Benasque 13/7/10

Staircases with Boundaries

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Papers $\subset$ { hep-th/0404014 ; hep-th/0512337 ; 0911.4969 ; <to appear> }

... & see also Balázs' talk tomorrow

The main problem:

- 2d QFTs (without boundaries) have a very rich set of renormalisation group fixed points - the 2d Conformal Field Theories.

- The picture becomes even richer when we add a boundary - each bulk fixed point splits into many bulk-and-boundary fixed points - the 2d Boundary Conformal Field Theories.

To organise this zoo we'd like to have...

(a) Some simple, well-defined, quantities or 'probes' to (partially) characterise the different 2d CFT and BCFT;

(b) Off-critical versions of the probes from (a) which will indicate how the different fixed points are linked by renormalisation group flows;

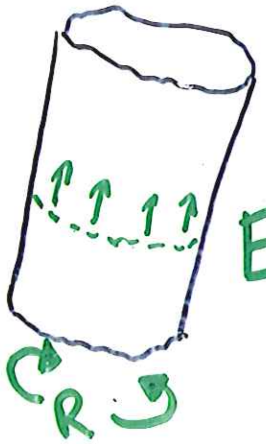
(c) Better still would be if the flows from (b) could be embedded in larger multiparameter families so they could all be treated simultaneously;

(d) Finally – and in line with the theme of this workshop – would like the families of flows from (c) to be integrable so they can be treated using <u>exact</u> equations.

Many of these ingredients are standard...

(a) : critical bulk & boundary –

- In the bulk, the first characteristic of a CFT is the central charge c, which via a mapping from the plane to a cylinder is related to the energy of the ground state on a circle of circumference R:



$$E_0(R) = -\frac{\pi}{6R}c$$

Here, focus on minimal models: [BPZ]

$$c<1 \ \& \ \text{unitary} \Rightarrow c = 1 - \frac{6}{m(m+1)} \qquad m=3,4\ldots$$

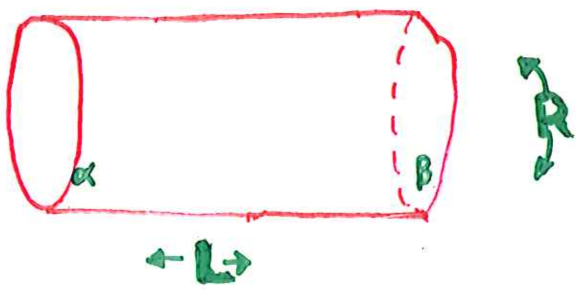
call this model $\mathcal{M}_m$

- At the boundary the analogue is the boundary entropy, or g-function.

g-functions in critical theories [Affleck & Ludwig 1991]

[also, in lattice context, Tsvelick 1985

- Put a CFT on a cylinder of length L and circumference R with conformal boundary conditions α and β at the ends:



- Ask about the cylinder partition function in the limit $L \to \infty$ to get some universal info about the system.

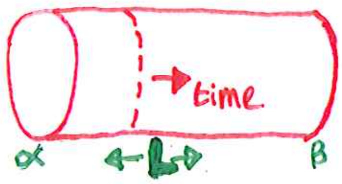
$$\log Z_{\alpha\beta} \sim \underbrace{\frac{\pi c}{6}\frac{L}{R}}_{\text{"bulk" piece (linear in } L)} + \underbrace{\text{constant}}_{\text{boundary bit}}$$

"bulk" piece (linear in L)

boundary bit - write it as $\log g_\alpha g_\beta$ and call g the "universal noninteger groundstate degeneracy", or g-function.

∃ 2 Hamiltonian Descriptions:

①



("closed string")

(boundary state)

$$Z_{\alpha\beta} = \langle \alpha | e^{-L H^{circ}(R)} | \beta \rangle$$

$$= \sum_n \langle \alpha | \psi_n \rangle e^{-L E_n^{circ}(R)} \langle \psi_n | \beta \rangle$$

where $H^{circ} |\psi_n\rangle = E_n |\psi_n\rangle$

$$\sim \langle \alpha | \psi_0 \rangle \langle \psi_0 | \beta \rangle e^{-L E_0^{circ}(R)}$$

Since $E_0(R) = -\frac{\pi c}{6R}$, taking logs

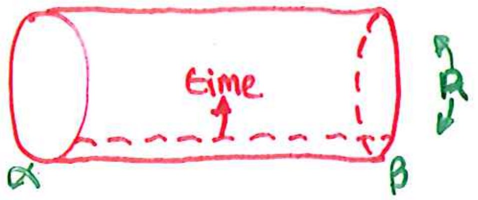
recovers $\log Z_{\alpha\beta} \sim \frac{\pi c}{6R} L + \log g_\alpha g_\beta$

with $g_\alpha = \langle \alpha | \psi_0 \rangle$

(or $\frac{\langle \alpha | \psi_0 \rangle}{\sqrt{\langle \psi_0 | \psi_0 \rangle}}$ if $|\psi_0\rangle$ isn't normalised)

... to evaluate g_α, it's best to flip for a moment to the other description...

(2)



("open string")

$$Z_{\alpha\beta} = \mathrm{tr}\left(e^{-RH_{\alpha\beta}(R)}\right)$$

$$= \sum_a N^a_{\alpha\beta}\, \chi_a\left(e^{-\pi R/L}\right)$$

Multiplicity of the rep a

character: $\chi_a(e^{-\frac{\pi R}{L}}) = \mathrm{tr}_a\, e^{-\frac{\pi}{L}\overbrace{(L_0 - \frac{c}{24})}^{\text{'}H_{\alpha\beta}\text{' in CFT-speak}}R}$

- To take the limit $L \to \infty$, first make a 'modular' transformation using $\chi_a(e^{-\pi\frac{R}{L}}) = \sum_b S^b_a\, \chi_b(e^{-4\pi\frac{L}{R}})$

  modular S matrix

  - As $L \to \infty$ only the gnd state, sitting inside the $b=0$ rep, matters – ie $\chi_b \to \delta_{b0}\,\chi_0(e^{-4\pi\frac{L}{R}}) \sim \delta_{b0}\, e^{\frac{\pi c}{6R}L}$

- Hence $$Z_{\alpha\beta} \sim e^{\frac{\pi c}{6R}L} \sum_a N^a_{\alpha\beta}\, S^0_a$$

... which matches the general form seen earlier & shows that $g_\alpha g_\beta = \sum_a N^a_{\alpha\beta} S^0_a$ — solve to find the g's ...

(the method works by secretly going back to the '(1)' channel, but at the level of the characters where everything is well under control)

For the minimal model $\mathcal{M}_m$ with central charge $c_m = 1 - \frac{6}{m(m+1)}$ there is a basic ("Cardy") conformal boundary condition for each primary field ϕ_{ab}, labelled by two integers a and b with $1 \leq a \leq m-1$, $1 \leq b \leq m$, and $(a,b) \sim (m-a, m+1-b)$. Its boundary entropy is

$$\boxed{g(m,a,b) = \left(\frac{8}{m(m+1)}\right)^{1/4} \frac{\sin\frac{a\pi}{m} \sin\frac{b\pi}{m+1}}{\sqrt{\sin\frac{\pi}{m} \sin\frac{\pi}{m+1}}}}$$

For superpositions of Cardy boundaries the boundary entropies add.

