Correlations in Quantum States and Decoherent Capabilities of Operations

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Correlations (classical, quantum, total) in bipartite quantum states.

Main theme:

Classicality/Quantumness

Non-disturbance/Disturbance

Commutativity/Non-commutativity

Outline

1. Correlations in Bipartite States
   - Decomposition of bipartite states
   - Separability/entanglement
   - Classicality/quantumness
   - Quantum discord
   - No-broadcasting theorems
2. Decoherent Capabilities of Operations
3. Questions

1. Correlations in Bipartite States

Correlations are encoded in multipartite states. We only consider bipartite case.

• Classical world: bivariate probability distribution p^{ab} with marginals p^a and p^b

Correlations in p^{ab} are usually quantified by the Shannon mutual information:

$$I(p^{ab}) := H(p^a) + H(p^b) - H(p^{ab})$$

• Quantum world: bipartite state ρ^{ab}, a density matrix in the tensor product Hilbert space $H^a \otimes H^b$, with marginals ρ^a and ρ^b

Total correlations in ρ^{ab} are usually quantified by the quantum mutual information:

$$I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$$

Fundamental difference between classical and quantum worlds

Perfect correlations

• Classical world: $p_{ij}^{ab} = p_i \delta_{ij}$

Shannon mutual information:

$I(p^{ab}) = H(\{p_i\})$.

• Quantum world: $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ with

$|\Psi^{ab}\rangle = \sum_i \sqrt{p_i}|i\rangle \otimes |i\rangle$

Quantum mutual information:

$I(\rho^{ab}) = 2H(\{p_i\})$.

Key issues: How to classify and quantify correlations in a bipartite state?

- Classification issue: Different correlations, Separate total correlations into classical part and quantum part

- Quantification issue: Measures of various correlations

Decomposition of bipartite states

- Any bipartite state ρ^{ab} can always be represented as

$$\rho^{ab} = \sum_i X_i^a \otimes X_i^b$$

with $\{X_i^a\}$ a set of quantum states on H^a, and $\{X_i^b\}$ a set of linearly independent self-adjoint operators (not necessarily non-negative) on H^b.

- Entanglement/separability (Werner, 1989):

A bipartite state ρ^{ab} is called separable if

$$\rho = \sum_i p_i \rho_i^a \otimes \rho_i^b.$$

Here $p = \{p_i\}$ is a probability distribution, ρ_i^a and ρ_i^b are quantum states for H^a and H^b, respectively.

Otherwise, it is called entangled.

- Detection (Hard problem): How to tell if a bipartite state is separable or entangled?

  Various Bell inequalities
  Peres' positive partial transposition
  Methods based on uncertainty relations
  ……

- Quantification (Complicated problem):

How to quantify the entanglement of a bipartite state?

Entanglement of formation
Entanglement cost
Relative entropy of entanglement
Squashed entanglement
Negativity
......

A "paradox" for entanglement of formation Li and Luo, *Phys. Rev. A*, 2007

Werner state

$$w = \theta \frac{P_-}{d_-} + (1-\theta)\frac{P_+}{d_+},$$

where P_- (P_+) is the projection from $C^d \otimes C^d$ to the anti-symmetric (symmetric) subspace of $C^d \otimes C^d$, and $d_\pm = \frac{d^2 \pm d}{2}$.

Entanglement of formation

$$E(w) = H(\frac{1}{2} - \sqrt{\theta(1-\theta)}).$$

Quantum mutual information

$$I(w) = \log \frac{2d^2}{(d^2-d)^{\theta}(d^2+d)^{1-\theta}} - H(\theta).$$

When d is sufficiently large,

$$E(w) > I(w).$$

"Entanglement" > "Total correlations"?

Entanglement is not the only kind of quantum correlation

Certain quantum advantage is not based on quantum entanglement, but rather on separable states which still possess certain quantum correlations.

