Stabilization for the semilinear wave equation with geometric control condition

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The damped nonlinear wave equation

\[
\begin{align*}
\square u = \partial_t^2 u - \Delta u &= -\gamma(x) \partial_t u + f(u) \\
(u(0), \partial_t u(0)) &= (u_0, u_1) \in X = H^1 \times L^2.
\end{align*}
\]

(1)

\$\Omega$ is a connected bounded open set with boundary in dimension 3 (for simplicity)

\$f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

\[f(0) = 0, \quad 0 \text{ is an equilibrium solution} \]

\[sf(s) \geq 0, \quad f \text{ is defocusing} \]

\[|f(s)| \leq C(1 + |s|)^p, \quad |f'(s)| \leq C(1 + |s|)^{p-1} \text{ with } 1 \leq p < 5 \quad f \text{ is subcritical}.\]
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\begin{cases}
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\[
E(t) = \frac{1}{2} \left( \int_\Omega |\partial_t u|^2 + \int_\Omega |\nabla u|^2 \right) + \int_\Omega V(u)
\]

where \(V(u) = \int_0^u f(s) ds\).

local theory by Strichartz estimates (Burq-Lebeau-Planchon 2006)
Bibliography

**Linear results** with Geometric Control Condition: Rauch-Taylor (75), Bardos-Lebeau-Rauch (92)

**Assumption (Geometric Control Condition)**

*There exists $T_0 > 0$ such that every ray of geometric optic travelling at speed 1 meets $\omega$ in a time $t < T_0$.*
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Nonlinear stabilization results: If $\omega$ is the exterior of a ball of $\mathbb{R}^d$:

- Dehman-Lebeau-Zuazua (03) (subcritical case, controllability in large time)
- Dehman-Gérard (02) (critical case on $\mathbb{R}^3$ using profile decomposition)
- Aloui-Ibrahim-Nakanishi (09) (any nonlinearity for weak solutions, uses Morawetz-type estimates)
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Other type of nonlinear result, controllability at **high frequency**:
Dehman-Lebeau (09): in subcritical case, (same time and geometrical assumption as linear case),
C.L. (10) in critical case with non-focusing assumptions
Stabilization theorem

Theorem (R.J., C.L.)

Let $R_0 > 0$, $\omega$ satisfying assumption Geometric Control Condition and $\gamma \in C^\infty(\Omega, \mathbb{R}^+)$ satisfying $\gamma(x) > \eta > 0$ for all $x \in \omega$. Assume moreover that $f$ satisfies the previous assumptions and is analytic. Then, there exist $C, \lambda > 0$ such that for any $(u_0, u_1)$ in $H^1 \times L^2$, with

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0;$$

the unique strong solution of (1) satisfies $E(u)(t) \leq Ce^{-\lambda t} E(u)(0)$ for $t \geq 0$. 
Idea of proof in the subcritical case (Dehman-Lebeau-Zuazua)

We have

\[ E(T) = E(0) - \int_0^T \int_\Omega \gamma(x) |\partial_t u|^2. \]

So to get exponential decay, we need to prove an observability estimate

\[ \int_0^T \int_\Omega \gamma(x) |\partial_t u|^2 \geq CE(0) \]

for solutions of the damped wave equation bounded in energy by \( R_0 \).
Idea of proof in the subcritical case (DLZ)

By contradiction: let $u_n$ be a bounded sequence of solutions with:

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\int_0^T \int_{\Omega} \gamma(x) |\partial_t u_n|^2 \leq \frac{1}{n} E(u_n)(0). \quad (2)
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Assume $E(u_n)(0) \to \alpha > 0$ (otherwise linear behavior: easier)

We prove $u_n \to 0$ in energy which is a contradiction.
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- Using propagation of regularity and then unique continuation, we get $u \equiv 0$. That is $u_n \rightharpoonup 0$. 
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Contradiction to $\alpha > 0$. 
Main assumptions

Their method of proof could allow to prove the stabilization in more general domains under the more general assumptions

- **Geometric Control Condition**
- **Unique Continuation** \( u \equiv 0 \) is the unique strong solution in the energy space of

\[
\begin{align*}
\square u + f(u) &= 0 \quad \text{on} \quad [0, T] \times \Omega \\
\partial_t u &= 0 \quad \text{on} \quad [0, T] \times \omega.
\end{align*}
\]
The problem of unique continuation

Classical technique: use Carleman estimate for $w = \partial_t u$ solution of

$$\begin{cases}
\Box w + Vw &= 0 \quad \text{on} \quad [0, T] \times \Omega \\
w &= 0 \quad \text{on} \quad [0, T] \times \omega.
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with $V = f'(u)$. 
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Problem: the geometric assumptions are not so natural, we often need to check them "by hands" on each geometric situation.
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Problem: the geometric assumptions are not so natural, we often need to check them "by hands" on each geometric situation.

There are some other available unique continuation results using partial analyticity.
Unique continuation under partial analyticity

Theorem (Zuily-Robbiano, a particular case)

Let $\nu$ be a solution on an open set $\mathcal{U}$ of

$$\Box \nu + d(x, t)\nu = 0$$

where $d$ is smooth, analytic in time.

Let $\varphi \in C^2(\mathcal{U}, \mathbb{R})$ such that $\varphi(x_0, t_0) = 0$ and $(\nabla \varphi, \partial_t \varphi)(x, t) \neq 0$ for all $(x, t) \in \mathcal{U}$.

Moreover, assume

- $\nu \equiv 0$ in $\{(x, t) \in \mathcal{U}, \varphi(x, t) \leq 0\}$.
- $\varphi$ not characteristic at $(x_0, t_0)$.