Note: up to a $\mathbb{Z}_2$ "spin flip" ambiguity $g(m,a,b) = g(m,m-a,b)$ and some sporadic coincidences, the g-function is powerful enough to pin down conformal boundary conditions uniquely.

... so it's a good probe of boundary r.g. flows ...

Next step:

(b): off-critical versions of c and g –

Prelude:

- An off-critical theory will generally exhibit a characteristic distance scale (correlation length) and on scales much shorter than this (ie, in its U.V. limit) its properties will be those of some conformal field theory. The off-critical theory is then a perturbation of that CFT by some relevant operator (or operators):

relevant bulk field in the spectrum of the UV CFT.

$$\mathcal{A}_{PCFT} = \mathcal{A}_{CFT} + \lambda \int \phi(x,\bar{x})\, d^2x$$

off-critical action

CFT action (eg that for $\mathcal{M}_m$)

bulk coupling

- At long distances (IR) $\exists$ two possibilities:
  - Some massless degrees of freedom remain, so the IR theory is another CFT & there is a flow $CFT_1 \to CFT_2$
  - Only massive degrees of freedom are left: $CFT \to$ massive scattering theory

- In the presence of boundaries the story is further complicated by the possibility to add a boundary perturbation:

$$\mathcal{A}_{PBCPT} = \mathcal{A}_{BCPT} + \lambda \int \phi(x,\bar{x})\, d^2x + \mu \int \phi(x)\, dx$$

bulk coupling

boundary coupling

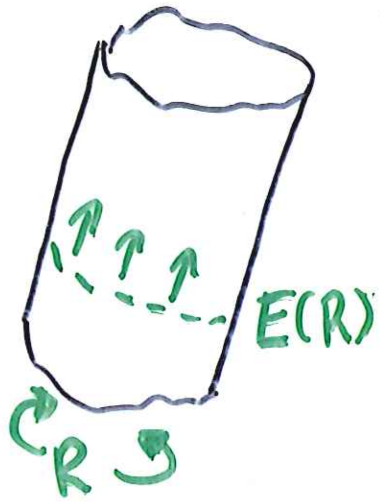
relevant boundary field in the spectrum of the UV CFT with the given conformal boundary condition

- For cases where the <u>bulk</u> flow is to another CFT, ie $CFT_1 \to CFT_2$, we therefore expect there to be mappings between their respective conformal boundary conditions: $\{CBC_s \text{ of } CFT_1\} \to \{CBC_s \text{ of } CFT_2\}$.

- But to unravel this structure is surprisingly hard, even in simple cases of minimal models $\mathcal{M}_m$. Most results have been for pure-bulk flows, or for boundary flows where the bulk remains critical (ie $\lambda = 0$).

<u>Claim</u> is that <u>exact</u>, <u>off-critical</u>, g-functions can help...

• For the bulk flow the story is well-known. [A1. 3]
The previous characterisation of the central charge in terms of the finite-size ground state energy generalises naturally:



"effective central charge"

$$E(R) = -\frac{\pi}{6R}\, c_{eff}(r)$$

$(r = MR)$

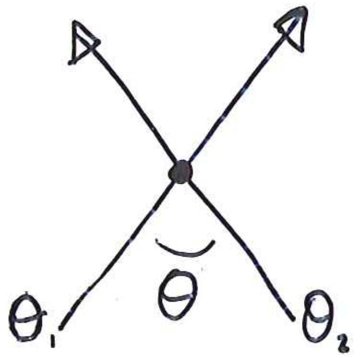
mass/crossover scale, related to the bulk coupling λ

• At fixed points of the RG, $c_{eff} = c$ (at least in unitary cases).

• Furthermore, in cases where the off-critical theory is integrable (the only cases I'll worry about here) $c_{eff}(r)$ can be found exactly using the Thermodynamic Bethe Ansatz, or TBA, which I'll treat as something of a black box for now. Special interest in the flows $\mathcal{M}_m \to \mathcal{M}_{m-1}$, originally found, perturbatively, by Sasha Zamolodchikov for m large.

Simplest cases:

If the IR theory is not a CFT, so no massless degrees of freedom remain, then it will be an integrable theory of massive particles which scatter with some S-matrix:



$$= S(\theta_1 - \theta_2)$$

the rapidities of particles 1 & 2: $(p, \bar{p}) = (Me^{\theta}, Me^{-\theta}$

To find $c_{eff}(r)$ $(r = MR)$, first solve

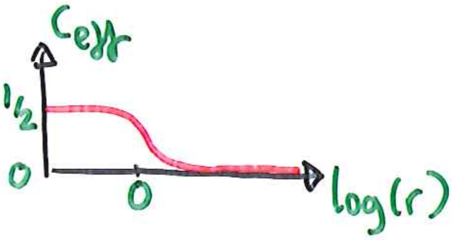
$$\boxed{\varepsilon(\theta) = r\cosh\theta - \int_{\mathbb{R}} \phi(\theta - \theta') L(\theta') \, d\theta'}$$

for $\varepsilon(\theta)$ (the pseudoenergy) where $\phi(\theta) = -\frac{i}{2\pi} \frac{d}{d\theta} \log S$
$L(\theta) = \log(1 + e^{-\varepsilon(\theta)}$

Then
$$\boxed{c_{eff}(r) = \frac{3}{\pi^2} \int_{\mathbb{R}} r\cosh\theta \, L(\theta) \, d\theta}$$

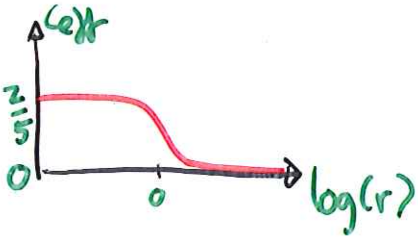
Examples:

$$S(\theta) = -1$$



(Ising)

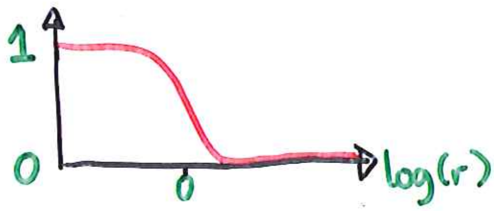
$$S(\theta) = \frac{\sinh(\theta) + i\sin(\pi/3)}{\sinh(\theta) - i\sin(\pi/3)}$$



(Lee-Yang)

$$S(\theta) = \frac{\sinh(\theta) - i\sin(\gamma)}{\sinh(\theta) + i\sin(\gamma)}, \quad \gamma \in [0, \pi]$$

(sinh-Gordon)



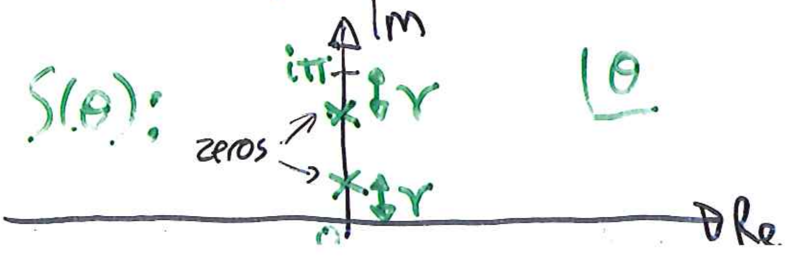
The sinh-Gordon case also has a simple Lagrangian formulation

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - 2\mu\cosh(\beta\phi) \qquad \gamma = \frac{\beta^2/8}{(1+\beta^2/8\pi)}$$