A. Datta, A. Shaji, C. M. Caves, Quantum discord and the power of one qubit, *Phys. Rev. Lett.* 100, 050502 (2008).

Classicality/quantumness (of correlations)

- A state ρ^{ab} shared by two parties a and b is called classical (w.r.t. correlations, or more precisely, classical-classical) if it is left undisturbed by certain local von Neumann measurement $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$, that is,

$$\rho^{ab} = \Pi(\rho^{ab}) := \sum_{ij} (\Pi_i^a \otimes \Pi_j^b)\rho^{ab}(\Pi_i^a \otimes \Pi_j^b).$$

- Otherwise, it is called quantum (w.r.t. correlations).

- ρ^{ab} is called classical-quantum, if it is left undisturbed by a local von Neumann measurement $\Pi^a = \{\Pi_i^a\}$ for party a, that is,

$$\rho^{ab} = \Pi^a(\rho^{ab}) := \sum_i (\Pi_i^a \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i^a \otimes \mathbf{1}^b).$$

Characterizations

- ρ^{ab} is classical-classical iff

$$\rho^{ab} = \sum_{ij} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|.$$

Here $\{p_{ij}\}$ is a bivariate probability distribution, $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthogonal sets for parties a and b, respectively.

- ρ^{ab} is classical-quantum iff

$$\rho^{ab} = \sum_{i} p_{i} |i\rangle\langle i| \otimes \rho_i^b.$$

Separability vs. classicality
Li and Luo, *Phys. Rev. A*, 2008

- ρ^{ab} is separable iff it can be viewed as a local reduction of a classical state in a larger system:

$$\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{aa'bb'}.$$

Here $\rho^{aa'bb'}$ is a classical state (w.r.t. the cut $aa' : bb'$) on $(H^a \otimes H^{a'}) \otimes (H^b \otimes H^{b'})$.

Quantum Discord
Ollivier and Zurek, *Phys. Rev. Lett.* 2001

- The quantum discord of ρ^{ab} is defined as

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab}).$$

Here

$$C(\rho^{ab}) := \sup_{\Pi^a} I(\Pi^a(\rho^{ab})),$$

$\Pi^a = \{\Pi_i^a\}$ is a von Neumann measurement for party a, and

$$\Pi^a(\rho^{ab}) := \sum_i (\Pi_i^a \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i^a \otimes \mathbf{1}^b).$$

Luo, Quantum discord for two-qubit systems, *Phys. Rev. A*, 2008

- For two-qubit state

$$\rho^{ab} = \frac{1}{4}(\mathbf{1}^a \otimes \mathbf{1}^b + \sum_{j=1}^{3} c_j \sigma_j^a \otimes \sigma_j^b),$$

we have

$$Q(\rho^{ab}) = \frac{1}{4}\sum_{j=0}^{3} \lambda_j \log \lambda_j - \frac{1+c}{2}\log(1+c) - \frac{1-c}{2}\log(1-c)$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$, and

$$\lambda_0 = 1 - c_1 - c_2 - c_3, \quad \lambda_1 = 1 - c_1 + c_2 + c_3$$
$$\lambda_2 = 1 + c_1 - c_2 + c_3, \quad \lambda_3 = 1 + c_1 + c_2 - c_3.$$

Recent interests in quantum discord

- D. Girolami, M. Paternostro, G. Adesso, arXiv:1008.4136
- V. Madhok and A. Datta, arXiv:1008.4135
- D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter, arXiv:1008.3205
- M. D. Lang and C. M. Caves, arXiv:1006.2775
- B. Dakic, V. Vedral and C. Brukner, arXiv:1004.0190

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Experimental investigation of classical and quantum correlations

J. S. Xu et al., *Nature Communications* **1**, 7 (2010).

An alternative quantum discord
Dakic, Vedral and Brukner, arXiv:1004.0190
Luo and Fu, *Phys. Rev. A*, 2010

- Geometric measure of quantum discord

$$D(\rho^{ab}) := \min_{\Pi^a} ||\rho^{ab} - \Pi^a(\rho^{ab})||^2,$$

where the min is over von Neumann measurements $\Pi^a = \{\Pi^a_k\}$ on system H^a, and $\Pi^a(\rho^{ab}) := \sum_k (\Pi^a_k \otimes \mathbf{1}^b)\rho^{ab}(\Pi^a_k \otimes \mathbf{1}^b)$.

