The Proximity Force approximation as a Derivative Expansion

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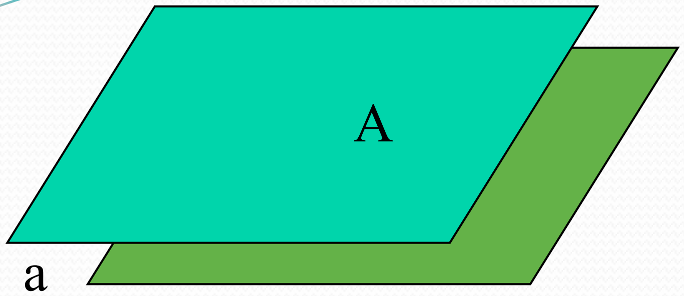
Introduction

Until recent development of theoretical methods that allowed for the exact evaluation of the Casimir energy for several geometries, the interaction between different bodies has been mostly computed using the so called proximity force approximation (PFA)

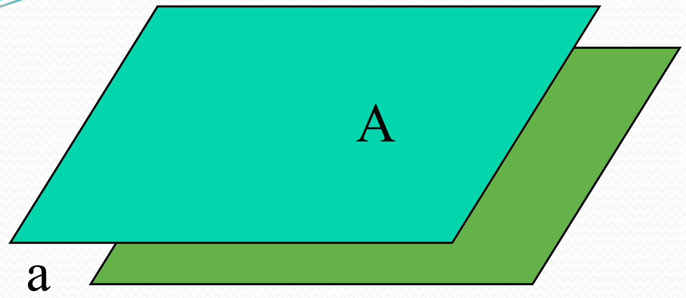
PFA is suposed to be reliable as long as the interacting surfaces are smooth, almost parallel, and very close



PFA makes use of Casimir's expression for the energy per unit area for two parallel plates at a distance "a" apart

For a massless-scalar field

$$E_{\mathrm{pp}}(a) = -\frac{\pi^2}{1440\, a^3}$$



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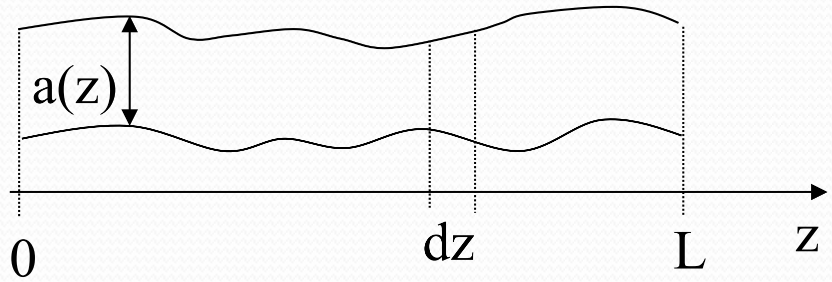
The PFA then approximates the interaction between two Dirichlet surfaces separated by a gap of spatially varying width z



$$E_{\mathrm{PFA}} = \int_{\Sigma} d\sigma\, E_{\mathrm{pp}}(z)$$

Its accuracy has been assessed only in some of the particular geometries where it was possible to compute the Casimir energy numerically or analytically

On general grounds, one expects that

$$E_{\mathrm{C}} = E_{\mathrm{PFA}} \left\{ 1 + \gamma \frac{a}{\mathcal{L}} + \mathcal{O}\left[\left(\frac{a}{\mathcal{L}} \right)^2 \right] \right\}$$

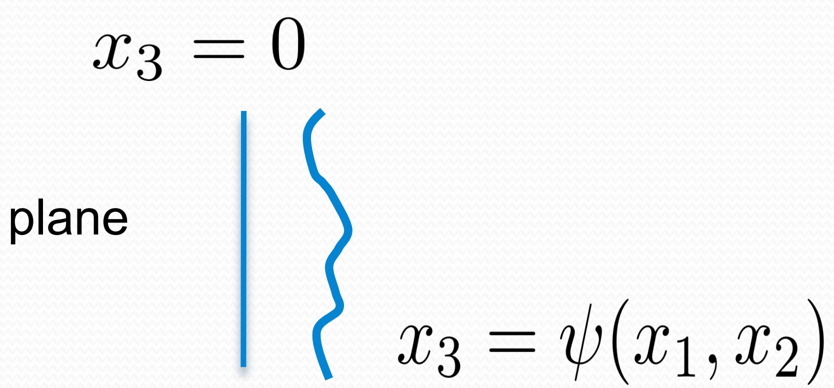
$\mathcal{L}$ typical length associated to the curvature of one of the surfaces (assumed much smaller than the curvature of the second one)

