A system of Schrödinger equations modeling two trapped ions. 
Some controllability results and open problems.

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Trapped ions or Qubits

- The goal is to create quantum logic gates like the phase gate or the C-Not gate. See S.Haroche lectures at College de France on Quantum Information Theory (available on the web) and experiments by the group S.Haroche, J.M.Raimond and collaborators at ENS Paris.
- Experiments are based on trapped ions (qubits) with the case of one single trapped ion (one qubit problem) or two coupled trapped ions (two qubits problem).
Trapped ions or Qubits

- Each ion is a two level system, trapped in an electromagnetic cavity, all ions are stabilized by the same spatial oscillations, here a harmonic oscillator with vibration quantum $\omega$ (phonon).
- The system is submitted to a superposition of electromagnetic waves of complex amplitude $u_1$ and $u_2$. The phases depend on the spatial coordinate in order to be able to conserve the impulse: when an ion absorbs a photon, its energy changes and its impulse captures the photon impulse and excites the (quantized) vibration modes (phonon) inside the trap.
Mathematical model

- Two ions.
- Each ion is a two level system.
- Coupled to the same quantized harmonic oscillator

\[ A = \frac{1}{2}(-\partial_{xx}^2 + x^2) \]

with vibration quantum \( \omega \).

We have

\[ A = a^\dagger a + \frac{1}{2} = aa^\dagger - \frac{1}{2} \]

where

\[ a = \frac{1}{\sqrt{2}}(x + \partial_x) \]

is the annihilation operator and

\[ a^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x) \]

is the creation operator.
Mathematical model

- Controls: two electromagnetic waves of complex amplitude $u_1$ and $u_2$ and phases depending on spatial coordinate:

$$u_j(t)e^{i(\Omega_j t - k_j x)}, \quad j = 1, 2,$$

- State of the system: 4-d vector-wave function

$$|\psi\rangle = \psi^t (\psi_{gg}, \psi_{ge}, \psi_{eg}, \psi_{ee})$$

- Dynamics of the system described by the Hamiltonian $H$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle,$$

where

$$\frac{H}{\hbar} = \omega A + \frac{\Omega}{2}(\sigma_{1,z} + \sigma_{2,z}) + \left(u_1 e^{i(\Omega_1 t - k_1 x)} + u_1^* e^{-i(\Omega_1 t - k_1 x)}\right)\sigma_{1,x}$$

$$+ \left(u_2 e^{i(\Omega_2 t - k_2 x)} + u_2^* e^{-i(\Omega_2 t - k_2 x)}\right)\sigma_{2,x}.$$

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### Mathematical model

Pauli matrices:

\[
\sigma_{1,z} = (|e><e| - |g><g|)_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\sigma_{2,z} = (|e><e| - |g><g|)_2 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\sigma_{1,x} = (|g><e| + |e><g|)_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
\sigma_{2,x} = (|g><e| + |e><g|)_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

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Schrödinger system

\[ i \frac{\partial \psi}{\partial t} = \omega A \psi + \frac{\Omega}{2} \sigma_{1,z} \psi + \frac{\Omega}{2} \sigma_{2,z} \psi \]
\[ + (u_1 e^{i(\Omega_1^L t - k_1 x)} + u_1^* e^{-i(\Omega_1^L t - k_1 x)}) \sigma_{1,x} \psi \]
\[ + (u_2 e^{i(\Omega_2^L t - k_2 x)} + u_2^* e^{-i(\Omega_2^L t - k_2 x)}) \sigma_{2,x} \psi, \]
\[ \psi(0) = \psi^0. \]

Question: Given an initial configuration \( \psi^0 \) and a final configuration \( \psi^1 \), can we find control amplitudes \( u_1 \) and \( u_2 \) in order to drive the system at time \( T \) “close” to \( \psi^1 \)?