A formula

Let $\rho^{ab} = \sum_{ij} c_{ij} X_i \otimes Y_j$ be a state on $C^m \otimes C^n$ expressed in local orthonormal operator bases, then

$$D(\rho^{ab}) = \mathrm{tr}(CC^t) - \max_A \mathrm{tr}(ACC^tA^t),$$

where $C = (c_{ij})$, and the max is over $m \times m^2$ dimensional matrices $A = (a_{ki})$ such that $a_{ki} = \mathrm{tr}|k\rangle\langle k|X_i$, and $\{|k\rangle\}$ is any orthonormal base for H^a.

In particular,

$$D(\rho^{ab}) \geq \mathrm{tr}(CC^t) - \sum_{i=1}^{m} \lambda_i = \sum_{i=m+1}^{m^2} \lambda_i,$$

where λ_i are the eigenvalues of CC^t listed in decreasing order (counting multiplicity).

Conjecture. $D(\rho^{ab}) \leq \sum_{i=m}^{m^2} \lambda_i,$

Observable correlations: symmetric version of quantum discord

- The observable correlations of ρ^{ab} is defined as

$$C_2(\rho^{ab}) := \sup_{\Pi} I(\Pi(\rho^{ab})).$$

Here $\Pi(\rho^{ab}) := \sum_{ij}(\Pi_i^a \otimes \Pi_j^b)\rho^{ab}(\Pi_i^a \otimes \Pi_j^b)$. The quantity

$$Q_2(\rho^{ab}) := I(\rho^{ab}) - C_2(\rho^{ab})$$

may be interpreted as a measure of quantum correlations in ρ^{ab}.

Lindblad conjecture, 1991

- The Lindblad conjecture states that

$$C_2(\rho^{ab}) \geq Q_2(\rho^{ab}).$$

- Supporting evidence:

(1) It can happen $C_2(\rho^{ab}) > 0$ and $Q_2(\rho^{ab}) = 0$, but never $C_2(\rho^{ab}) = 0$ and $Q_2(\rho^{ab}) > 0$.

(2) For any classical state, we have $C_2(\rho^{ab}) = I(\rho^{ab})$ and $Q_2(\rho^{ab}) = 0$.

(3) For any pure state, we have $C_2(\rho^{ab}) = Q_2(\rho^{ab}) = \frac{1}{2} I(\rho^{ab})$.

Unfortunately, the Lindblad conjecture is false
Luo and Zhang, *J. Stat. Phys.* 2009

Counterexamples in two-qubit systems!

- Species of correlations:

Scheme 0	Total correlations			
Scheme 1 Werner, 1989	Separable			Entanglement
Scheme 2 Ollivier ans Zurek, 2001 Henderson and Vedral, 2001	Classical I		Quantum I	
Scheme 3 Lindblad, 1991 Piani et al, Luo, 2008	Classical II	Quantum II		
Scheme 4 Modi et al, 2010	Classical II	Dissonance		Entanglement
……	……			

Broadcasting for quantum states

- A quantum state ρ on Hilbert space H is broadcastable if there exists an operation $\mathcal{E} : \mathcal{S}(H) \longrightarrow \mathcal{S}(H \otimes H)$ such that both the reduced states of $\mathcal{E}(\rho)$ are identical to ρ.
- A family of quantum states is called broadcastable if all states in the family can be broadcast by the *same* operation.

Broadcasting for correlations

- The correlations in ρ^{ab} are locally broadcastable if there exist two operations $\mathcal{E}^a : \mathcal{S}(H^a) \longrightarrow \mathcal{S}(H^{a_1} \otimes H^{a_2})$ and $\mathcal{E}^b : \mathcal{S}(H^b) \longrightarrow \mathcal{S}(H^{b_1} \otimes H^{b_2})$ such that

$$I(\rho^{a_1 b_1}) = I(\rho^{a_2 b_2}) = I(\rho^{ab}).$$

Here $I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab})$ is the quantum mutual information, $\rho^{a_1 a_2 b_1 b_2} := \mathcal{E}^a \otimes \mathcal{E}^b(\rho^{ab})$ and

$$\rho^{a_1 b_1} := \mathrm{tr}_{a_2 b_2} \rho^{a_1 a_2 b_1 b_2}, \ \rho^{a_2 b_2} := \mathrm{tr}_{a_1 b_1} \rho^{a_1 a_2 b_1 b_2}.$$