γ fixes the accuracy of the PFA in each particular geometry

The Casimir energy can be thought as a functional of the shape of the surfaces of the interacting bodies. As the PFA should be adequate for almost plane surfaces, **a derivative expansion** of this functional should reproduce, to lowest order, the PFA

Terms involving derivatives of the functions that describe the shape of the surfaces should contain the **corrections to the PFA**

Our goal: to find a general formula to compute the first corrections to PFA for rather arbitrary surfaces



On general grounds, one expects

$$E_{\mathrm{DE}} = - \int d^2\mathbf{x}_{\|} \ \left[V(\psi) \ + \ Z(\psi) \ (\partial_\alpha \psi)^2 \right]$$

+ ….

Dimensional analysis

$$\boxed{E_{\rm DE} = -\frac{\pi^2}{1440}\int d^2\mathbf{x}_{\|}\ \frac{1}{\psi^3}\left[\beta_1 + \beta_2(\partial_\alpha\psi)^2\right]}$$

ψ Describes one of the two surfaces

1 or 2

+ ….

??

The mirrors occupy two surfaces, denoted by L and R, defined by the equations

$$x_3 = 0 \qquad \psi(x_1, x_2) = x_3$$

Derivative Expansion

Functional approach to the Casimir effect

$$\mathcal{Z} \;=\; \int \mathcal{D}\varphi \; \delta_L(\varphi)\, \delta_R(\varphi)\; e^{-S_0(\varphi)}$$

$$S_0(\varphi) \;=\; \frac{1}{2} \int d^4x \, (\partial\varphi)^2$$

Auxiliary fields $\mathcal{Z} \;=\; \int \mathcal{D}\varphi \mathcal{D}\lambda_L \, \mathcal{D}\lambda_R \; e^{-S(\varphi;\lambda_L,\lambda_R)}$

After integrating out the scalar field φ

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\lambda_L \mathcal{D}\lambda_R e^{-\frac{1}{2}\int d^3x_{\parallel} \int d^3y_{\parallel} \sum_{\alpha,\beta} \lambda_\alpha(x_{\parallel}) \mathbb{T}_{\alpha\beta} \lambda_\beta(y_{\parallel})}$$

$$\mathbb{T}_{LL}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | y_{\parallel}, 0 \rangle$$
$$\mathbb{T}_{LR}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | y_{\parallel}, \psi(\mathbf{y}_{\parallel}) \rangle$$
$$\mathbb{T}_{RL}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, \psi(\mathbf{x}_{\parallel}) | (-\partial^2)^{-1} | y_{\parallel}, 0 \rangle$$
$$\mathbb{T}_{RR}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, \psi(\mathbf{x}_{\parallel}) | (-\partial^2)^{-1} | y_{\parallel}, \psi(\mathbf{y}_{\parallel}) \rangle$$

$$\mathbb{T}_{LL}(x_{\|}, y_{\|}) = \langle x_{\|}, 0|(-\partial^2)^{-1}|y_{\|}, 0\rangle$$

$$\langle x|(-\partial^2)^{-1}|y\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2}$$

The vacuum energy of the system (after subtracting the zero-point energy of the free field)

$$E_{\mathrm{vac}} = \lim_{T\to\infty} \left(\frac{\Gamma}{T}\right) = \frac{1}{2T} \mathrm{Tr}\log \mathbb{T}$$

$$\Gamma \equiv -\log \frac{\mathcal{Z}}{\mathcal{Z}_0}$$

The first terms in the derivative expansion

To evaluate β_1 and β_2 it is enough to consider

$$\psi(\mathbf{x}_{||}) = a + \eta(\mathbf{x}_{||})$$

Up to second order

$$\Gamma(a,\eta) = \Gamma^{(0)}(a) + \Gamma^{(1)}(a,\eta) + \Gamma^{(2)}(a,\eta) + \ldots$$

Then $$\Gamma^{(l)}(\psi) = \Gamma^{(l)}(a,\eta)\Big|_{a\to\psi,\eta\to\psi}$$

Expanding first the matrix $\mathbb{T}$ in powers of η

$$\mathbb{T} = \mathbb{T}^{(0)} + \mathbb{T}^{(1)} + \mathbb{T}^{(2)} + \ldots$$

One obtains the expansion of the effective action

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots$$

Each term:

$$\begin{aligned}
\Gamma^{(0)} &= \frac{1}{2}\mathrm{Tr}\log\mathbb{T}^{(0)} \\
\Gamma^{(1)} &= \frac{1}{2}\mathrm{Tr}\log\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}\right] \\
\Gamma^{(2)} &= \frac{1}{2}\mathrm{Tr}\log\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(2)}\right] \\
&\quad -\frac{1}{4}\mathrm{Tr}\log\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}\right]
\end{aligned}$$

