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CQUEST

Liouville gravity & Matrix model

- I Brief introduction (Bulk) ① + ② ('80)
- II Bulk correspondence ③ + ④ + ⑤ ('90 - 2008)
- III boundary " ⑥ (2010)

$g_{ab} \to e^{\varphi} \hat{g}_{ab}$

1 Liouville gravity in 2D

Polyakov (PI)

$$S = \overbrace{S_L + S_{gh} + S_M}^{S_0} + \Delta S_{int}$$

$$S_L = \frac{\sqrt{\hat{g}}}{4\pi}\left[(\partial_a \varphi)^2 \hat{g}^{ab} + 4\pi\mu\, e^{2b\varphi} + Q\hat{R}\varphi\right]$$

background charge $Q = b + \frac{1}{b}$

$$\begin{cases} c_L = 1 + 6Q^2 \\ c_{gh} = -26 \\ c_M \end{cases}$$

$$Z = \sum_h Z_h$$

$2b\varphi \to 2b\varphi - \ln\mu$

$$Z_g(\mu) = \int e^{-S_0} = \mu^{\frac{Q}{b}\frac{\chi}{2}}\, Z(\mu = 1)$$

$\chi = \int R = 2 - 2h$

$$\boxed{\begin{aligned} Z_A &= A \int \frac{d\mu}{2\pi i}\, Z_g(\mu)\, e^{\mu A} \\ &= Z_g(\mu=1)\, A^{-Q\chi/2b} \cdot \Gamma(-Q/b) \end{aligned}}$$

2 Matrix model

$$e^{Z_N} = \int [DM]\, e^{-\frac{N}{g}\mathrm{Tr}\, V(M)}$$

$M = N \times N$ hermitian matrix

large N expansion $N^V \left(\frac{1}{N}\right)^E N^F = N^\chi$

$\chi = V - E + F = 2 - 2h$

$Z_N(g) =$ connected diagrams

$= N^2 \mathcal{Z}_0(g) + N^0 \mathcal{Z}_1(g) + N^{-2} \mathcal{Z}_2(g) + \cdots$

$\boxed{2}$

$z_h(g) = \sum \frac{g^n}{n} f_n$

$$\boxed{z_0(g) \simeq \sum n^{\gamma-2}\left(\frac{g}{g_c}\right)^n \sim (g_c-g)^{2-\gamma}}$$

NB

$\langle n\rangle_h = \frac{\partial}{\partial g}\ln z_h(g)$

$\langle n\rangle_0 \sim \frac{1}{g_c-g}$: $g \to g_c$ $\langle n\rangle_h$

"A" $\Leftrightarrow \langle n\rangle \to \infty$ rescale the area

$\left\langle \frac{\#\text{ of faces}}{g} \right\rangle$ continuum limit

(*) $\boxed{\text{double scaling limit}}$

$Z_h(g) \sim (g_c-g)^{(2-\gamma_{st})\chi/2}$ + regular

$Z_N(g) = \sum N^{\chi} Z_h(g)$

$= \sum_h \kappa^{-\chi} f_h(g) + \cdots$

$\kappa^{-1} = N(g_c-g)^{\frac{2-\gamma_{st}}{2}}$ = finite

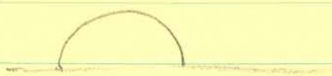
$\boxed{\text{NB}}$ $e^Z = \int [\prod d\lambda_i]\, \Delta(\lambda)^2 \exp -\frac{N}{g}\sum_i V(\lambda_i)$

$\Delta(\lambda) = \prod_{i<j}(\lambda_i-\lambda_j)$

B & Z (1978)

($V(M) = \frac{M^2}{2} + M^4$ "pure gravity")

$g\to 0$ $u(\lambda) = \frac{1}{\pi}\sqrt{1-\lambda^2/4}$



"Wigner's semi-circle law"

$g \to g_c$ $u(\lambda) \sim \left(1-\frac{\lambda^2}{8}\right)^{3/2}$

$Z_0 \sim (g_c-g)^{2-\gamma}$ $g_c = -\frac{1}{48}$, $\boxed{\gamma = -1/2}$ ✓

$\boxed{3}$

$$Z \sim x^{(2-\gamma_{str})\frac{\chi}{2}} \qquad x = (g - g_c)$$

(*) (?) $\langle A \rangle \sim \frac{\partial \ln Z_N}{\partial x} \sim -\frac{\partial \ln Z_{LG}}{\partial \mu}$ for pure gravity

$$\boxed{(2-\gamma_{str})\frac{\chi}{2} = 1 + \frac{Q}{b}\frac{\chi}{2}}$$

$\boxed{\gamma = -\frac{1}{2}}$ $\boxed{\frac{1}{b^2} = \frac{3}{2}}$

$2 + \frac{1}{2} = \frac{5}{2} = 1 + \frac{1}{b^2}$

$(\mu = -x)$?