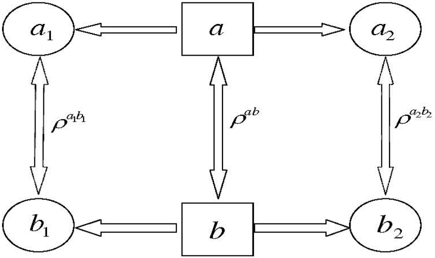


FIG. 1: Local broadcasting of correlations. Local operations $\mathcal{E}^a : \mathcal{S}(H^a) \to \mathcal{S}(H^{a_1} \otimes H^{a_2})$ and $\mathcal{E}^b : \mathcal{S}(H^b) \to \mathcal{S}(H^{b_1} \otimes H^{b_2})$ are performed by parties a and b, respectively, with a resulting four-partite state $\rho^{a_1a_2b_1b_2} = \mathcal{E}^a \otimes \mathcal{E}^b(\rho^{ab})$. The two reduced states $\rho^{a_1b_1} = \mathrm{tr}_{a_2b_2}\rho^{a_1a_2b_1b_2}$ and $\rho^{a_2b_2} = \mathrm{tr}_{a_1b_1}\rho^{a_1a_2b_1b_2}$ are expected to reproduce the same amount of correlations as those in ρ^{ab}. The amount of correlations is quantified by the quantum mutual information.

No-broadcasting for non-commuting states

- **Theorem 1** (Barnum et al. *PRL*, 1996)
A family of quantum states $\{\rho_i\}$ can be simultaneously broadcast if and only if the states are commutative.

No-local-broadcasting for quantum correlations

- **Theorem 2** (Piani et al. *PRL*, 2008)
The correlations in a bipartite state ρ^{ab} can be locally broadcast if and only if the correlations are classical.

Question

What are the relations between Theorem 1 and Theorem 2?

Unification for no-broadcasting theorems

- Quantum discord helps to build a bridge between Theorems 1 and 2.

Theorem 1 $\Leftrightarrow$ Zero discord $\Leftrightarrow$ Theorem 2.

- Three key ingredients in the proof:

  Decomposition of bipartite states

  Classical-quantum states

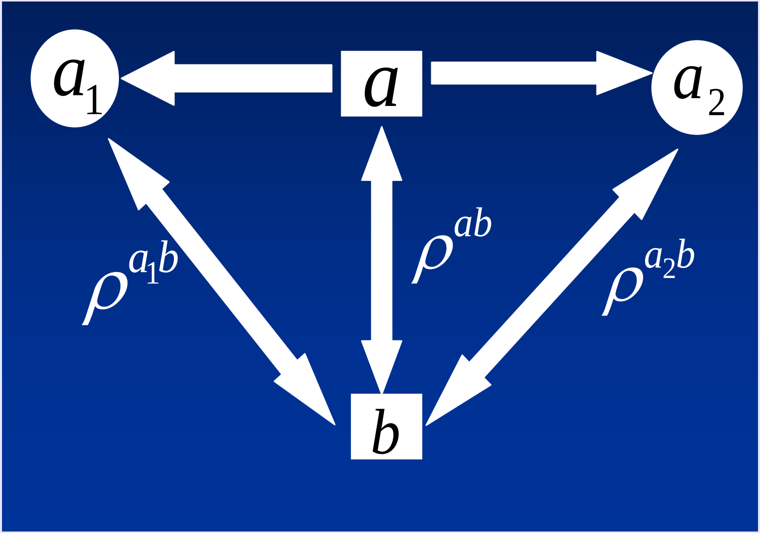
  Monotonicity of relative entropy

Unilocal broadcasting for correlations

- The correlations in ρ^{ab} is locally broadcast by party a, if there exists an operation

$$\mathcal{E}^a : \mathcal{S}(H^a) \longrightarrow \mathcal{S}(H^{a_1} \otimes H^{a_2})$$