Zeroth-order: $\psi \to a$ in $\mathbb{T}^{(0)}$

$$\Gamma^{(0)}(a) \;=\; \frac{1}{2}\,\mathrm{Tr}\log\left[1-(T^{(0)}_{LL})^{-1}T^{(0)}_{LR}(T^{(0)}_{RR})^{-1}T^{(0)}_{RL}\right]$$

Evaluating the trace $\Gamma^{(0)} \;=\; \frac{T}{2}\int d^2\mathbf{x}_{\|}\int\frac{d^3k_{\|}}{(2\pi)^3}\log[1-e^{-2k_{\|}a}]$

We then replace $a \to \psi$ to extract the zeroth-order Casimir energy

$$E^{(0)}_{\mathrm{vac}} \;=\; \frac{1}{2}\int d^2\mathbf{x}_{\|}\int\frac{d^3k_{\|}}{(2\pi)^3}\log[1-e^{-2k_{\|}\psi(\mathbf{x}_{\|})}]$$

$$= \; -\frac{\pi^2}{1440}\int d^2\mathbf{x}_{\|}\,\frac{1}{\psi(\mathbf{x}_{\|})^3}$$

PFA to the vacuum energy

Second order: two contributions

$$\Gamma^{(2)} = \Gamma^{(2,1)} + \Gamma^{(2,2)}$$

$$\Gamma^{(2,1)} = \frac{1}{2}\mathrm{Tr}\log\left[\left(\mathbb{T}^{(0)}\right)^{-1}\mathbb{T}^{(2)}\right]$$

$$\Gamma^{(2,2)} = -\frac{1}{4}\mathrm{Tr}\log\left[\left(\mathbb{T}^{(0)}\right)^{-1}\mathbb{T}^{(1)}\left(\mathbb{T}^{(0)}\right)^{-1}\mathbb{T}^{(1)}\right]$$

With two derivatives of η

The second order effective action is

$$\Gamma^{(2)}(a,\eta) = -\frac{T}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel \, \frac{1}{a^3}\,(\partial_\alpha \eta)^2$$

Finally

$$E^{(2)}_{\text{vac}} = \frac{\Gamma^{(2)}(\psi)}{T} = \frac{1}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel \, \frac{(\partial_\alpha \psi)^2}{\psi^3}$$

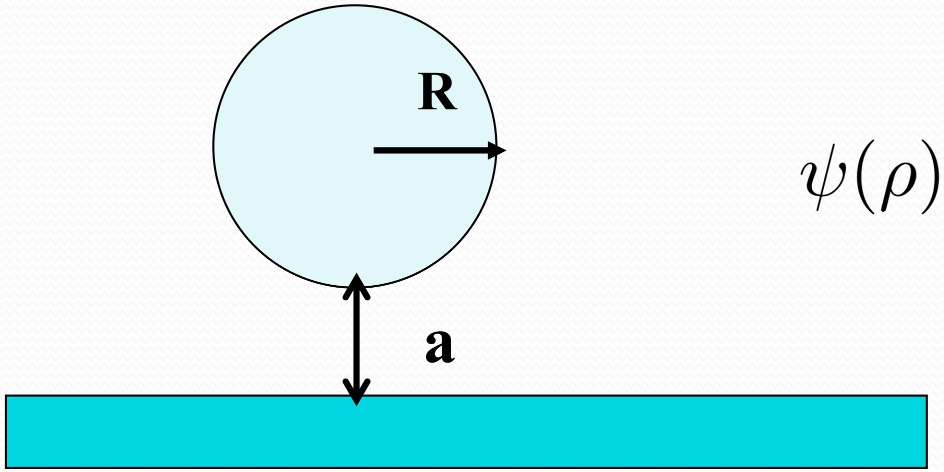
First correction to PFA

The formula

$$E_{\rm DE} \equiv E_{\rm vac}^{(0)} + E_{\rm vac}^{(2)} = -\frac{\pi^2}{1440} \int d^2\mathbf{x}_\parallel \frac{1}{\psi^3} \left[1 + \frac{2}{3} (\partial_\alpha \psi)^2 \right]$$