(c.f. Sine-Gordon)

$$M \propto (\mu)^{1/(1+\beta^2/8\pi)}$$

$S(\theta)$:

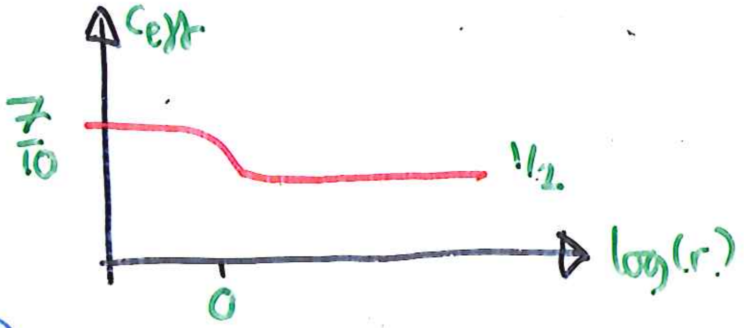


← (Analytic structure of the S-matrix)

The Staircase

- Examples so far are <u>massive</u> field theories with finite correlation length and $c_{eff}(\infty) = 0$ - not good for treating the $CFT_1 \to CFT_2$ flows we had before.

- Subsequently more complicated sets of TBA equations have been found which <u>do</u> describe these situations, eg the $\mathcal{M}_m \to \mathcal{M}_{m-1}$ flows. For $m=4$, this is the Tricritical Ising ($c = 7/10$) $\to$ Ising ($c = 1/2$) flow, & the TBA eqn involves <u>two</u> pseudoenergy functions ε_1 & ε_2; in general you'll need $m-2$ of them.



These eqns are exact & non<u>perturbative</u>, & confirm large-m picture.

However they become quite complicated...

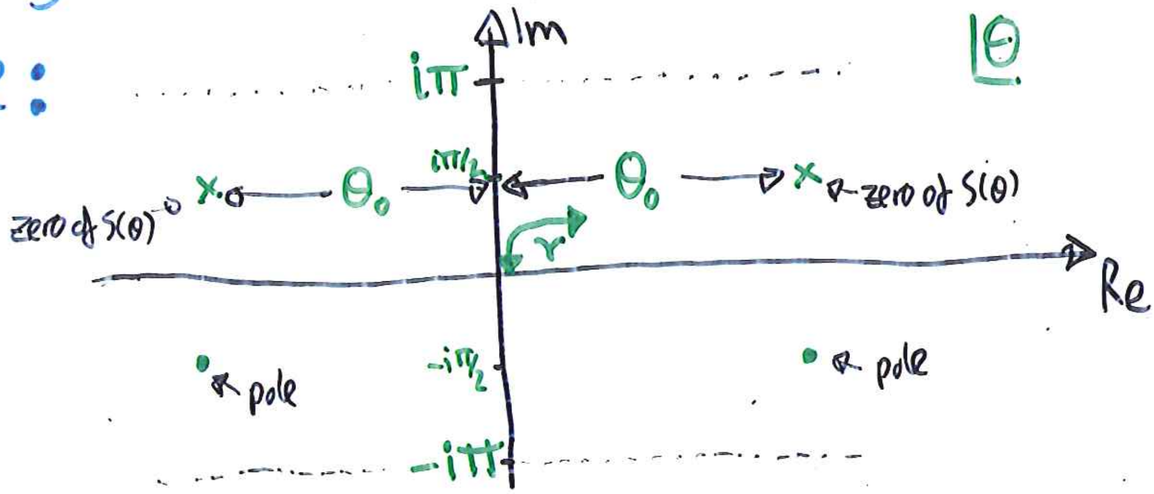
- But later Al. Zamolodchikov found an alternative - the staircase model.

Idea: look at the Sinh-Gordon S-matrix at complex values of the coupling γ, in particular

$$\gamma = \frac{\pi}{2} \pm i\theta_0$$

for θ_0 real and (eventually) large

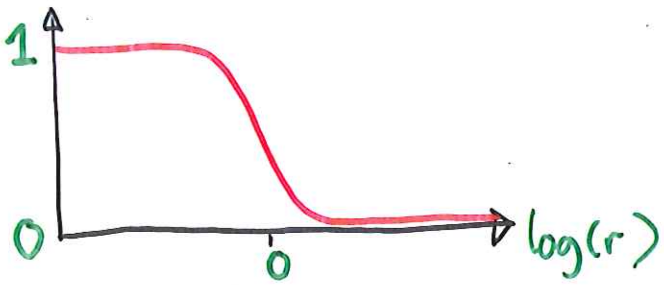
- Purely as an S-matrix this still makes sense:



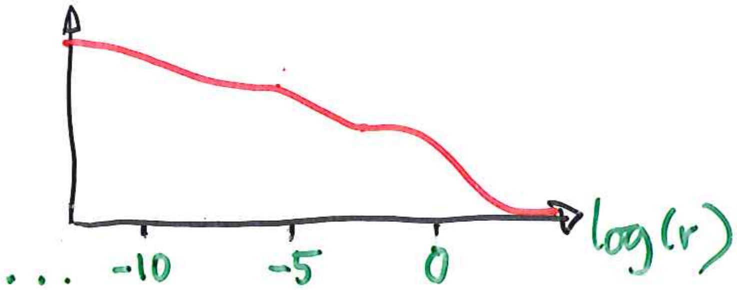
(the poles look like "resonances" of unstable particles)

... but it has a dramatic effect on the behaviour of the effective central charge...

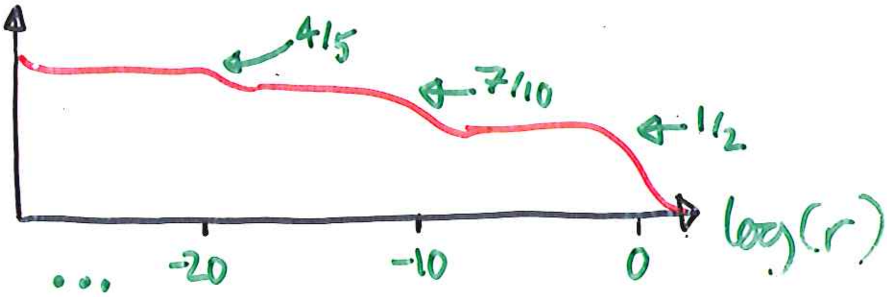
$\theta_0 = 0$:



$\theta_0 = 10$:



$\theta_0 = 20$:

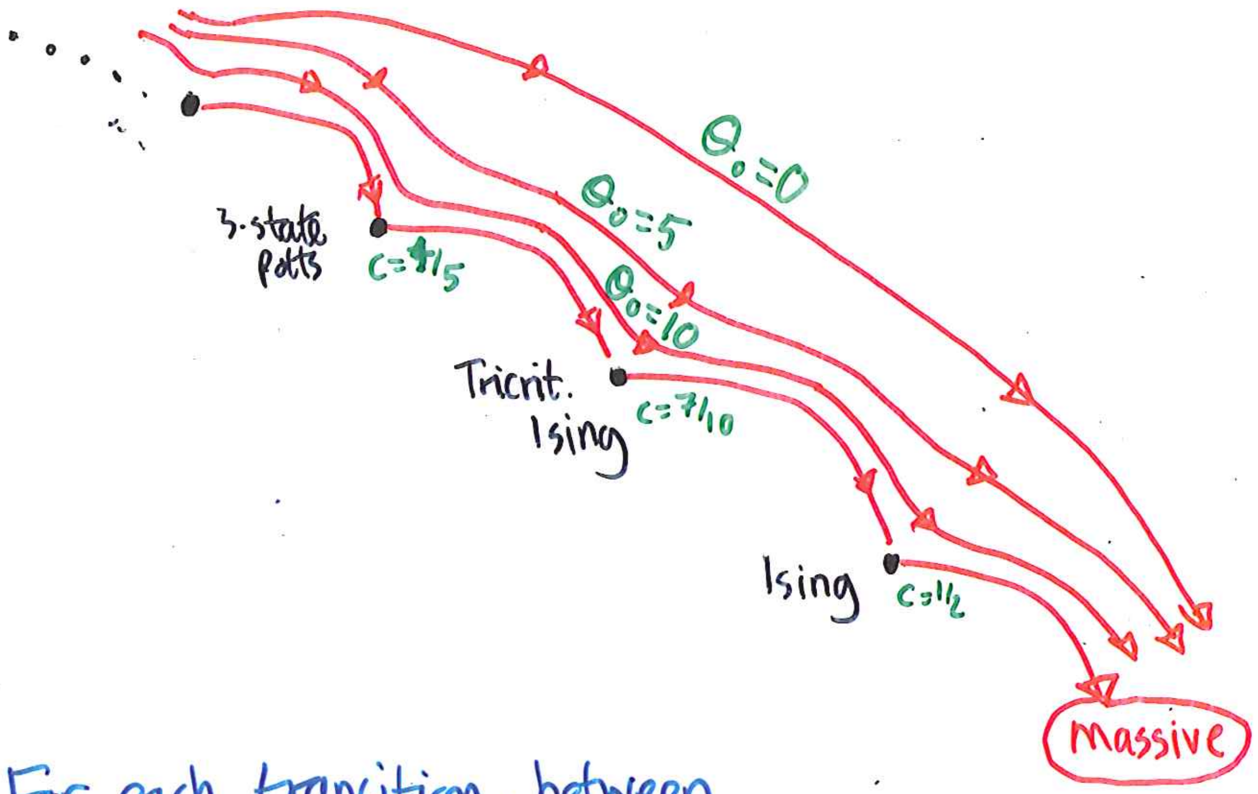


As $\theta_0 \to \infty$ see "plateaux" in the evolution of the effective central charge of width $\theta_0/2$, with steps at the central charges of all the $c<1$ unitary minimal models $\mathcal{M}_m$, ie

$$c_m = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4 \ldots$$

Conjectured interpretation:

$\exists$ a one-parameter family of integrable QFT whose RG trajectories (in the parameter $\to \infty$ limit) "visit" each $c<1$ minimal CFT in turn:



For each transition between plateaux, one can prove that the limiting flow of $c_{eff}(r)$ as $\theta_0 \to \infty$ exactly matches the previously-conjectured flows between minimal models.

Remarkable that the quantum sinh-Gordon model "knows" about all the unitary $c<1$ CFTs in such a simple way...

... back to the boundary:

- Given that the existence of bulk flows $\mathcal{M}_m \to \mathcal{M}_{m-1}$ between nearest-neighbour minimal models has been established both perturbatively ($m \gg 1$, Sasha Zamolodchikov) and by TBA (all m, Aliosha Zamolodchikov) it is natural to ask what happens to their boundary conditions during such flows.

- The boundary generalisation of Sasha Zamolodchikov's large m computations is delicate (g is a subleading effect) but was recently [0907.2560] acheived by Fredenhagen, Gaberdiel and Schmidt-Colinet (FGS). (Though even at large m FGS had to 'bolt on' some non-perturbative results for pure-boundary flows to get a full picture.)

- It would be more satisfactory to have <u>exact</u> equations, à la TBA, to describe the g-function flow and thereby pin down how the boundary conditions change.

- For purely massive flows with diagonal bulk scattering, a set of exact g-function equations had previously been proposed and tested [PED, DF, CR, RT 2004; PED, AL, CR, RT 2005] but it was not obvious how to generalise these equations to cover theories with massless IR limits.

- Instead we "cheat" by studying the staircase model, with a boundary. In the far IR this is <u>always</u> a massive scattering theory, so the exact g-function equations should work fine...

... but at intermediate scales, if we tune the parameter Θ_0 to large enough values, it can look as similar to one of the $\mathcal{M}_m \to \mathcal{M}_{m-1}$ "interpolating" theories as we like.

As a bonus, we will see whether the set of bulk-and-boundary flows between minimal models (or a least a large integrable subset of them) can be "unified" via the boundary staircase model, as happened for the bulk flows through Aliosha's original staircase.

One small mystery: the boundary staircase model naturally inherits <u>two</u> independent boundary parameters from its sinh-Gordon parent (or even more, if we add defects). But we might expect boundary flows between minimal models to have just <u>one</u> boundary parameter. Rather than worry about this, just try & see ...

Exact g-functions for the boundary staircase:

Return to the staircase model, and the general formula* for an exact g-function in a massive scattering theory, with S-matrix $S(\theta)$.

(* treat this as another "black box")

TBA eq: $$\varepsilon(\theta) = r\cosh\theta - \int_{\mathbb{R}} \phi(\theta-\theta')L(\theta')\,d\theta'$$

where $\phi(\theta) = \frac{-i}{2\pi}\frac{d}{d\theta}\log S(\theta)$

$L(\theta) = \log(1+e^{-\varepsilon(\theta)})$

g-function: $$\ln g(r) = \ln g_0(r) + \ln g_b(r)$$

where

$$\ln g_b(r) = \frac{1}{2}\int_{\mathbb{R}} \left(\phi_b(\theta) - \tfrac{1}{2}\delta(\theta) - \phi(2\theta)\right)L(\theta)\,d\theta$$

(boundary-condition dependent part)

where $\phi(\theta)$, $L(\theta)$ as above,

and $\phi_b(\theta) = \frac{-i}{2\pi}\frac{d}{d\theta}\log R(\theta)$

with $R(\theta)$ the reflection amplitude for the particle in the theory on the given boundary:

and $$\ln g_0(r) = \sum_{n=1}^{\infty}\frac{1}{2n}\int_{\mathbb{R}^n}\frac{d\theta_1}{(1+e^{\varepsilon(\theta_1)})}\frac{d\theta_2}{(1+e^{\varepsilon(\theta_2)})}\cdots\frac{d\theta_n}{(1+e^{\varepsilon(\theta_n)})}\phi(\theta_1+\theta_2)\phi(\theta_2-\theta_3)\ldots\phi(\theta_n-\theta_1)$$

(correction term for bulk flow)

For $S(\theta)$ we take the staircase S-matrix;

for $R(\theta)$ we borrow the reflection amplitude for the sinh-Gordon model, continued in an analogous fashion to the S-matrix.

