Hilbert Uniqueness Method and Regularity: Applications to the order of convergence of discrete controls for the wave equation

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Outline of the talk

1. Introduction: The Hilbert Uniqueness Method

2. An alternate HUM type method

3. Application: the order of convergence of discrete controls
1. **Introduction: The Hilbert Uniqueness Method**

2. An alternate HUM type method

3. Application: the order of convergence of discrete controls
An abstract control problem

Let $\mathcal{A}$ be a skew-adjoint operator defined on a Hilbert space $\mathcal{X}$. Consider the following model:

$$y'(t) = \mathcal{A} y(t) + B v(t), \quad y(0) = y^0 \in \mathcal{X},$$

where $B \in \mathcal{L}(\mathcal{Y}, \mathcal{D}(\mathcal{A})^*)$ and $v \in L^2(0, T; \mathcal{Y})$.

**Assumption**

For all $v \in L^2(0, T; \mathcal{Y})$, solutions can be defined in the sense of transposition in $C^0([0, T]; \mathcal{X})$.

**Goal : Exact controllability**

Fix a time $T > 0$ and $y^0 \in \mathcal{X}$. Can we find $v \in L^2(0, T; \mathcal{Y})$ such that $y(T) = 0$?
Hypotheses

- \( \mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X} \) is a **skew-adjoint** operator.
  \[ \implies \text{The energy } \|z(t)\|^2_\mathcal{X} \text{ of solutions is constant.} \]

- \( \mathcal{A} \) has **compact resolvent**.
  \[ \implies \text{Its spectrum is discrete.} \]

\[ \leadsto \text{Spectrum of } \mathcal{A}: \]

\[ \sigma(\mathcal{A}) = \{ i\mu^j : j \in \mathbb{N} \}, \]

where \((\mu^j)_{j \in \mathbb{N}}\) is an increasing sequence of real numbers, corresponding to an **orthonormal basis** \((\psi^j)_{j \in \mathbb{N}}\)

\[ \mathcal{A}\psi^j = i\mu^j\psi^j. \]
Examples

- **Wave equation** in a bounded domain + BC with distributed control

\[
\begin{cases}
  u'' - \Delta u = \chi_\omega v, & (t, x) \in \mathbb{R} \times \Omega, \\
  u|_{\partial \Omega} = 0, \\
  (u(0), \dot{u}(0)) = (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega),
\end{cases}
\]

\[
A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix}, \quad X = H^1_0(\Omega) \times L^2(\Omega),
\]

\[
B = \begin{pmatrix} 0 \\ \chi_\omega \end{pmatrix}, \quad Y = L^2(\omega).
\]

- **Wave equation** in a bounded domain + BC with boundary control

- **Schrödinger equation** \( A = -i\Delta + \text{BC} \), Linearized KdV

\( A = \partial_{xxx} + \text{BC} \), Maxwell equation, . . .
Use the adjoint system to characterize the controls!

For all $z$ solution of

$$z' = A\, z, \quad z(0) = z^0 \in \mathcal{X},$$

we have

$$\langle y(T), z(T) \rangle_{\mathcal{X}} - \langle y^0, z^0 \rangle_{\mathcal{X}} = \int_0^T \langle v(t), B^* z(t) \rangle_{\mathcal{Y}} \, dt.$$  

In particular, $v$ is a control if and only if $\forall z^0 \in \mathcal{X}$

$$0 = \int_0^T \langle v(t), B^* z(t) \rangle_{\mathcal{Y}} \, dt + \langle y^0, z^0 \rangle_{\mathcal{X}}.$$
Fundamental hypotheses

- $\mathcal{B}^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{Y}$, $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}), \mathcal{Y})$.

**Definition**

$\mathcal{B}^*$ is **admissible** if $\forall T > 0$, $\exists K_T > 0$,

$$\int_0^T \| \mathcal{B}^* z(t) \|_\mathcal{Y}^2 \, dt \leq K_T \| z^0 \|_\mathcal{X}^2,$$

$\forall z^0 \in \mathcal{D}(\mathcal{A})$.