(interaction theory)

(3) Coupled to matter (LG)

$$\Delta S \sim \int_M e^{2a\varphi} \cdot \Phi_\alpha$$

* Staudacher (1990) ($b^2 = \frac{2}{5}$ Lee-Yang)

* $\Delta_a = a(Q-a)$

$$c_M = c_{(p/p')} = 1 - 6q^2 = 1 - \frac{6(p-p')^2}{pp'} \qquad q = \beta^{-1} - \beta, \quad \beta = \sqrt{\frac{p}{p'}} < 1$$

$$\Delta_{m,n} = \alpha_{mn}(\alpha_{mn} - q) \qquad \alpha_{mn} = \left(\frac{m-1}{2}\right)\beta - \frac{(n-1)}{2}\beta^{-1}$$

$$\alpha_{11} = 0 \Rightarrow \Delta_{11} = 0$$

CFT & Minimal gravity

* $c_M + c_L - 26 = 0 \Rightarrow 0 < \beta = b < 1$ "real b"

* $\Delta_a + \Delta_\Phi = 1 \Rightarrow \boxed{a = \alpha + b}$

(4) Al. Zam (05). $M_{(2,5)}$ on sphere ($\beta^2 = \frac{2}{5} = b^2$)

$$LG: \Delta S = \frac{ih}{2\pi}\int_M \Phi_{1,2}\, e^{3b\varphi} = \frac{ih}{2\pi} U_{12}$$

$$Z_{LG} = Z_L + \frac{1}{2}\left(\frac{-ih}{2\pi}\right)^2 \langle U_{12} U_{12} \rangle + \frac{1}{3!}\left(\frac{-ih}{2\pi}\right)^3 \langle U_{12}^3 \rangle$$

$$= Z_L \left[1 + \frac{1}{2}\left(\frac{ih}{2\pi}\right)^2 \frac{105}{4}(2\pi L_{GLY})^2 + \frac{1}{3!}\left(\frac{ih}{2\pi}\right)^3 \frac{105}{8}(2\pi L_{GLY})^3 + \cdots \right]$$

$$L_{GLY} = \frac{-1}{(\pi\mu)^{3b}} \cdot \frac{\sqrt{\gamma(4/5)}}{4\,\gamma(2/5)}$$

$\boxed{4}$

$\boxed{\text{One-matrix}}$

$$\frac{\partial^2 Z_N}{\partial x^2} = u_*(t, x) \qquad P(u_*)=0 \quad \text{where}$$

$$P(u) = u^3 - tu - x$$

"string eq"

$$Z_N(t,x) = x^{7/2} \sum_{\substack{n=0\\(n\neq 1)}}^{\infty} \left(\frac{1}{2}\right) \frac{\Gamma(3n/2 - 7/2)\, y^n}{n!\ \Gamma(n/2 - 1/2)} \qquad y = x t^{-3/2}$$

$\propto y < 1$

$$= Z_{GLY}(t,0)\left(1 + \frac{105}{8} y^2 - \frac{35}{16} y^3 - \frac{105}{128} y^4 + \cdots\right)$$

$$\left.\begin{aligned} \left(\frac{ih}{2\pi}\right)^2 (2\pi L_{GLY})^2 = y^2 = \left(\frac{x}{t^{3/2}}\right)^2 \\ \left(\frac{ih}{2\pi}\right)^3 (2\pi L_{GLY})^3 = y^3 = \left(\frac{x}{t^{3/2}}\right)^3 \end{aligned}\right\} \Rightarrow \frac{h\ell}{(\pi\mu)^{3/2}} = \frac{x}{t^{3/2}}$$

$\boxed{t = \mu, \quad x \sim h}$ $\cancel{\#t}$ μ

The role of parameter? $\begin{cases} \text{KdV flow, } \cancel{\text{coupled}} \\ \text{generating function of operators} \end{cases}$

D(90) D&S(90), G&M(90)

⑤ $M(2, 2p+1)$ series

$t_{p-1} \sim x$

$$P(u, \{t_k\}) = u^{p+1} - \mu u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-1-k}, \quad \boxed{\frac{\partial^2 Z_N}{\partial x^2} = u_*}$$

$$Z_N = \frac{1}{2}\int_0^{u_*} P^2(u)\, du,$$

P(u) has μ-scale inv
$u \to \alpha u, \quad \mu \to \alpha^2 \mu$
$t_k \to \alpha^{k+1} t_k$

- (t_k) describes the KdV flow

$$Z = Z_N(\mu, \{t_k\}) = \left\langle e^{\sum_k t_k O_k} \right\rangle \qquad \frac{\partial}{\partial t_k}\frac{\partial^2 Z}{\partial x^2} = \langle O_x O_x O_k \rangle = \frac{\partial u_*}{\partial t_k}$$

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What is the relation between $\underline{\underline{t_k}}$ and λ_k CFT flow in LG.

KdV flow in Matrix model

(MSS (91) $t_k \sim \lambda_k$)

$$t_k \sim \lambda_k + \sum C_k^{l_1 l_2 \cdots l_n} \lambda_{k_1} \cdots \lambda_{k_n}$$

"resonance term".