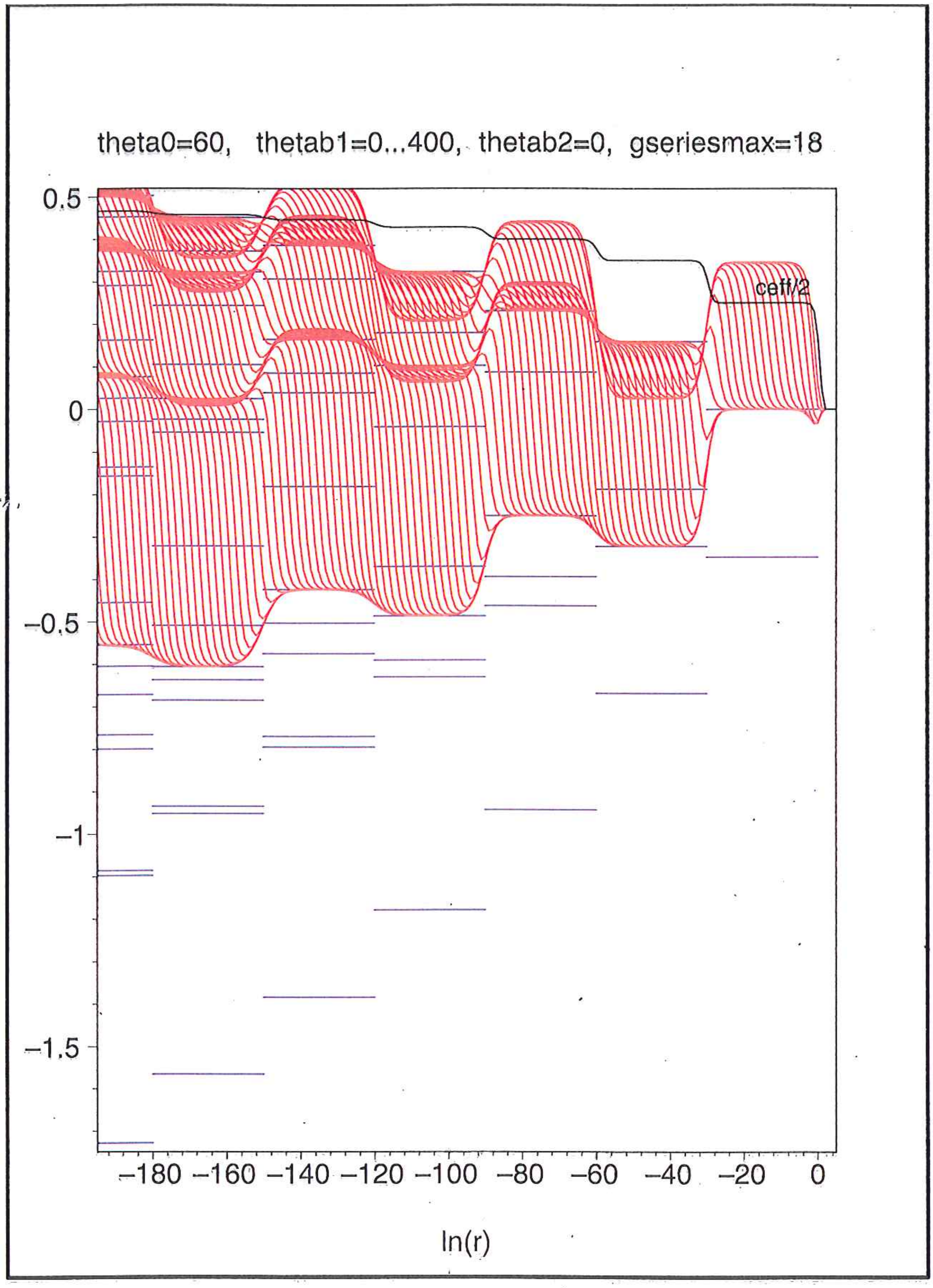
<u>Sinh-G reflection amplitudes:</u>

$\exists$ a two-parameter family, corresponding to a two-parameter family of integrable boundary conditions for the model.

$$R = \frac{(1)\left(\frac{3}{2}+i\frac{\theta_0}{\pi}\right)\left(\frac{3}{2}-\frac{i\theta_0}{\pi}\right)}{(1-E)(1+E)(1-F)(1+F)} \quad , \quad (a) = \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi a}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi a}{4}\right)}$$

E, F : the two boundary parameters, usually taken real. But to see interesting boundary transitions we'll set $E = \frac{i}{\pi}\theta_{b_1}$, $F = \frac{i}{\pi}\theta_{b_2}$ with θ_{b_1} and θ_{b_2} real, boundary analogues of θ_0.

Insert these ingredients into the general formula to find...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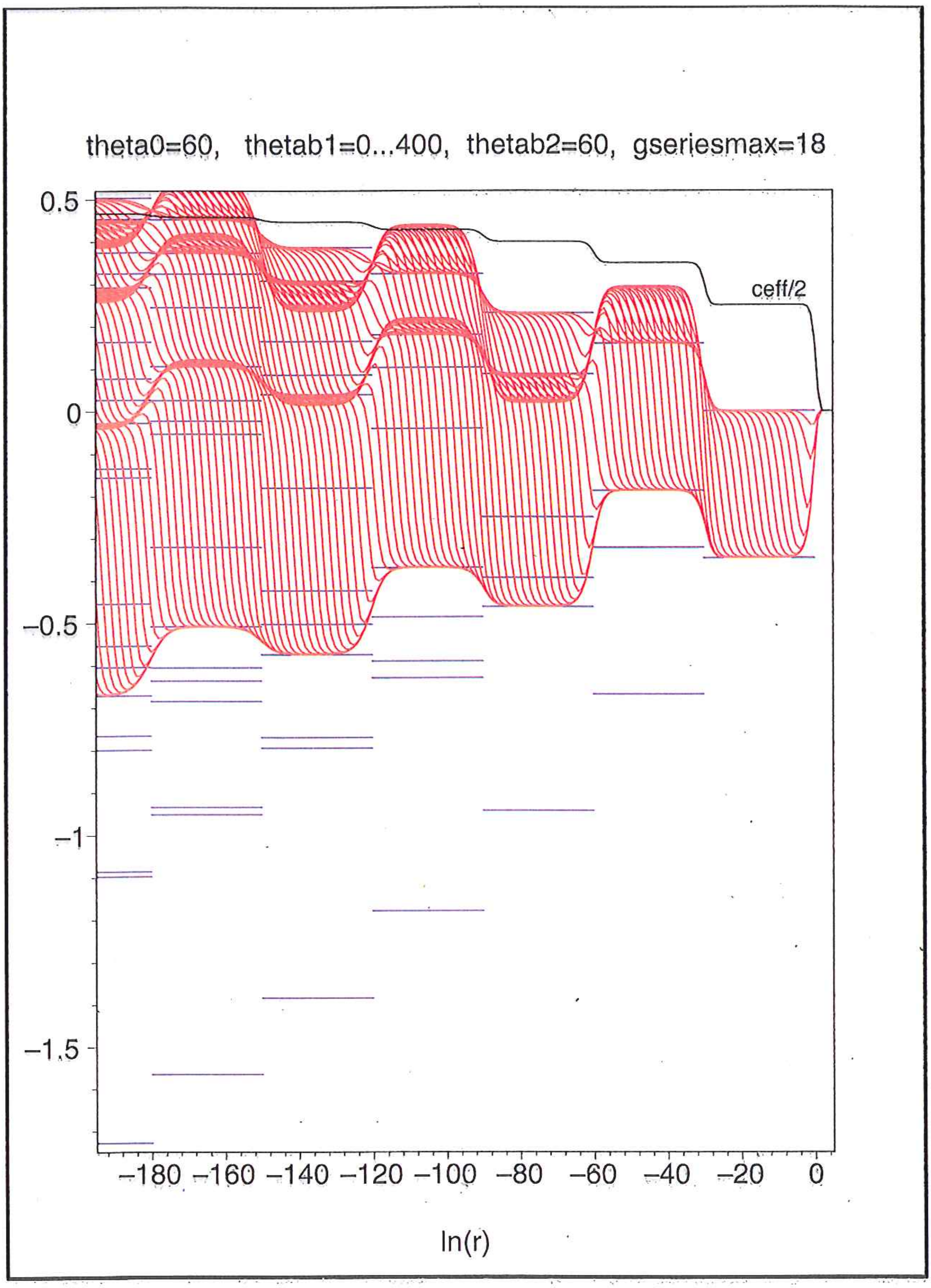
ln(r)