**Definition**

$\mathcal{B}^*$ is **exactly observable** at time $T^* > 0$ if $\exists k_* > 0$,

$$k_* \| z^0 \|_\mathcal{X}^2 \leq \int_0^{T^*} \| \mathcal{B}^* z(t) \|_\mathcal{Y}^2 \, dt,$$

$\forall z^0 \in \mathcal{X}$.
Let $T \geq T^*$. Define, for $z^0 \in X$, 

$$J(z^0) = \frac{1}{2} \int_0^T \|B^* z(t)\|_Y^2 \, dt + \langle y^0, z^0 \rangle,$$

where $z$ satisfies $z' = Az$, $z(0) = z^0$. **Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z^0$.** Then $v = B^* Z$ is such that the solution $y$ of 

$$y' = Ay + Bv, \quad y(0) = y^0,$$

satisfies $y(T) = 0$. Besides, $v$ is the control of minimal $L^2(0, T; Y)$-norm.
A regularity problem

On the regularity

If \( y^0 \in \mathcal{D}(A) \),

- Does the function \( Z^0 \) computed that way belongs to \( \mathcal{D}(A) \)?
- Is the controlled solution \((y, v)\) a strong solution?
  i.e. \( y \in C^1([0, T]; \mathcal{X}) \)

General Answer: NO!
Consider the wave equation

\[
\begin{cases}
    w_{tt} - w_{xx} = 0, & 0 < x < 1, \ 0 < t < T, \\
    w(0, t) = 0, \ w(1, t) = v(t), & 0 < t < T, \\
    (w(x, 0), w_t(x, 0)) = (w^0(x), w^1(x)) & \in L^2(0, 1) \times H^{-1}(0, 1).
\end{cases}
\]

The adjoint problem is

\[
q_{tt} - q_{xx} = 0, \quad q(0, t) = q(1, t) = 0, \quad (q^0, q^1) \in H^1_0(0, 1) \times L^2(0, 1),
\]

and the solutions write

\[
q = \sqrt{2} \sum_{k \geq 1} \left( \hat{q}^0_k \cos(k \pi t) + \frac{\hat{q}^1_k}{k \pi} \sin(k \pi t) \right) \sin(k \pi x),
\]

**Controllability in time \( T = 4 \):**

If \((w^0(x), w^1(x)) = \sqrt{2} \sum_{k \geq 1} (\hat{w}^0_k, \hat{w}^1_k) \sin(k \pi x),\)

\[
\hat{Q}^0_k = \frac{\hat{w}^1_k}{4k^2 \pi^2}, \quad \hat{Q}^1_k = -\frac{\hat{w}^0_k}{4}.
\]
In particular, the HUM control can be computed explicitly

\[ v(t) = Q_x(1, t) = \frac{1}{4} \sum_{k \geq 1} (-1)^k k \pi \left( \frac{\hat{w}_k^1}{k^2 \pi^2} \cos(k \pi t) - \frac{\hat{w}_k^0}{k \pi} \sin(k \pi t) \right). \]

\[ \Rightarrow v(0) = \frac{1}{4} \sum_{k \geq 1} (-1)^k \frac{\hat{w}_k^1}{k \pi} \neq 0 ! \]

\[ \Rightarrow \text{If } w^0 \in H^1_0(0, 1), \text{ the controlled solution is not a strong solution in general because of the failure of the compatibility conditions } w^0(1) = v(0) = 0. \]
Main question

How to construct a control method which respects the regularity of the solutions?

If \( y^0 \in \mathcal{D}(\mathcal{A}) \), we want

- \( Z^0 \in \mathcal{D}(\mathcal{A}) \)
- the controlled equation \( y' = \mathcal{A}y + \mathcal{B}v \) is satisfied in the strong sense.

Related result - Dehman Lebeau 2009:
The wave equation with distributed control \( \mathcal{B} = \chi_\omega \) where \( \chi_\omega \) is smooth, and where the HUM operator is modified by a function \( \eta(t) \) vanishing at \( t \in \{0, T\} \).
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The modified HUM method

Let $y^0 \in \mathfrak{X}$, and $\delta > 0$ such that $T - 2\delta \geq T^*$, where $T^*$ is the time of observability. Define, for $z^0 \in \mathfrak{X}$,

$$J(z^0) = \frac{1}{2} \int_0^T \eta(t) \|B^*z(t)\|_Y^2 \ dt + \langle y^0, z^0 \rangle,$$

where $z$ satisfies $z' = Az$, $z(0) = z^0$ and

$$\eta \in C^\infty(\mathbb{R}), \quad \eta = \begin{cases} 
0 & \text{on } (-\infty, 0] \cup [T, \infty) \\
1 & \text{on } [\delta, T - \delta]
\end{cases} \quad \eta \geq 0.$$

Observability $\Rightarrow$ Existence and Uniqueness of a minimizer $Z^0$. Then $v = \eta B^*Z$ is such that the solution $y$ of

$$y' = Ay + Bv, \quad y(0) = y^0,$$

satisfies $y(T) = 0$.