Parameters:

\( \omega \) large and \( \Omega \) very large,

\[ |\Omega_1^L - \Omega| << \Omega, \quad |\Omega_2^L - \Omega| << \Omega, \quad \omega << \Omega, \]

\[ |u_1| << \Omega, \quad |u_2| << \Omega, \quad \left| \frac{du_1}{dt} \right| << \Omega, \quad \left| \frac{du_2}{dt} \right| << \Omega. \]
Laser frame

Set

\[ \psi = e^{-i \frac{\Omega_1}{2} t \sigma_1} e^{-i \frac{\Omega_2}{2} t \sigma_2} \varphi \]

or

\[ \varphi = e^{i \frac{\Omega_2}{2} t \sigma_2} e^{-i \frac{\Omega_1}{2} t \sigma_1} \psi. \]

And

\[ \Delta_1 = \frac{\Omega - \Omega_1^L}{2}, \quad \Delta_2 = \frac{\Omega - \Omega_2^L}{2}, \]

\[ k_1 x = \eta_1 (a + a^\dagger), \quad k_2 x = \eta_2 (a + a^\dagger), \]

where \( \eta_j, j = 1, 2 \) are the Lamb-Dicke parameters with

\[ \eta_j \ll 1. \]
Interaction frame

\[ A = a^\dagger a + \frac{1}{2}, \]
\[ S(t) = e^{-i\omega tA}.e^{-i\Delta_1 t\sigma_{1,z}}.e^{-i\Delta_2 t\sigma_{2,z}} \]

\( (A, \sigma_{1,z} \text{ and } \sigma_{2,z} \text{ commute.}) \)

\[ \xi(t) = S(-t)\varphi(t) \quad , \quad \varphi(t) = S(t)\xi(t). \]

\[ i \frac{\partial \xi}{\partial t} = S(-t) \left( u_1 e^{2i\Omega_1 t - i\eta_1(a+a^\dagger)} + u_1^* e^{i\eta_1(a+a^\dagger)} \right) (|e><g|)_1 S(t)\xi \]
\[ + S(-t) \left( u_1 e^{-i\eta_1(a+a^\dagger)} + u_1^* e^{-2i\Omega_1 t - i\eta_1(a+a^\dagger)} \right) (|g><e|)_1 S(t)\xi \]
\[ + S(-t) \left( u_2 e^{2i\Omega_2 t - i\eta_2(a+a^\dagger)} + u_2^* e^{i\eta_2(a+a^\dagger)} \right) (|e><g|)_2 S(t)\xi \]
\[ + S(-t) \left( u_2 e^{-i\eta_2(a+a^\dagger)} + u_2^* e^{-2i\Omega_2 t - i\eta_2(a+a^\dagger)} \right) (|g><e|)_2 S(t)\xi \]
Lamb-Dicke approximation

\[ |\eta_1|, |\eta_2| << 1. \]

\[ e^{i\eta_j(a + a^\dagger)} \sim \left( I_d + i\eta_j(a + a^\dagger) \right), \quad e^{-i\eta_j(a + a^\dagger)} \sim \left( I_d - i\eta_j(a + a^\dagger) \right). \]

We then have (for example)

\[ e^{i\omega t A} (e^{i\eta_1(a + a^\dagger)}) e^{-i\omega t A} \sim I_d + i\eta_1 (ae^{-i\omega t} + a^\dagger e^{i\omega t}). \]

We obtain

\[ i \frac{\partial \xi}{\partial t} = \left( u_1 e^{2i\Omega_1 t} \left( I_d - i\eta_1 (ae^{-i\omega t} + a^\dagger e^{i\omega t}) \right) \right. \]
\[ + u_1^* \left( I_d + i\eta_1 (ae^{-i\omega t} + a^\dagger e^{i\omega t}) \right) e^{2i\Delta_1 t} (|e > g|) \right) \xi \]
\[ + \left( u_1 (I_d - i\eta_1 (ae^{-i\omega t} + a^\dagger e^{i\omega t}) \right) \]
\[ + u_1^* e^{-2i\Omega_1 t} \left( I_d + i\eta_1 (ae^{-i\omega t} + a^\dagger e^{i\omega t}) \right) e^{-2i\Delta_1 t} (|g > e|) \xi \]
\[ + \ldots \]
First of all we take each control $u_j$ to be a superposition of 3 monochromatic waves, two of them having a pulsation shifted by $\pm$ a vibration quantum $\omega$. In fact we take

