet i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow \begin{pmatrix} R \\ B \\ G \end{pmatrix} \tag{39}$$

where indices R, B, and G are the three colour of quark flavors. Here we attempt to introduce a local phase transformation in color space. Under $SU(3)_c$ symmetry, the spinor ψ transforms as

$$\psi \longmapsto \psi^{'} = U\psi = \exp\{i\lambda_a \alpha^a(x)\}\psi \tag{40}$$

where λ are Gellmann matrices, $a = 1, 2, ...8$ and the parameter α is space time dependent.

Octonionic Reformulation of QCD

We may develop accordingly the octonionic reformulation of quantum chromo dynamics (QCD) on replacing the Gellmann λ matrices by octonion basis elements e_A. The value of $\lambda_a \alpha^a(x)$ as

$$\sum_{a=1}^{8} \lambda_a \alpha^a(x) = -i \sum_{q=1}^{7} e_q \beta^q(x) \tag{41}$$

Since $\psi \longmapsto \psi^{\prime} = U\psi = \exp\{e_q \beta^q(x)\}$, So we may write the locally gauge invariant $SU(3)_c$, Lagrangian density in the following form;

$$L_{local} = \left(i\overline{\psi}\gamma_\mu D_\mu \psi - m\overline{\psi}\psi\right) \quad -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{42}$$

where $D_\mu \psi = \partial_\mu \psi + \boldsymbol{e}\, e_a A^a_\mu \psi + \mathrm{g}\, e_a B^a_\mu \psi$. and
$G^a_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathbf{e}\, f_{abc} A^b_\mu A^c_\nu\right) + \left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu - \mathbf{g}\, f_{abc} B^b_\mu B^c_\nu\right)$.

Octonionic Reformulation of QCD

Where the **e** and **g** are the coupling constants due to the occurrence of respectively the electric and magnetic charges on dyons. Hence the locally gauge covariant Lagrangian density is written as

$$L_{local} = \left(i\overline{\psi}\gamma_\mu\partial_\mu\psi - m\overline{\psi}\psi\right) - \mathbf{e}\left(\overline{\psi}\gamma^\mu\psi\right)e_a A^a_\mu - \mathbf{g}\left(\overline{\psi}\gamma^\mu\psi\right)e_a B^a_\mu - \frac{1}{4}G^a_{\mu\nu} \tag{43}$$

which leads to the following expression for the gauge covariant current density of coloured dyons

$$J^a_\mu = \mathbf{e}\left(\overline{\psi}\gamma^\mu\psi\right)e_a + \mathbf{g}\left(\overline{\psi}\gamma^\mu\psi\right)e_a. \tag{44}$$

which leads to the conservation of Noetherian current in octonion formulation of $SU(3)_c$ gauge theory of quantum chromodynamics (QCD) i.e.

$$D_\mu J^\mu = 0 \;\; where \;\; J^\mu = J^{\mu a}\lambda_a. \tag{45}$$

Octonion Formulation of SU(3) Flavor Group

- The Lie algebra of $SU(3)$ exhibits most of the features of the larger Lie algebras.
- $SU(3)$ may play a special role connected with its description in terms of octonions.
- The elements of $SU(3)$ group may be obtained in terms of 3×3 Hermitian Gell Mann λ Hermitian matrices related to octonions where first three matrices describe the familiar isotopic spin generators from the $SU(2)$ subgroup of $SU(3)$.
- The fourth and fifth generators and the sixth and seventh generators are denoted as the V - spin and the U - spin. V - spin connects the up (u) and strange quarks (s) while U - spin connects the down (d) and strange quarks (s).
- $SU(3)$ flavor group contains fundamental building blocks in isospin space along with the strangeness.
- The eighth generator is diagonal in nature responsible for hyper charge.

Octonion Formulation of SU(3) Flavor Group

- The I, U and $V-$ spin algebra fulfills the angular momentum algebra and turn out to the sub algebras of $SU(3)$.
- The $SU(3)$ multiplets are constructed in form of a $I-$ multiplets, $V-$ multiplets and an $U-$ multiplets.
- The $I-$ spin, $U-$ spin and $V-$spin algebra are closely related and are the elements of sub algebra of $SU(3)$. $SU(3)$ multiplets described as

$$\begin{aligned} I_1 &= \frac{ie_1}{2}; \quad I_2 = \frac{ie_2}{2}; \quad I_3 = \frac{ie_3}{2} \, (I - \ Spin) \\ V_1 &= \frac{ie_4}{2}; \quad V_2 = \frac{ie_5}{2}; \quad V_3 = \frac{ie_3}{4}\left(8\sqrt{3}+1\right) \, (U - \ Spin) \\ U_1 &= \frac{ie_6}{2}; \quad U_2 = i\frac{e_7}{2}; \quad U_3 = \frac{ie_3}{4}\left(8\sqrt{3}-1\right) \, (V - \ Spin) \end{aligned} \tag{46}$$