Besides, $v$ is the control of minimal $L^2((0, T), dt/\eta; Y)$-norm.
Main result

Theorem (SE Zuazua)

Assume that admissibility and observability property hold. If \( y^0 \in D(A) \), then the minimizer \( Z^0 \) computed by the above method and the control function \( v = \eta B^* Z \) are more regular:

- \( Z^0 \in D(A) \),
- \( v \in H^1_0(0, T; \mathcal{Y}) \).

In particular, the controlled solution \( y \) with control \( v \) is a strong solution of the controlled equation. Moreover, there exists a constant \( C = C(\eta) \) such that

\[
\| Z^0 \|_{D(A)} + \| v \|_{H^1_0(0, T; \mathcal{Y})} \leq C \| y^0 \|_{D(A)}.
\]
Before the proof

First remark that, due to the classical observability property,

\[ \|Z^0\|_\mathcal{X} + \|v\|_{L^2(0,T;\mathcal{Y})} \leq C \|y^0\|_\mathcal{X}. \]

Also remark that admissibility and observability properties yield

\[ k \|Z^0\|_{\mathcal{D}(\mathcal{A})} \leq \int_0^T \eta(t) \|B^*Z'(t)\|_{\mathcal{Y}}^2 \, dt \leq K \|Z^0\|_{\mathcal{D}(\mathcal{A})}. \]

\[ \rightarrow \] It is sufficient to prove that

\[ \int_0^T \eta(t) \|B^*Z'(t)\|_{\mathcal{Y}}^2 \, dt < \infty. \]

Indeed, this implies \( Z^0 \in \mathcal{D}(\mathcal{A}) \) and \( v \in H^1_0(0,T;\mathcal{Y}) \).
Idea of the proof

Write the characterization of the control $v = \eta B^* Z$:

$$0 = \int_0^T \eta(t) \langle B^* Z(t), B^* z(t) \rangle_Y \, dt + \langle y^0, z^0 \rangle_x,$$

for all $z$ solution of $z' = Az$, $z(0) = z^0$. Then take formally $z = Z'' = A^2 Z$:

$$\int_0^T \eta(t) \| B^* Z'(t) \|^2_Y \, dt = -\langle Ay^0, AZ^0 \rangle_x$$

$$- \int_0^T \eta'(t) \langle B^* Z'(t), B^* Z(t) \rangle_Y \, dt.$$

Using observability,

$$\int_0^T \eta(t) \| B^* Z'(t) \|^2_Y \, dt \leq C(\| \eta' \|_{\infty}) \| y^0 \|_{DA}^2.$$
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The 1d wave equation

\[
\begin{cases}
    w_{tt} - w_{xx} = 0, & 0 < x < 1, 0 < t < T, \\
    w(0, t) = 0, \ w(1, t) = v(t), & 0 < t < T, \\
    (w(x, 0), w_t(x, 0)) = (w^0(x), w^1(x)) \in L^2(0, 1) \times H^{-1}(0, 1).
\end{cases}
\]

The adjoint problem is

\[
q_{tt} - q_{xx} = 0, \quad q(0, t) = q(1, t) = 0, \quad (q^0, q^1) \in H^1_0(0, 1) \times L^2(0, 1),
\]

Controllability is OK for \( T \geq T^* = 2 \).

Assume $T > T^* = 2$ and $\eta$ vanishing at $t = 0, T$.

Initial data to be controlled: $(w^0, w^1) \in H^{-1}(\Omega) \times L^2(\Omega)$.

Minimize the functional

$$J(q^0, q^1) = \frac{1}{2} \int_0^T \eta |\partial_x q(1, t)|^2 \, dt + \langle w^1, q^0 \rangle_{H^{-1} \times H^1_0} - \int_\Omega w^0 q^1.$$ 

over $(q^0, q^1) \in H^1_0(\Omega) \times L^2(\Omega)$, $q$ solution of the adjoint problem.
Minimizer $= (Q^0, Q^1)$.