$$u_1(t)e^{-2i\Delta_1 t} = v_0(t) + \tilde{v}_r(t)e^{-i\omega t} + \tilde{v}_b(t)e^{i\omega t}$$

$$u_2(t)e^{-2i\Delta_2 t} = w_0(t) + \tilde{w}_r(t)e^{-i\omega t} + \tilde{w}_b(t)e^{i\omega t}.$$ 

Then, using the averaging approximation, we can show that we can neglect the rapidly oscillating terms as $\omega$, $\Omega_1^L$, $\Omega_2^L$ and $\Omega$ are very large.
Similar to Law-Eberly equations in the case of one qubit.

\[
i \frac{\partial y}{\partial t} = (v_0 - i\eta_1 \tilde{v}_r a^\dagger - i\eta_1 \tilde{v}_b a)(|g><e|)_1 y
\]
\[
+ (v_0^* + i\eta_1 \tilde{v}_r^* a + i\eta_1 \tilde{v}_b^* a^\dagger)(|e><g|)_1 y
\]
\[
+ (w_0 - i\eta_2 \tilde{w}_r a^\dagger - i\eta_2 \tilde{w}_b a)(|g><e|)_2 y
\]
\[
+ (w_0^* + i\eta_2 \tilde{w}_r^* a + i\eta_2 \tilde{w}_b^* a^\dagger)(|e><g|)_2 y.
\]

Writing

\[v_r = -i\eta_1 \tilde{v}_r, \quad v_b = -i\eta_1 \tilde{v}_b,\]

\[w_r = -i\eta_1 \tilde{w}_r, \quad w_b = -i\eta_1 \tilde{w}_b,\]

and

\[y =^t (y_{gg}, y_{ge}, y_{eg}, y_{ee}),\]

we obtain
Approximate model

\[ i \frac{\partial y_{gg}}{\partial t} = (v_0 + v_r a^\dagger + v_b a)y_{eg} + (w_0 + w_r a^\dagger + w_b a)y_{ge} \]

\[ i \frac{\partial y_{ge}}{\partial t} = (v_0 + v_r a^\dagger + v_b a)y_{ee} + (w_0^* + w_r^* a + w_b^* a^\dagger)y_{gg} \]

\[ i \frac{\partial y_{eg}}{\partial t} = (v_0^* + v_r^* a + v_b^* a^\dagger)y_{gg} + (w_0 + w_r a^\dagger + w_b a)y_{ee} \]

\[ i \frac{\partial y_{ee}}{\partial t} = (v_0^* + v_r^* a + v_b^* a^\dagger)y_{ge} + (w_0^* + w_r^* a + w_b^* a^\dagger)y_{eg} \]

\[ y(0) = y^0. \]
Strategy

- Find a control (exact if possible) for the approximate system which drives an initial configuration to a desired one in time $T$. We would like to have only one of the controls ($v_0, v_r, v_b$ or $w_0, w_r, w_b$) being active at each time these controls being piecewise constant (not mandatory...).

- Take this control in the original system. This will provide an approximate control for the real system in time $T$. This can be proved due to approximation properties for the Lamb-Dicke and the averaging approximations mentioned above.

- Approximate control is relevant here because when we switch off control we keep close to the target (property of Schrödinger system).

- Both the original and the approximate systems are reversible and preserve the $(L^2)^4$-norm.
Control of the approximate system

It remains to study the control properties for the approximate system. Here we have only partial results at the moment and a (strong) conjecture for obtaining the global result.