along with the hyper charge is written in terms of octonions as follows

$$Y = \frac{1}{\sqrt{3}}\lambda_8 = -\frac{8ie_3}{\sqrt{3}}. \tag{47}$$

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$$V_1 = \frac{ie_4}{2}; \quad V_2 = \frac{ie_5}{2}; \quad V_3 = \frac{ie_3}{4} \left(8\sqrt{3}+1\right) \, (U - \; Spin)$$
$$U_1 = \frac{ie_6}{2}; \quad U_2 = i\frac{e_7}{2}; \quad U_3 = \frac{ie_3}{4} \left(8\sqrt{3}-1\right) \, (V - \; Spin) \quad (46)$$

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Octonion Formulation of SU(3) Flavor Group

- Here I_1, I_2 and I_3 contain the 2×2 isospin operators (i. e. quaternion units).
- U_3, V_3, I_3 and Y are linearly independent generators and are simultaneously diagonalized.
- It will to be noted that $\lambda_1, \lambda_2, \lambda_3$ agree with $\sigma_1, \sigma_2, \sigma_3$.
- The complexified variants contain the third operators $I_\pm$, $U_\pm$, $V_\pm$ which characterizes the states of $SU(3)$ multiplets. The operators $I_\pm$, $U_\pm$, $V_\pm$ are defined as

$$I_\pm = I_x \pm i I_y = \frac{1}{2}(\lambda_1 \pm i\lambda_2) = \frac{i}{2}(e_1 \pm i e_2). \quad (48)$$

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Octonion Formulation of SU(3) Flavor Group

- Here I_1, I_2 and I_3 contain the 2×2 isospin operators (i. e. quaternion units).
- U_3, V_3, I_3 and Y are linearly independent generators and are simultaneously diagonalized.
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With the help of shift operators and their properties, we may derive the quark states of these multiplets as $|q_1\rangle, |q_2\rangle, |q_3\rangle$. So the quark states of I ,U and V spin are described as

$$I_- \mid q_1 >=\mid q_2 >; \quad I_+ \mid q_2 >=\mid q_1 > . \tag{51}$$

$$U_- \mid q_2 >=\mid q_3 >; \quad U_+ \mid q_3 >=\mid q_2 > . \tag{52}$$

$$V_- \mid q_1 >=\mid q_3 >; \quad V_+ \mid q_3 >=\mid q_1 > . \tag{53}$$

Thus, the operators $I_\pm$, $U_\pm$, $V_\pm$ are viewed as operators which transforms one flavor into another flavor of quarks

$$\begin{aligned} I_\pm (I_3) &\longmapsto I_3 \pm 1; \\ V_\pm (V_3) &\longmapsto V_3 \pm 1; \\ U_\pm (U_3) &\longmapsto U_3 \pm 1. \end{aligned} \tag{54}$$

It means the action of $I_\pm$,$U_\pm$ and $V_\pm$ shifts the values of I_3 ,V_3 and U_3 by ± 1.

Commutation relations for Octonion Valued Shift Operetors

The I, U and V - spin algebras are closed. Let us obtain the commutation relations of shift operators $I_{\pm}$, $U_{\pm}$, $V_{\pm}$ for $SU(3)$ group in terms of octonions as

$$[U_{+},\ U_{-}] = \frac{ie_3}{2}\left(8\sqrt{3}-1\right) = 2U_3;$$
$$[V_{+},\ V_{-}] = \frac{ie_3}{2}\left(8\sqrt{3}+1\right) = 2V_3;$$
$$[I_{+},\ I_{-}] = ie_3 = 2I_3.$$
$$[I_{+},\ V_{+}] = [I_{+}, U_{+}] = [U_{+}, V_{+}] = 0; \quad (55)$$

Commutation relations for Octonion Valued Shift Operetors

$$[Y,\ U_{\pm}] = \pm\left[\frac{i}{2}(e_6 \pm ie_7)\right] = \pm\, U_{\pm};$$
$$[Y,\ V_{\pm}] = \pm\left[\frac{i}{2}(e_4 \pm ie_5)\right] = \pm\, V_{\pm};$$
$$[Y,\ I_{\pm}] = \pm\left[\frac{i}{2}(e_1 \pm ie_2)\right] = \pm\, I_{\pm}. \tag{56}$$

$$[I_+,\ V_-] = -\left[\frac{i}{2}(e_6 - ie_7)\right] = -U_-;$$
$$[T_+, U_+] = \left[\frac{i}{2}(e_4 + ie_5)\right] = V_+;$$
$$[U_+,\ V_-] = \frac{i}{2}(e_1 - ie_2) = I_-. \tag{57}$$