such that $I(\rho^{a_1 b}) = I(\rho^{a_2 b}) = I(\rho^{ab})$. Here $\rho^{a_1 b} := \mathrm{tr}_{a_2} \rho^{a_1 a_2 b}$, $\rho^{a_2 b} := \mathrm{tr}_{a_1} \rho^{a_1 a_2 b}$, and $\rho^{a_1 a_2 b} := \mathcal{E}^a \otimes \mathcal{I}^b(\rho^{ab}) \in \mathcal{S}(H^{a_1} \otimes H^{a_2} \otimes H^b)$.


a_1
a
a_2
$\rho^{a_1 b}$
ρ^{ab}
$\rho^{a_2 b}$
b

No-unilocal-broadcasting for quantum correlations

- **Theorem** $1\frac{1}{2}$ (Luo and Sun, *Phys. Rev. A*, 2010)
  Correlations in a bipartite state ρ^{ab} can be locally broadcast by party a if and only if the the quantum discord vanishes (i.e., correlations are classical-quantum).

In some sense, Theorem $1\frac{1}{2}$ interpolates between Theorem 1 and Theorem 2.

A unified picture:

Theorem 1 $\Leftrightarrow$ Theorem $1\frac{1}{2}$ $\Leftrightarrow$ Theorem 2

2. Decoherent Capabilities of Operations

$\mathcal{E}^b$: quantum operation on H^b

ρ^b: a quantum state on H^b, regarded as a partial state of a *pure* state $|\Psi^{ab}\rangle$ of a composite system $H^a \otimes H^b$ such that

$$\rho^b = \mathrm{tr}_a |\Psi^{ab}\rangle\langle\Psi^{ab}|.$$

Fundamental question: How to quantify the decoherence caused by $\mathcal{E}^b$?

Let $\rho^{ab} := |\Psi^{ab}\rangle\langle\Psi^{ab}|$ and $\mathcal{I}^a$ be the identity operation on the system H^a. Now consider the final bipartite state

$$\rho^{a'b'} := \mathcal{I}^a \otimes \mathcal{E}^b(\rho^{ab}).$$

Idea: We may study the decoherent effects of $\mathcal{E}^b$ by investigating the changes of classical and quantum correlations between the initial state ρ^{ab} and the final state $\rho^{a'b'}$.

$$\begin{tikzcd}
\rho^{a} \arrow[r, "\mathcal{I}^{a}"] \arrow[d, dash, "\rho^{ab} := |\Psi^{ab}\rangle\langle\Psi^{ab}|"] & \rho^{a'} \arrow[d, dash, "\rho^{a'b'} := \mathcal{I}^{a} \otimes \mathcal{E}^{b}(\rho^{ab})"] \\
\rho^{b} \arrow[r, "\mathcal{E}^{b}"] & \rho^{b'}
\end{tikzcd}$$

The decoherent information of $\mathcal{E}^b$ with respect to ρ^b is defined as

$$D(\rho^b, \mathcal{E}^b) := I(\rho^{ab}) - I(\rho^{a'b'}),$$

which is the loss of total (classical + quantum) correlations of the purification ρ^{ab} of ρ^b caused by the quantum operation $\mathcal{E}^b$.

We may decompose the decoherent information $D(\rho^b, \mathcal{E}^b)$ of $\mathcal{E}^b$ into a classical part and a quantum part.
The classical decoherent information (quantifies the loss of classical correlations):

$$D_{\mathrm{c}}(\rho^b, \mathcal{E}^b) := C(\rho^{ab}) - C(\rho^{a'b'}).$$

The quantum decoherent information (quantifies the loss of quantum correlations):

$$D_{\mathrm{q}}(\rho^b, \mathcal{E}^b) := Q(\rho^{ab}) - Q(\rho^{a'b'}).$$

By the definitions, we apparently have,

$$D(\rho^b, \mathcal{E}^b) = D_{\mathrm{c}}(\rho^b, \mathcal{E}^b) + D_{\mathrm{q}}(\rho^b, \mathcal{E}^b).$$

We may interpret $D(\rho^b, \mathcal{E}^b)$ as the total decoherence of ρ^b under the operation $\mathcal{E}^b$, which is separated into the classical decoherence $D_{\mathrm{c}}(\rho^b, \mathcal{E}^b)$ and the quantum decoherence $D_{\mathrm{q}}(\rho^b, \mathcal{E}^b)$.