We use the spectral decomposition of operator $A$. Its eigenfunctions $\phi_n$ are the Hermite functions associated with eigenvalues $n + \frac{1}{2}$ that, for convenience, we may write $\phi_n = |n \rangle$. We then have

$$A|n \rangle = \left(n + \frac{1}{2}\right)|n \rangle,$$

and

$$a|0 \rangle = |0 \rangle, \ a|n + 1 \rangle = \sqrt{n + 1}|n \rangle, \ a^\dagger|n \rangle = \sqrt{n + 1}|n + 1 \rangle.$$
Control of the approximate system

For instance, if we write $|gg, n\rangle = \langle n |y_{gg}\rangle$ and similar notations, and if only $v_r$ is active, $|gg, n\rangle$ and $|eg, n - 1\rangle$ form an independent system which solves

$$i\partial_t |gg, n\rangle = v_r \sqrt{n} |eg, n - 1\rangle, \quad i\partial_t |eg, n - 1\rangle = v_r^* \sqrt{n} |eg, n\rangle.$$

Of course, similar computations can also be done when the other controls are active.

We can represent these decompositions and their dynamics as follows:
Control of the approximate system

\begin{align*}
\nu_0 \left\{ \begin{array}{ll}
|gg, n\rangle \leftrightarrow_{|v_0|} |eg, n\rangle, \\
|ge, n\rangle \leftrightarrow_{|v_0|} |ee, n\rangle,
\end{array} \right.
\end{align*}

\begin{align*}
\nu_r \left\{ \begin{array}{ll}
|gg, n+1\rangle \leftrightarrow_{\sqrt{n+1}|v_r|} |eg, n\rangle, \\
|ge, n+1\rangle \leftrightarrow_{\sqrt{n+1}|v_r|} |ee, n\rangle,
\end{array} \right.
\end{align*}

\begin{align*}
\nu_b \left\{ \begin{array}{ll}
|gg, n\rangle \leftrightarrow_{\sqrt{n+1}|v_b|} |eg, n+1\rangle, \\
|ge, n\rangle \leftrightarrow_{\sqrt{n+1}|v_b|} |ee, n+1\rangle,
\end{array} \right.
\end{align*}

\begin{align*}
\omega_0 \left\{ \begin{array}{ll}
|gg, n\rangle \leftrightarrow_{|w_0|} |ge, n\rangle, \\
|eg, n\rangle \leftrightarrow_{|w_0|} |ee, n\rangle,
\end{array} \right.
\end{align*}

\begin{align*}
\omega_r \left\{ \begin{array}{ll}
|gg, n+1\rangle \leftrightarrow_{\sqrt{n+1}|w_r|} |ge, n\rangle, \\
|eg, n+1\rangle \leftrightarrow_{\sqrt{n+1}|w_r|} |ee, n\rangle,
\end{array} \right.
\end{align*}

\begin{align*}
\omega_b \left\{ \begin{array}{ll}
|gg, n\rangle \leftrightarrow_{\sqrt{n+1}|w_b|} |ge, n+1\rangle, \\
|eg, n\rangle \leftrightarrow_{\sqrt{n+1}|w_b|} |ee, n+1\rangle.
\end{array} \right.
\end{align*}
One can go from any pure state $|ee, n>$ to any pure state $|gg, m>$. Let us take the case $m < n$.

\[
|ee, n> \xrightarrow{\sqrt{n}|v_b|} |ge, n-1> \xrightarrow{|v_0|} |ee, n-1> \cdots |ee, m+1>
\]

\[
|ee, m+1> \xrightarrow{\sqrt{m+1}|v_b|} |ge, m> \xrightarrow{|w_0|} |gg, m>.
\]
To go from $|gg, 0\rangle$ to $(|gg, 0\rangle + |ee, 0\rangle)/\sqrt{2}$, we use 4 steps: $v_b, w_0, w_b, w_0$:

$$|gg, 0\rangle \xrightarrow{v_b} \frac{1}{\sqrt{2}}(|gg, 0\rangle + |eg, 1\rangle) \xrightarrow{w_0} \frac{1}{\sqrt{2}}(|ge, 0\rangle + |ee, 1\rangle)$$

$$\xrightarrow{w_b} \frac{1}{\sqrt{2}}(|gg, 0\rangle - |eg, 0\rangle) \xrightarrow{w_0} \frac{1}{\sqrt{2}}(|gg, 0\rangle + |ee, 0\rangle).$$
To go from $a_0|gg, 0 > + b_0|ge, 0 > + c_0|eg, 0 > + d_0|ee, 0 >$ with $|a_0|^2 + |b_0|^2 + |c_0|^2 + |d_0|^2 = 1$ to $|gg, 0 >$.