Commutation relations for Octonion Valued Shift Operetors

Accordingly, we may write the hyper charge as

$$Y = \frac{1}{\sqrt{3}}\lambda_8 = -\frac{8ie_3}{\sqrt{3}} = \frac{2}{3}(U_3 + V_3) = \frac{2}{3}(2U_3 + I_3) = \frac{2}{3}(2V_3 - I_3). \quad (58)$$

and the term hyper charge Y commutes with third component of I, U and V - spin multiplets of $SU(3)$ flavor group

$$[Y,\, I_3] = [Y,\, U_3] = [Y,\, V_3] = 0. \quad (59)$$

and it also

$$I_+ = \frac{i}{2}(e_1 \pm ie_2) = (I_-)^\dagger$$
$$V_+ = \frac{i}{2}(e_4 \pm ie_5) = (V_-)^\dagger$$
$$U_+ = \frac{i}{2}(e_6 \pm ie_7) = (U_-)^\dagger. \quad (60)$$

Commutation relations for Octonion Valued Shift Operetors

The commutation relation between the third components of I, U and V with $I_{\pm}$ are given as

$$\begin{aligned}
[I_3,\ I_{\pm}] &= \pm\left[\frac{i}{2}\left(e_1 \pm ie_2\right)\right] = \pm\, I_{\pm};\\
[U_3,\ I_{\pm}] &= \mp\frac{1}{2}\left[\frac{i}{2}\left(e_1 \pm ie_2\right)\right] = \mp\,\frac{1}{2}I_{\pm};\\
[V_3,\ I_{\pm}] &= \pm\frac{1}{2}\left[\frac{i}{2}\left(e_1 \pm ie_2\right)\right] = \pm\frac{1}{2}I_{\pm};
\end{aligned} \tag{61}$$

The octonions are related to raising I_+, U_+, V_+ and lowering I_-, U_-, V_- operators as

$$\begin{aligned}
e_1 = i\left(I_+ + I_-\right);\quad e_2 = \left(I_+ - I_-\right);\quad e_3 = 2I_3;&\\
e_4 = i\left(V_+ + V_-\right);\ \ e_5 = \left(V_+ - V_-\right);&\\
e_6 = i\left(U_+ + U_-\right);\ \ e_7 = \left(I_+ + I_-\right).&
\end{aligned} \tag{62}$$

Commutation relations for Octonion Valued Shift Operetors

The commutation relations between I_+ and I_-, U_+ and $U_-$and V_+ and V_- are described as

$$\begin{aligned} [I_+,\, I_-] &= ie_3 = 2I_3; \\ [U_+,\, U_-] &= \frac{ie_3}{2}\left(8\sqrt{3}-1\right) = 2U_3; \\ [V_+,\, V_-] &= -\frac{ie_3}{4}\left(8\sqrt{3}+1\right) = 2V_3. \end{aligned} \tag{63}$$

Conclusion

- Quaternions are used to study the successful gauge theory of electro - weak unification and octonions are used to describe the *QCD* i.e. the theory of strong interaction.
- We have tried to reformulate the duality and gauge theories in terms of hyper complex numbers over the fields of real, complex and quaternion number system.
- The quaternion formulation be adopted in a better way to understand the explanation of the duality conjecture and gauge theories as the candidate for the existence of monopoles and dyons where the complex parameters are described as the constituents of quaternion.

Conclusion

- The isospin symmetries are good approximation to simplify the interaction among hadrons.
- The motivation behind the present theory was to develop a simple compact and consistent algebraic formulation of $SU(2)$ and $SU(3)$ symmetries in terms of normed algebras namely quaternions and octonions.
- We have described the compact simplified notations instead of using the Pauli and Gell mann matrices.
- In this study, We have obtained $SU(2)$ and $SU(3)$ groups, and many commutating generators as simple roots, a feature that generalizes to all lie algebras.
- Octonion representation of $SU(3)$ flavor group directly establishes the one to one mapping between the non - associativity and the theory of strong interactions.

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Conclusion

- It also shows that the theory of hadrons (or quark - colour etc.) has the direct link with non - associativity (octonions) while the isotopic spin leads to non commutativity (quaternions).
- The flavor properties of quarks play an important role in the weak interaction of hadrons while color property distinguishes quarks from leptons.
- As such, normed algebras namely the algebra of complex numbers, quaternions and octonions play an important role for the physical interpretation of quantum electrodynamics (QED), standard model of EW interactions and quantum chromo dynamics (QCD).

THANKS