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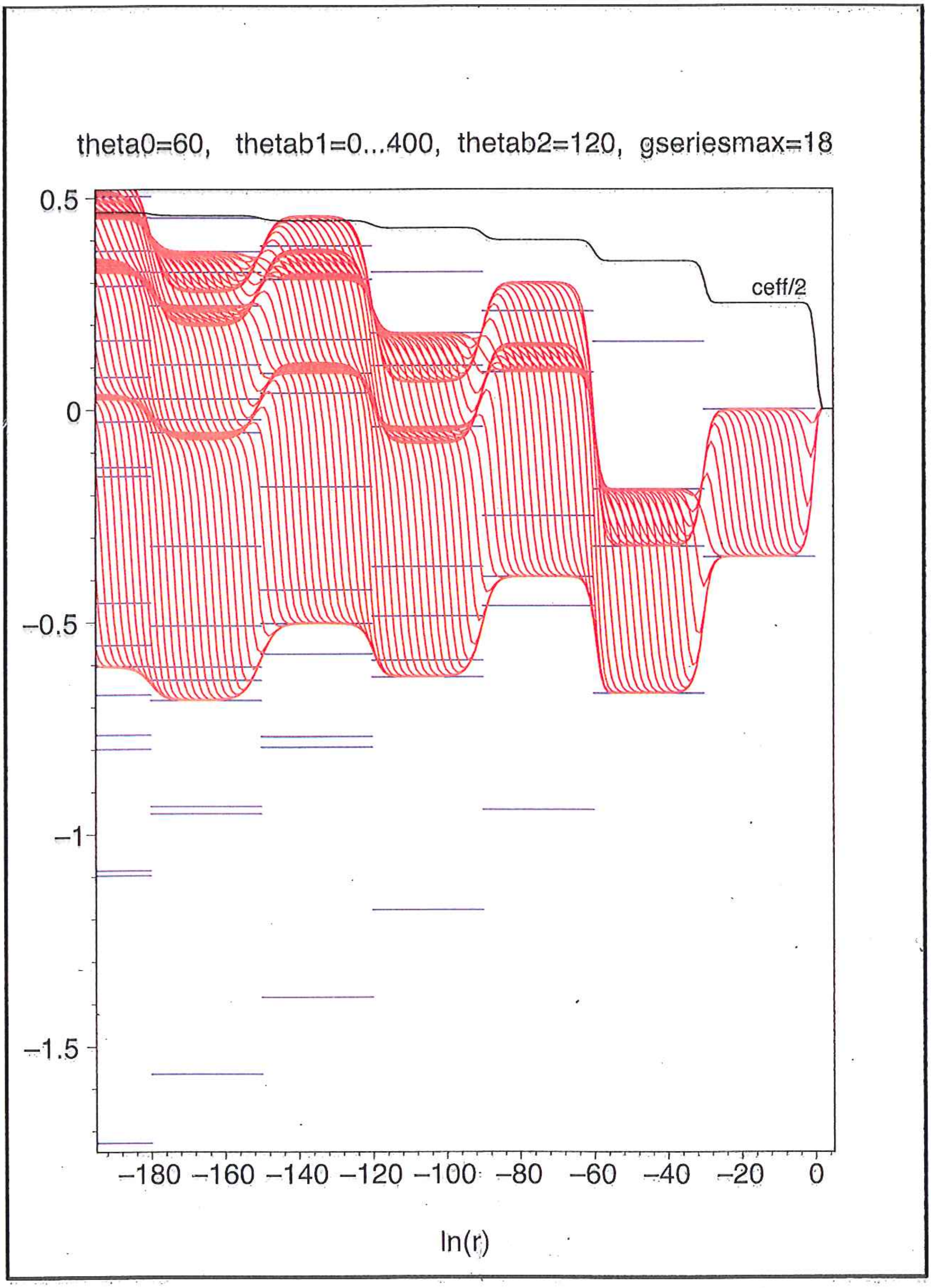
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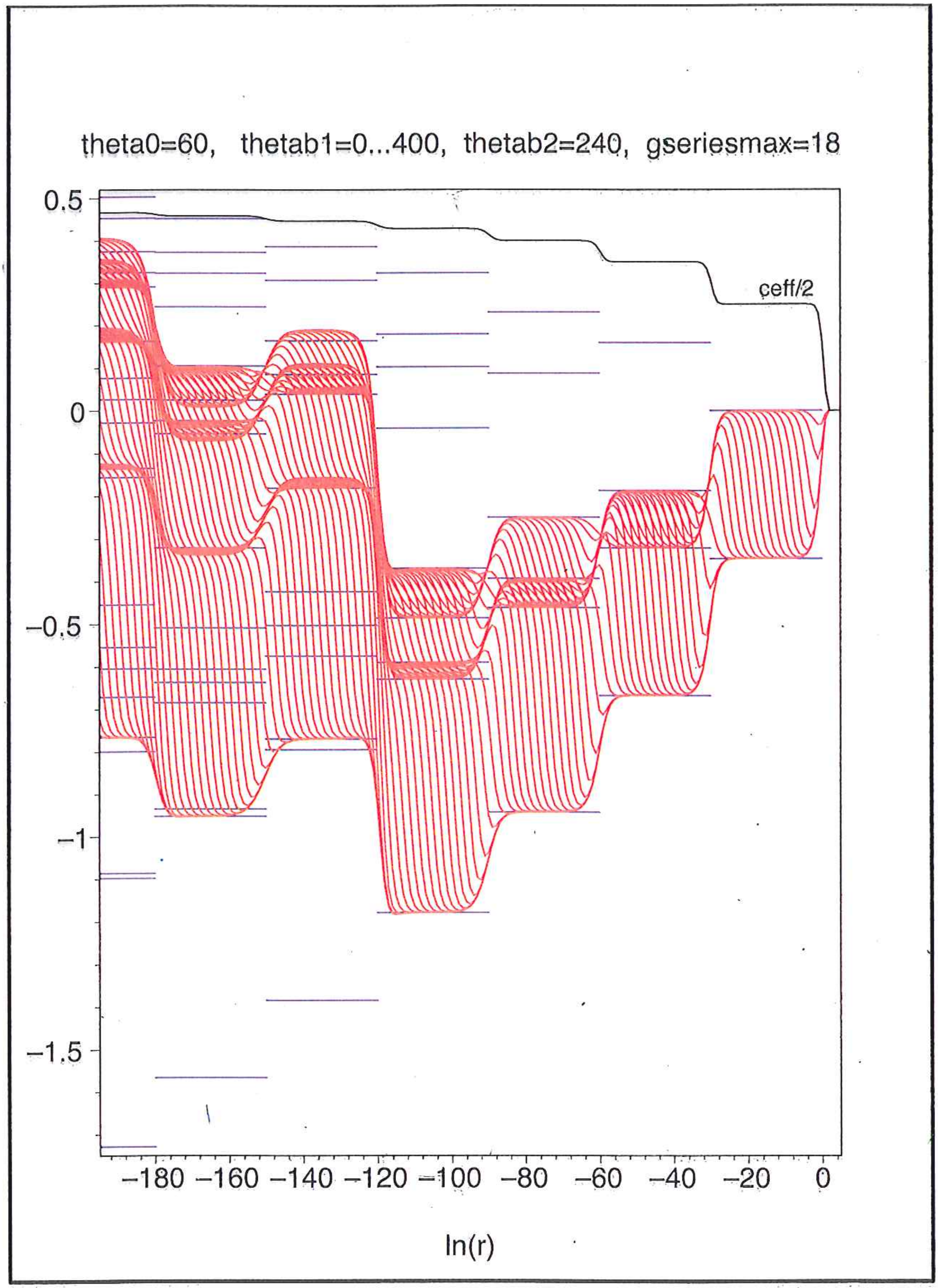