B & Z (2008)

$$P(u, \{t_k\}) = u_0^{p+1} \, Q(\tilde{u}, \{\lambda_k\}) \qquad \tilde{u} = \frac{u}{u_0}. \qquad \boxed{u_0 \sim \sqrt{\mu}}$$

$$Q = Q_0 + \sum \lambda_k Q_k + \sum \frac{\lambda_{k_1} \lambda_{k_2}}{2} Q_{k_1 k_2} + \cdots$$

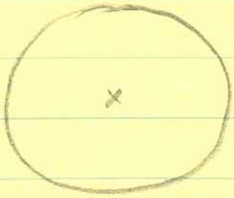
$$Q_0(\tilde{u}) = \int_1^{\tilde{u}} L_p(x)\, dx = \frac{L_{p+1}(\tilde{u}) - L_{p-1}(\tilde{u})}{2p+1}$$

$$Q_k = \frac{d^{n-1}}{d\tilde{u}^{n-1}} L_{p-s_{k_1}-n}(\tilde{u})$$

(checked up to 4-pt funct.)

$$\boxed{Q_0(1) = 0 \quad \text{fixes} \quad u_0 \sim \sqrt{\mu}}$$

6 Boundary effect



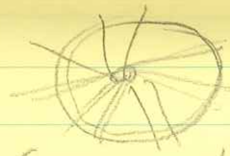
LG : add boundary action

$$\boxed{\mu_B \int_{\partial M} e^{b\varphi}}$$

$$\Delta S_B = \xi_k \int_{\partial M} \left[e^{\beta_k \varphi} \right] \Phi_k^0$$

1. Bulk effect in the presence of bound. — B & R (2010)
2. boundary flow — I & R (2010)

$\boxed{6}$



macroscopic loop operator

$$W(L) \sim \langle \mathrm{tr}(M^{L}) \rangle \xrightarrow[\text{cont}]{} \frac{1}{L}\int dy \, \langle y | \, e^{L(\partial_y^2 - u_*)} | y \rangle$$

large N limit $\partial_y \to p_y$

$$\longrightarrow \frac{1}{\sqrt{L}} \int dy \, e^{-L u_*(y)}$$

$\boxed{\text{GSS(91)}}$

Disk partition function BR(2010)

$$Z_L(\{\lambda_k\}) \sim \frac{1}{\sqrt{L}} \int_x^\infty dy \, e^{-L u_*(y)}$$

$$\sim \frac{1}{\sqrt{L}} \int_{u_*}^\infty e^{-Lu} \left(\frac{dy}{du}\right) du$$

use $\quad P(u, \mu, \lambda_1 \cdots \lambda_{p-2}, \lambda_{p-1} = y) = 0$

$$\langle O_k \rangle_L = \frac{\partial}{\partial \lambda_k} Z_L(\{\lambda_k\}) \Big|_{\{\lambda_k\}=0}$$

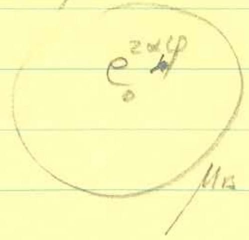
$$= u_0^{p-k} \sqrt{L} \int_{\xi_*=1}^\infty L_{p-1-k}(\xi) \, e^{-L u_0 \xi} \, d\xi$$

$$= \sqrt{\frac{2}{\pi}} \, u_0^{p-k-1/2} \, K_{p-k-1/2}(u_0 L)$$

check with Boundary Liouville theory FZZ(2000)

$$U(\alpha | \mu_B) \propto \mu^{(Q-2\alpha)/2b} \, \mathrm{ch}((2\alpha - Q)\pi s)$$

$$\cosh \pi b s = \mu_B \sqrt{\frac{\sin \pi b^2}{\mu}}$$



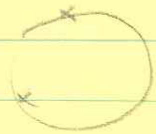
$$\Rightarrow U(\alpha | \mu_B) = \int_0^\infty \frac{dl}{l} \, e^{-\mu_B l} \, W_\alpha(l)$$

$$W_\alpha(l) \sim \mu^{(Q-2\alpha)/2b} \, K_{(Q-2\alpha)/b}(\kappa l) \qquad \kappa = \sqrt{\frac{\mu}{\sin \pi b^2}}$$

$\boxed{7}$

$$\left.\begin{aligned} \frac{1}{b^2} &= p + 1/2 \\ \alpha_k &= \frac{k+2}{2b} \end{aligned}\right\} \Rightarrow \quad \frac{Q-2\alpha_k}{b} = p-k-\frac{1}{2} \ \checkmark$$

Q :



- boundary operator? in Matrix model.
- boundary flow?
- Multi-matrix model?

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future works

① Supersymmetry

② boundary flow (p, q) / tricritical /

③ O(n) model

④ boundary resonance transf

⑤ S-W singularity resolution