- Turn on $w_0$ to kill term in $|ee, 0 >$.
- Turn on $v_r$ during $t_1$ with $|v_r|t_1 = \frac{\pi}{2}$ to obtain $a_1|gg, 0 > + b_1|ge, 0 > + c_1|gg, 1 >$.
- Turn on $w_r$ during time $t_2$ to kill term in $|gg, 1 >$. We obtain $a_2|gg, 0 > + b_2|ge, 0 >$.
- Turn on $w_0$ to obtain $|gg, 0 >$. 

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 invariant spaces

Let us introduce the spaces

\[ X_n^0 = \text{Span} \{ |gg, n\rangle, |ge, n\rangle, |eg, n\rangle, |ee, n\rangle \} / \mathbb{C}, \ n \in \mathbb{N}, \]
\[ X_{n+1}^b = \text{Span} \{ |gg, n\rangle, |ge, n+1\rangle, |eg, n+1\rangle, |ee, n+2\rangle \} / \mathbb{C}, \ n \in \mathbb{N}, \]
\[ X_{n+1}^r = \text{Span} \{ |ee, n\rangle, |ge, n+1\rangle, |eg, n+1\rangle, |gg, n+2\rangle \} / \mathbb{C}, \ n \in \mathbb{N}, \]

and

\[ X_0^b = \text{Span} \{ |ge, 0\rangle, |eg, 0\rangle, |ee, 1\rangle \} / \mathbb{C}. \]
\[ X_0^r = \text{Span} \{ |ge, 0\rangle, |eg, 0\rangle, |gg, 1\rangle \} / \mathbb{C}. \]

- \( X_n^0 \) is invariant under the action of the controls \( v_0, w_0 \);
- \( X_n^b \) is invariant under the action of the controls \( v_b, w_b \);
- \( X_n^r \) is invariant under the action of the controls \( v_r, w_r \).
We also define the spaces

\[ Y_n^0 = \text{Span} \{ |gg, k\rangle, |ge, k\rangle, |eg, k\rangle, |ee, k\rangle, \, k \leq n \} / C, \]
\[ Y_n^r = \text{Span} \{ |ee, k\rangle, |ge, k+1\rangle, |eg, k+1\rangle, |gg, k+2\rangle, \, k+1 \leq n \} / C. \]

We have

\[ Y_n^0 = \bigcup_{k \leq n} X_0^k, \]
\[ Y_n^r = \bigcup_{k+1 \leq n} X_{k+1}^r. \]
It can be shown, as for the “easy examples”, that:

- $X_0^r$ is controllable with controls $v_r$ and $w_r$.
- $Y_0^r$ is controllable with controls $v_r$, $w_r$ and $w_0$.
- $Y_0^0$ is controllable with controls $v_r$, $w_r$ and $w_0$. 
How to obtain a general result?

In order to obtain a general result, it would be enough to show that we can drive $Y^r_n$ to $Y^r_{n-1}$ in a controlled time. We have

$$Y^r_n = Y^r_{n-1} \cup X^r_n,$$

and we know that both $Y^r_{n-1}$ and $X^r_n$ are invariant under the action of $v_r$ and $w_r$. Therefore we would like to use only the controls $v_r$ and $w_r$. Then we want to show that with these controls, any element of $X^r_n$ can be driven to $|ee, n>$ for example in a controlled time.

This question is still open at the moment. We are trying (without success until now) to give an explicit construction and this is a problem in $X^r_n$ only and therefore in finite dimension (4) !! .....