Examples

A sphere in front of a plane



$$\psi(\rho) = a + R\left(1 - \sqrt{1 - \frac{\rho^2}{R^2}}\right)$$

$$0 \leq \rho \leq R$$

We describe an hemisphere

The derivative expansion will be well defined if we restrict the integrations to the region

$$0 \leq \rho \leq \rho_{\mathrm{M}} < R$$

Inserting the expression for ψ into the derivative expansion of the Casimir energy, it is possible to perform the integrations

$$\Downarrow$$

$$E_{\rm DE}(\rho_{\rm M}, a, R)$$

Expanding in powers of a/R

$$E_{\rm vac}^{(0)} \simeq -\frac{\pi^3}{1440}\frac{R}{a^2}\left[1-\frac{a}{R}\right]$$

$$E_{\rm vac}^{(2)} \simeq -\frac{\pi^3}{1080\,a}$$

$$E_{\rm DE} \simeq -\frac{\pi^3}{1440}\frac{R}{a^2}\left(1+\frac{1}{3}\frac{a}{R}\right)$$

•The result does not depend on ρ_M

•It is in agreement with the asymptotic expansion obtained from the exact formula (Gies et al ‘06; Emig ‘08; P. Maia Neto et al ‘08; Bulgac et al ‘06; Bordag et al ‘08, ‘10)

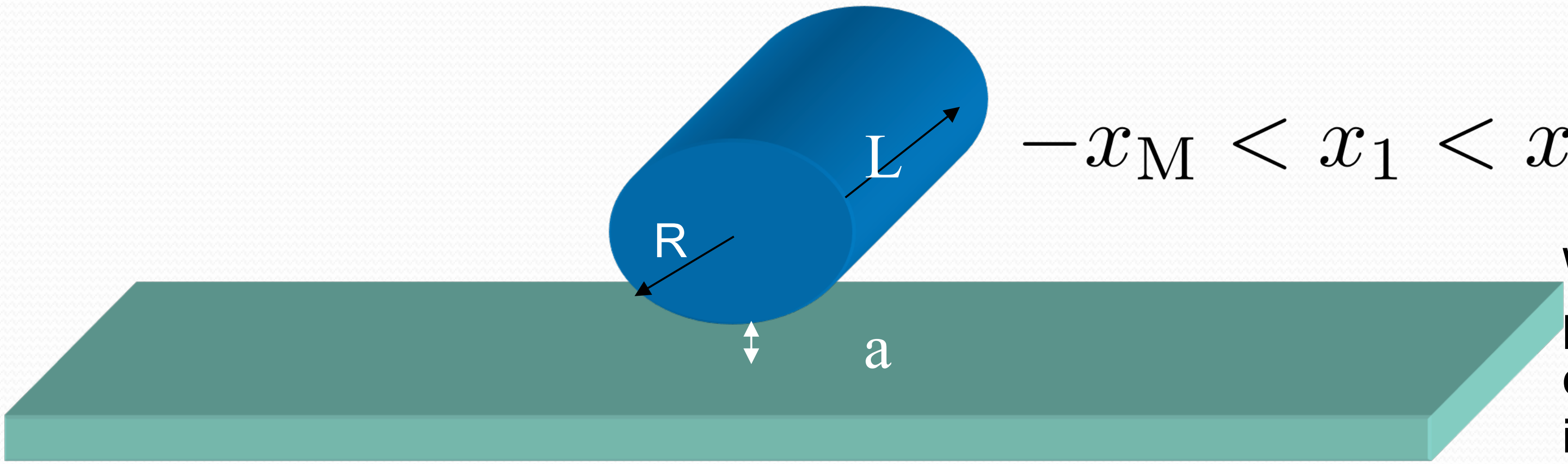
•The zeroth-order term includes part of the next to leading order correction

•It is correct to keep the sencond order term in the zeroth-order term only when the contribution coming from the order 2 is also taken into account

A cylinder in front of a plane

$$\psi(x_1) = a + R\left(1 - \sqrt{1 - \frac{x_1^2}{R^2}}\right)$$

$$-x_{\mathrm{M}} < x_1 < x_{\mathrm{M}} < R$$



We cover the part of the cylinder which is closer to the plane

A cylinder in front of a plane

After a similar calculation using the derivative expansion

$$E_{\mathrm{DE}} \simeq -\frac{\pi^3 L}{1920\sqrt{2}} \frac{R^{1/2}}{a^{5/2}} \left(1 + \frac{7}{36} \frac{a}{R}\right)$$