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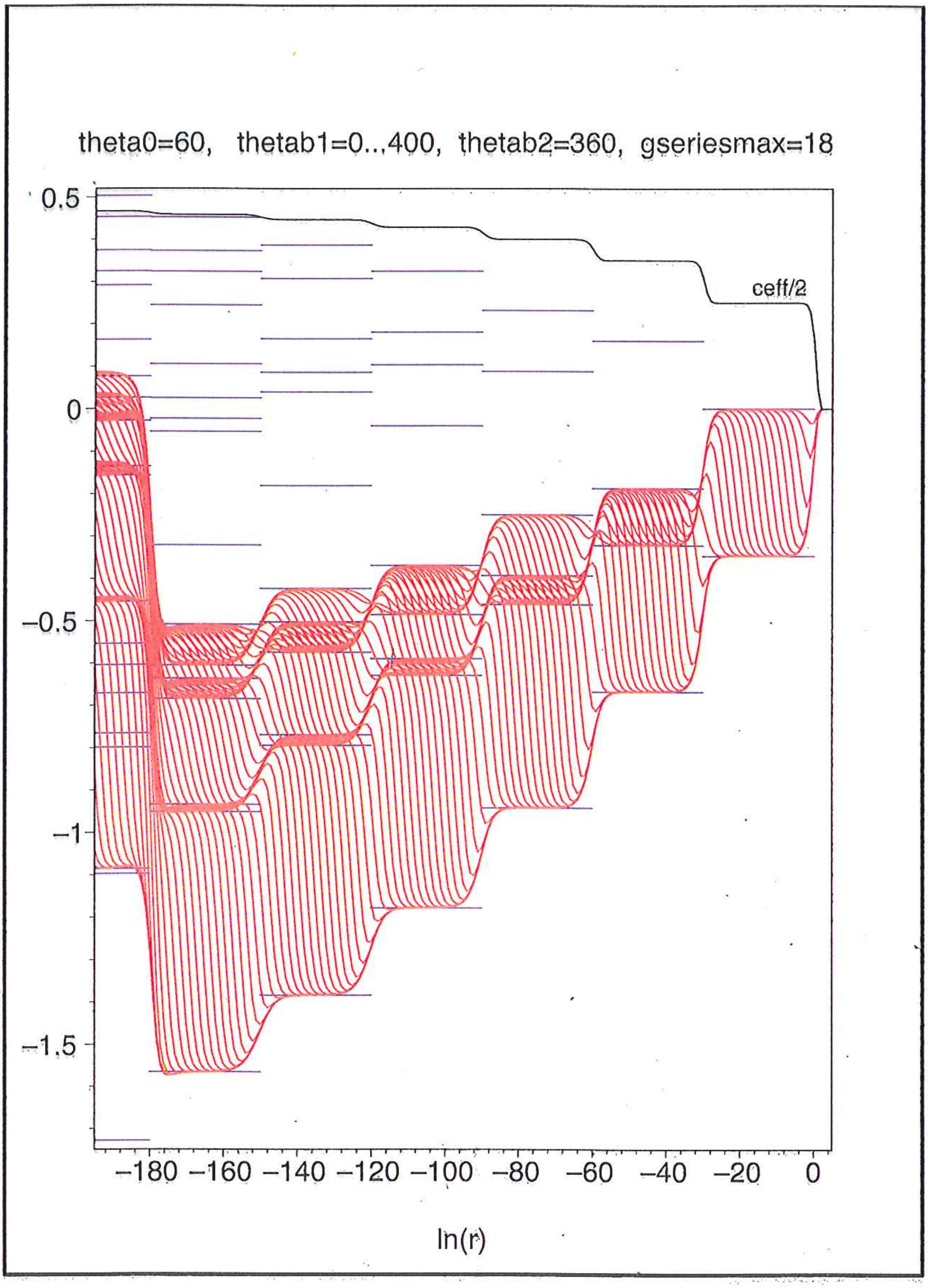
ln(r)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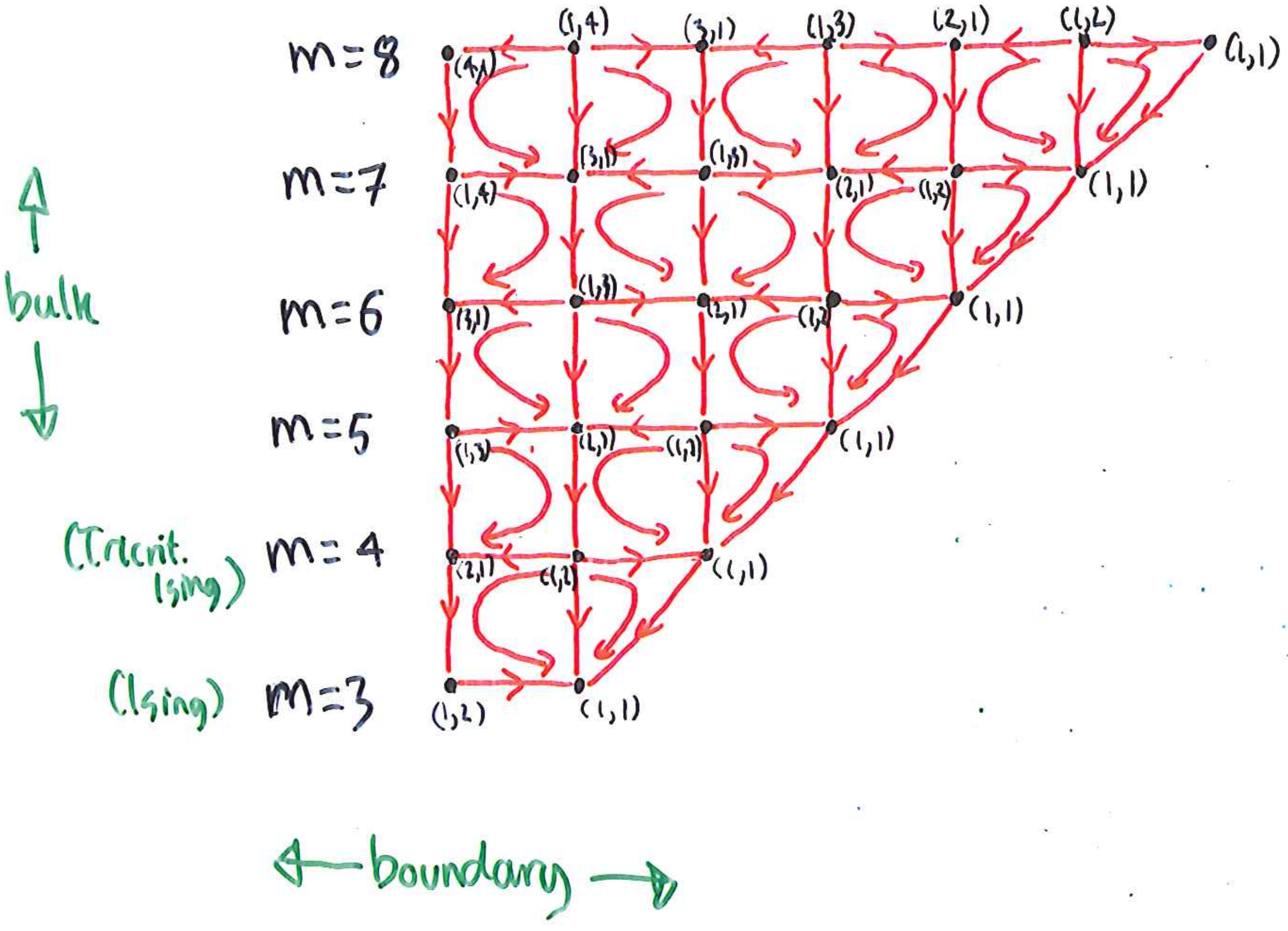
−180 −160 −140 −120 −100 −80 −60 −40 −20 0
ln(r)