To get some intrinsic quantities independent of ρ^b, we may define

$$D_{\mathrm{c}}(\mathcal{E}^b) := D_{\mathrm{c}}(\mathbf{1}^b/d, \mathcal{E}^b)$$

and

$$D_{\mathrm{q}}(\mathcal{E}^b) := D_{\mathrm{q}}(\mathbf{1}^b/d, \mathcal{E}^b)$$

as measures of classical decoherence and quantum decoherence, respectively, of $\mathcal{E}^b$. In this case, the initial state $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ is a maximally entangled pure state.

Alternatively, we may define

$$\overline{D}_{\mathrm{c}}(\mathcal{E}^b) := \max_{\rho^b} D_{\mathrm{c}}(\rho^b, \mathcal{E}^b)$$

and

$$\overline{D}_{\mathrm{q}}(\mathcal{E}^b) := \max_{\rho^b} D_{\mathrm{q}}(\rho^b, \mathcal{E}^b)$$

as measures of classical decoherence and quantum decoherence, respectively.

Open Question:

$$D_{\mathrm{c}}(\mathcal{E}^b) = \overline{D}_{\mathrm{c}}(\mathcal{E}^b), \quad D_{\mathrm{q}}(\mathcal{E}^b) = \overline{D}_{\mathrm{q}}(\mathcal{E}^b)?$$

The intuitive idea for guessing the above relations lies in that in order to evaluate the maximum decoherence of a quantum operation, it seems natural to start from a maximally entangled state, which is most vulnerable to decoherence.

Example 1. The Hadamard channel on a qubit:

$$\mathcal{E}^b(\rho^b) = M \circ \rho^b, \quad M = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}.$$

The classical and quantum decoherence are

$$D_{\mathrm{c}}(\mathcal{E}^b) = 0, \quad D_{\mathrm{q}}(\mathcal{E}^b) = H\Big(\frac{1-|\alpha|}{2}\Big).$$

The Hadamard channel is a purely quantum decoherent channel in the sense that its classical decoherence vanishes.

Example 2. The Pauli channel:

$$\mathcal{E}^b(\rho) = p_0\rho + \sum_{j=1}^{3} p_j\sigma_j\rho\sigma_j.$$

We have

$$D_{\mathrm{c}}(\mathcal{E}^b) = H(\frac{1-c}{2})$$

$$D_{\mathrm{q}}(\mathcal{E}^b) = -\sum_{j=0}^{3} p_j \log p_j - H(\frac{1-c}{2}).$$

The bit-flip channel is an example of the Pauli channel with
$\mathbf{p} = (p, 1-p, 0, 0),\ p \in [0, 1]$.
We have

$$D_{\mathrm{c}}(\mathcal{E}^b) = 0, \quad D_{\mathrm{q}}(\mathcal{E}^b) = H(p).$$

This means that the bit-flip channel is also a purely quantum decoherent channel.

Example 3. The amplitude damping channel:

$$\mathcal{E}^b(\rho) = E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger$$

with

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} D_{\mathrm{c}}(\mathcal{E}^b) &= H\left(\frac{1+\sqrt{1-p}}{2}\right), \\ D_{\mathrm{q}}(\mathcal{E}^b) &= 1 - H\left(\frac{1+p}{2}\right) - H\left(\frac{1+\sqrt{1-p}}{2}\right) \\ &\quad + H\left(\frac{p}{2}\right). \end{aligned}$$

The amplitude damping channel is hybrid.

Example 4. The phase damping channel

$$\mathcal{E}^b(\rho) = E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger$$

with

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

$$D_{\mathrm{c}}(\mathcal{E}^b) = 0, \quad D_{\mathrm{q}}(\mathcal{E}^b) = H\Big(\frac{1-\sqrt{1-p}}{2}\Big).$$

The phase damping channel is purely quantum.