Then, $\nu \equiv 0$ in a neighbourhood of $(x_0, t_0)$. 
Unique continuation under partial analyticity

Theorem (Zuily-Robbiano, a particular case)

Let $v$ be a solution on an open set $\mathcal{U}$ of

$$\Box v + d(x, t)v = 0$$

where $d$ is smooth, analytic in time.

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Pbm : in our case, $V = f'(u)$ has no reason to be analytic in time...
A result of analyticity


Let $U(t)$ be a global solution of

$$\partial_t U(t) = AU(t) + F(U(t)) \quad \forall t \in \mathbb{R}.$$ 

We assume that

(i) $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$.
(ii) $\{U(t), \ t \in \mathbb{R}\}$ is contained in a compact set $K$ of $X$.
(iii) $F$ is a compact and lipschitzian map and is analytic in a neighbourhood of $K$.
(iv) there exist projectors $(P_n)$ converging to the identity and commuting with the unbounded part of $A$.

Then, the solution $U(t)$ is analytic from $t \in \mathbb{R}$ into $X$. 
Idea of the proof

**Goal:** prove that \( t \mapsto U(t) \) is \( C^1 \) with \( t \) in a complex strip \( \mathbb{R} + i(-\epsilon, \epsilon) \).

**Idea:** use the fixed point theorem for contracting maps as in the proof of Cauchy-Lipschitz theorem.

\[
U(t) \mapsto e^{A(t-t_0)} U_0 + \int_{t_0}^t e^{A_s} F(U(t-s)) \, ds .
\]
Proof of our main result

Let \((u_n)\) solutions with \(E(u_n(0)) \leq E_0\) and \(T_n \to +\infty\) such that

\[
\int_0^{T_n} \int_\Omega \gamma(x) |\partial_t u_n|^2 \, dt\,dx \leq \frac{1}{n} E(u_n(0)) \leq \frac{1}{n} E_0.
\]

Assume that \(E(u_n(0)) \to \alpha > 0\) and set \(u^*_n(\cdot) = u_n(\cdot + T_n/2)\).

It remains to:

- show that \((u^*_n)\) converges strongly to a global solution \(u^*\) which does not dissipate energy.
- apply previous theorem to show that \(u^*\) is analytic in time and smooth in space.
- use the unique continuation property of Robbiano and Zuily to show that \(u^*\) is constant in time and so \(u^* \equiv 0\).
Asymptotic compactness

\[ U_n^*(0) = e^{A T_n/2} U_n(0) + \int_0^{T_n/2} e^{A(T_n/2 - \tau)} \begin{pmatrix} 0 \\ f(u_n(\tau)) \end{pmatrix} d\tau \]

We use that:

- \( \|e^{At}\| \leq Ce^{-\lambda t} \)
- \( U_n(t) \) is bounded in \( H^1(\Omega) \times L^2(\Omega) \)
- for \( f(u) = o(|u|^3) \), \( f \) maps a bounded set of \( H^1(\Omega) \) into a compact set of \( L^2(\Omega) \).
Compactness for $f(u) = o(|u|^5)$

For $f(u) \sim |u|^p$ with $p < 5$, we use


Let $s \in [0, 1)$, $R > 0$ and $T > 0$.

There exist $\epsilon > 0$ and $(q, r)$ satisfying $\frac{1}{q} + \frac{3}{r} = \frac{1}{2}$, $q \in [7/2, +\infty]$ and $C > 0$ such that,

if $v \in L^\infty([0, T], H^{1+s}(\Omega))$ has a Strichartz $L^q([0, T], L^r(\Omega))$ norm bounded by $R$, then

$$\|f(v)\|_{L^1([0, T], H^{s+\epsilon}(\Omega))} \leq C\|v\|_{L^\infty([0, T], H^{1+s}(\Omega))}.$$  

(proof by using Meyer’s multipliers)
Regularity of $u^*$

\[ U^*(t) = \int_{-\infty}^{t} e^{A(t-\tau)} F(U^*(\tau)) \, d\tau. \]

We use several times the result of Dehman, Lebeau and Zuazua until $u^*$ belongs to $H^2(\Omega)$ and then the usual Sobolev imbeddings are sufficient for the bootstrap argument.

$\implies u^*$ is $C^\infty$

$\implies$ we can apply the analyticity result of Hale and Raugel ($F(U^*)$ well defined)
Conclusion of the proof

We know that $u^*$ is analytic in time, smooth in space and does not dissipate energy. Set $\nu = u_t^*$, we have

$$\nu \equiv 0 \text{ on the support of } \gamma$$

$$\nu_{tt} = \Delta \nu + f'(u(t))\nu.$$  

A global version of the unique continuation result of Robbiano and Zuily *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients* shows that $\nu \equiv 0$. 
Control/Dynamical point of view

- Stabilisation / existence of a compact global attractor
- Propagation of compactness / asymptotic compactness
- Propagation of space regularity / asymptotic smoothness (regularity of the trajectories of the attractor)
- ? ? ? / asymptotic analyticity (the solutions of the attractor are analytic in time if the nonlinearity is analytic)
- Unique continuation properties / gradient structure (equilibria are the only trajectories which do not dissipate the energy)
Further results

- The stabilisation also holds for **unbounded manifolds** with bounded $C^\infty$—geometry if $\gamma \geq \alpha > 0$ outside a bounded set.
- The stabilisation also holds for **almost all the nonlinearities** $f$, even non-analytic ones (generic result).
- We get **control of the wave equation** by using the stabilisation and a local control near 0.
- Same kind of technics can be used to show **existence of a compact global attractor** for a more complex nonlinearity $f(x, u)$. 
THANK YOU FOR YOUR ATTENTION!!!!