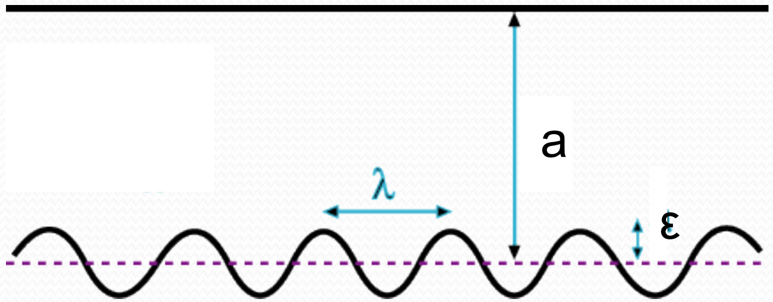
The result does not depends on the value of x_{M}

It is in agreement with the asymptotic expansion obtained from the exact formula for the cylinder-plane geometry and numerical findings (Dalvit et al ‘04; Emig et al ‘06; Bordag ‘06; Gies et al ’06; Rahi et al ’08; F.L, Mazzitelli & Villar ‘08)

More examples

A corrugated surface in front of a plane

$$\psi(x_1) = a + \epsilon \sin\left(\frac{2\pi x_1}{\lambda}\right)$$



Using the formula

$$E_{\mathrm{DE}} = -\frac{\pi^2}{1440}\left[\int d^2\mathbf{x}_\parallel \frac{1}{(a+\epsilon\sin\frac{2\pi x_1}{\lambda})^3} \times \left(1+\frac{2}{3}\left(\frac{2\pi}{\lambda}\right)^2 \epsilon^2\cos^2\frac{2\pi x_1}{\lambda}\right)\right]$$

Result for the corrugated surface in front of a plane

$$E_{\mathrm{DE}} \simeq -\frac{\pi^2 L^2}{1440 a^3}\left[1 + 3\left(\frac{\epsilon}{a}\right)^2 + \frac{4\pi^2}{3}\left(\frac{\epsilon}{\lambda}\right)^2\right]$$

This expression coincides with the small a/λ expansion of the result obtained by Emig, Hanke, Golestanian, and Kardar (2003)

Emig et al:

$$\frac{E_{\rm vac}}{L^2} = -\frac{\pi^2}{1440 a^3} - \frac{\epsilon^2}{a^5} G_{\rm TM}\left(\frac{a}{\lambda}\right)$$

For small argument

$$G_{\rm TM}(x) \simeq \frac{\pi^2}{480} + \frac{\pi^4 x^2}{1080}$$

Parabolic cylinder in front of a plane:

The surface is defined by $$\psi(x_1) = a + \frac{x_1^2}{R}$$

Expanding the result in powers of a/R we obtain

$$E_{\mathrm{DE}} \simeq -\frac{\pi^3 L}{3840} \frac{R^{1/2}}{a^{5/2}} \left(1 + \frac{8}{9} \frac{a}{R}\right)$$

The dependence of the energy on a and R is similar to that of the cylinder in front of a plane

Paraboloid in front of a plane:

The surface

$$\psi(\rho) = a + \frac{\rho^2}{R}$$

The approximation for the vacuum energy reads

$$E_{\mathrm{DE}} \simeq -\frac{\pi^3}{2880}\frac{R}{a^2}\left(1 + \frac{8}{3}\frac{a}{R}\right)$$

The dependence on a and R is similar to that of the sphere in front of a plane

Conclusions

- We have shown that the PFA can be thought as a derivative expansion of the Casimir energy with respect to the shape of the surfaces

- Main result

$$E_{\rm DE} \equiv E^{(0)}_{\rm vac} + E^{(2)}_{\rm vac} = -\frac{\pi^2}{1440}\int d^2\mathbf{x}_\parallel \frac{1}{\psi^3}\left[1 + \frac{2}{3}(\partial_\alpha\psi)^2\right]$$

shows that the lowest order reproduces the PFA. Moreover, when the first non trivial correction containing two derivatives of ψ is also included, the general formula gives the NTLO correction to PFA for a general surface

- For the surfaces considered, the PFA becomes a well defined and controlled approximation scheme: it is valid when $|\partial\alpha\psi| \ll 1$ or, more generally, when the scale of variation of the shape of the surface is much larger than the local distance between surfaces

- Corrections to PFA only contains local information about the geometry of the surface, and does not include correlations between different points of the surface

- For a scalar field satisfying Neumann or Robin boundary conditions, and also to the electromagnetic field satisfying perfect conductor boundary conditions on the surfaces, we expect the derivative expansion to be of the form

$$E_{\rm DE} = -\frac{\pi^2}{1440}\int d^2\mathbf{x}_\parallel \, \frac{1}{\psi^3}\left[\beta_1 + \beta_2(\partial_\alpha\psi)^2\right]$$