Interesting observations:

(1) The classical decoherence vanishes for the Hadamard channel, the bit-flip channel and the phase damping channel. In these channels, quantum correlations decohere without the decoherence of classical correlations, and consequently the decoherence is purely quantum.

(2) For amplitude damping channel, both the classical decoherence and quantum decoherence are nonzero. Moreover, the classical decoherence dominates the quantum decoherence, i.e., classical correlations decohere more rapidly than quantum correlations.

(3) Although the total decoherence $D_{\mathrm{c}}(\mathcal{E}^b) + D_{\mathrm{q}}(\mathcal{E}^b)$ of the amplitude channel is larger than that of the phase damping channel, the quantum decoherence is smaller. Moreover, the amplitude damping channel causes both classical and quantum decoherence, while the phase damping channel causes only quantum decoherence. The classical decoherence of the amplitude damping channel turns out to be equal to the quantum decoherence of the phase damping channel.

3. Questions

The Araki-Lieb inequality $|S(\rho^a) - S(\rho^b)| \leq S(\rho^{ab})$ is actually equivalent to

$$I(\rho^{ab}) \leq 2\min\{S(\rho^a), S(\rho^b)\}.$$

Conjecture 1. For the classical correlations $C(\rho^{ab})$ and the quantum correlations $Q(\rho^{ab})$,

$$\begin{aligned} C(\rho^{ab}) &\leq \min\{S(\rho^a), S(\rho^b)\}, \\ Q(\rho^{ab}) &\leq \min\{S(\rho^a), S(\rho^b)\}. \end{aligned}$$

Here $\rho^a = \mathrm{tr}_b \rho^{ab}$ and $\rho^b = \mathrm{tr}_a \rho^{ab}$ are the two marginals.

Supporting evidence:
(1) True for any pure state ρ^{ab} since $C(\rho^{ab}) = Q(\rho^{ab}) = S(\rho^{a}) = S(\rho^{b})$.
(2) True for any product state ρ^{ab} since $C(\rho^{ab}) = Q(\rho^{ab}) = 0$.
(3) True for any classical-quantum state ρ^{ab}.

Classical correlations may decohere either more slowly or more rapidly than quantum correlations.

There are channels with purely quantum decoherence ($D_{\mathrm{q}}(\mathcal{E}^b) > 0$) without classical decoherence ($D_{\mathrm{c}}(\mathcal{E}^b) = 0$).

Conjecture 2. There does not exist a channel $\mathcal{E}^b$ such that $D_{\mathrm{c}}(\mathcal{E}^b) > 0$ while $D_{\mathrm{q}}(\mathcal{E}^b) = 0$.

The point here is that both the classical decoherence $D_{\mathrm{c}}(\mathcal{E}^b)$ and quantum decoherence $D_{\mathrm{q}}(\mathcal{E}^b)$ are defined from initial joint states ρ^{ab} which, as purifications of $\rho^b = \mathbf{1}^b/d$, are *pure*.

When the initial joint state can be mixed, Xu *et al.* (Nat. Phys. 2010) have demonstrated experimentally, and Mazzola *et al.* (PRL, 2010) have shown theoretically, that there exists classical decoherence without quantum decoherence.

Quantum correlations cannot exits alone without classical correlations, they are actually parasitized on classical correlations: They can be wiped out without changing the underlying classical correlations, but whenever we wipe out certain amount of classical correlations, we have to wipe out some parasite (quantum correlations) on them. This is an intuitive underlying rationale leading to Conjecture 2.

Conjecture 2 naturally motivates the following closely related (and presumably equivalent) conjecture concerning the classical correlations and quantum correlations (as measured by the quantum discord) in any bipartite state.

Conjecture 3. There does not exist a bipartite state ρ^{ab} such that $Q(\rho^{ab}) = S(\rho^b)$ and $C(\rho^{ab}) < S(\rho^b)$.

All the three above conjectures have equivalent formulations when $C(\cdot)$ and $Q(\cdot)$ are replaced by $C_2(\cdot)$ and $Q_2(\cdot)$, respectively.