- Stationary values on the plots are always logs of Cardy boundary g-function values, or logs of sums thereof;

- The staircase g-function plots imply a rich set of flows. Eg for $\theta_{b_2} \to \infty$:



- These match & extend the predictions of Fredenhagen et al. But to be sure, the "effective" TBA systems describing single g-flows from $\mathcal{M}_m \to \mathcal{M}_{m-1}$ for $m \geq 4$ should be extracted and analysed analytically...

[m=4: 0911.4969 ; m>4: to appear]

Final comments:

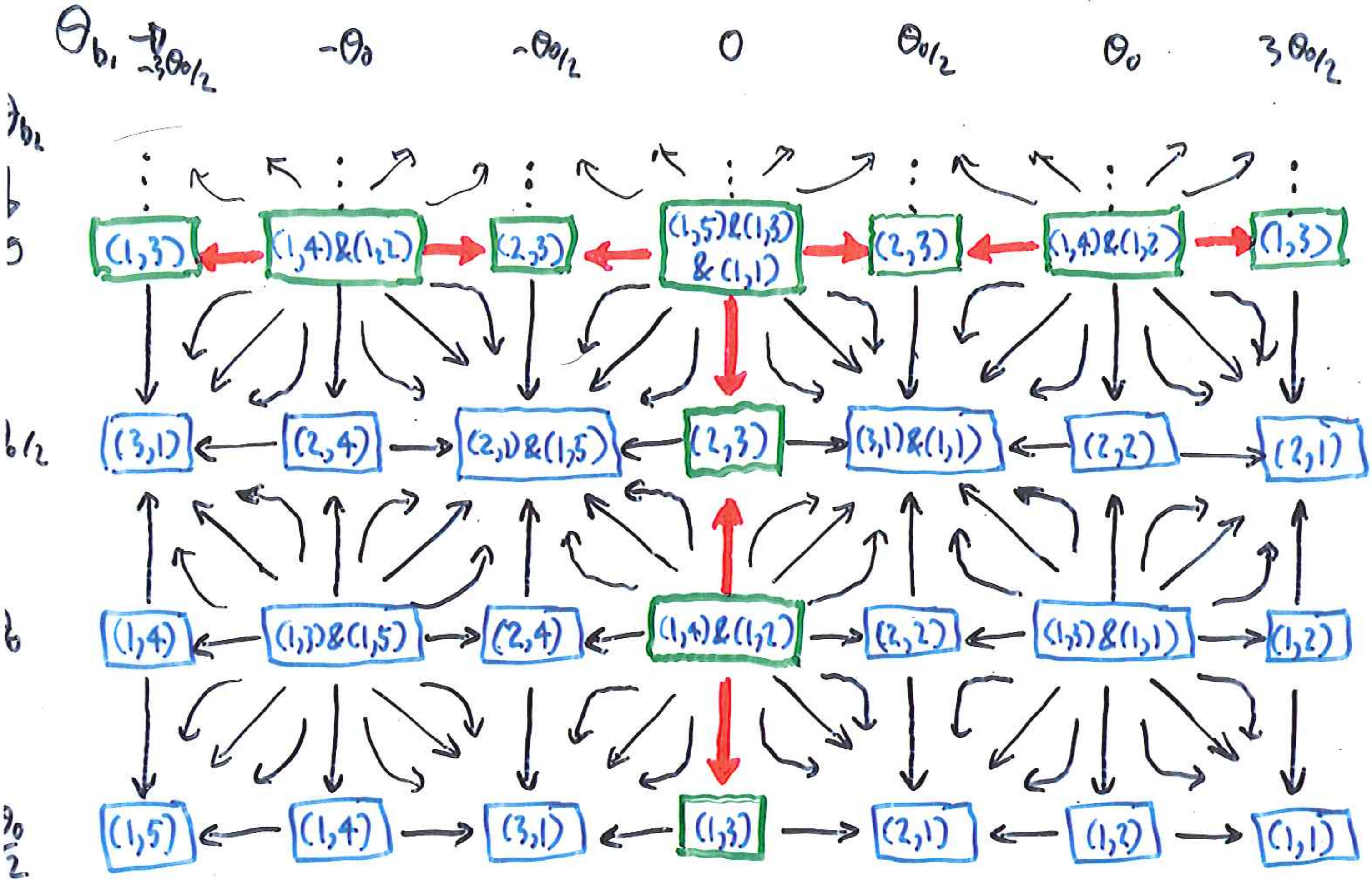
- The extra boundary parameter is incorporated by dint of the boundary staircase model visiting not just pure "Cardy" boundaries but also superpositions of such boundaries, which support multiple ϕ_{13} boundary operators and therefore multiparameter families of integrable boundary perturbations.[(*)] It would be interesting to study these in more detail.

  (* : see next page)

- $\exists$ many more staircases (Toda-related and beyond). So there's room for further generalisation!

- First-principles understanding of the g-function equations for all these flows would be good (recent progress by Pozsgay, 1003.5542)

Pure-boundary flows within $\mathcal{M}_5$: (as an example)



: RG flow

: flows self-dual under spin flip.

: self-dual CBCs

: other CBCs

- Can extend to more bndry parameters via defects, if you are so inclined...
- Lastly, it seems remarkable that the relatively-simple sinh-Gordon S-matrix and reflection factor (almost the simplest integrable QFT) should, after analytic continuation, know not only about <u>all</u> of the $c<1$ unitary conformal field theories, but also about <u>all</u> of their conformal boundary conditions.

<u>Moral</u>: the infrared "factorised scattering" description of an integrable QFT can be extremely economical...

<u>Second moral</u>: this is further evidence that the staircase model really exists. It deserves to be better understood! //