Then $v = \eta \partial_x Q(1, t)$ is the control of minimal $L^2((0, T), dt/\eta)$-norm.
Our result

Theorem

If \((w^0, w^1) \in H_0^1(0, 1) \times L^2(0, 1)\), then
\((Q^0, Q^1) \in H^2 \cap H_0^1(0, 1) \times H_0^1(0, 1)\) and \(v \in H_0^1(0, T)\).

Besides, there exists a constant \(C\) independant of \((w^0, w^1)\) such that

\[
\left\|(Q^0, Q^1)\right\|_{H^2 \cap H_0^1(0, 1) \times H_0^1(0, 1)} \leq C \left\|(w^0, w^1)\right\|_{H_0^1(0, 1) \times L^2(0, 1)},
\]

\[
\|v\|_{H_0^1(0, T)} \leq C \left\|(w^0, w^1)\right\|_{H_0^1(0, 1) \times L^2(0, 1)}.
\]
The 1-d discrete case

Space semi-discretization (finite difference, $h = \frac{1}{N+1}$)

$$
\begin{align*}
  w_j'' - \frac{1}{h^2}(w_{j-1} + w_{j+1} - 2w_j) &= 0, \quad j \in \{1, \cdots, N\}, \quad t \geq 0, \\
  w_0(t) &= 0, \quad w_{N+1}(t) = v(t), \quad t \geq 0.
\end{align*}
$$

\textbf{Figure:} Left, the initial data $u(0)$. Right, the HUM control for the continuous system for initial data $(u(0), 0)$. 
Numerical experiments

Figure: Discrete controls for different values of $N$. 

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Convergence rates of discrete controls
Discrete schemes are not uniformly observable.

Figure: Discrete Spectrum vs Continuous Spectrum.

 Filtering techniques are needed.
Spectrum of the discrete Laplace operator:

\[-\Delta_h \varphi = \lambda \varphi, \quad \varphi_0 = \varphi_{N+1} = 0\]

is given by the sequence \((\varphi^k, \lambda^k(h))\) (\(k \in \{1, \cdots, N\}\)):

\[\varphi_j^k = \sqrt{2} \sin(k\pi jh), \quad j \in \{1, \cdots, N\}, \quad \lambda^k(h) = \frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2}\right).\]

Define, for \(\gamma \in (0, 4)\),

\[C_h(\gamma) = \text{Span} \left\{ \varphi_k, \lambda^k(h) \leq \frac{\gamma}{h^2} \right\}\]

and the orthogonal projection \(\pi^h_\gamma\) over \(C_h(\gamma)\).
Let $\gamma \in (0, 4)$ and $T > 2/(1 - \gamma/4)$. Consider a sequence

$$(w_h^0, w_h^1) \xrightarrow{h \to 0} (w^0, w^1) \text{ in } L^2(0, 1) \times H^{-1}(0, 1).$$

Define the functionals

$$J_h(q_h^0, q_h^1) = \frac{1}{2} \int_0^T \eta(t) \left| \frac{q_N}{h} \right|^2 dt + \langle w_h^1, q_h^0 \rangle_{H^{-1} \times H^1} - \int_{\Omega} w_h^0 q_h^1,$$

where $q$ is the solution of

$$\begin{cases}
q''_j - \frac{1}{h^2} (q_{j-1} + q_{j+1} - 2q_j) = 0, & j \in \{1, \cdots, N\}, \ t \geq 0, \\
q_0(t) = 0, \quad q_{N+1}(t) = 0, & t \geq 0. \\
(q_j(0), q'_j(0)) = (q_j^0, q_j^1).
\end{cases}$$
The functionals

\[ J_h(q_h^0, q_h^1) = \frac{1}{2} \int_0^T \eta(t) \left| \frac{q_N}{h} \right|^2 dt + \langle w_h^1, q_h^0 \rangle_{H^{-1}_h \times H^1_h} - \int_{\Omega} w_h^0 q_h^1, \]

have a unique minimizer \((Q_h^0, Q_h^1)\) on \(C_h(\gamma)^2\). The functions

\[ v_h(t) = -\eta(t) \frac{Q_N(t)}{h} \]

are such that the solution \(y_h\) of the discrete wave equation with initial data \((y_h^0, y_h^1)\) and control function \(v_h\) satisfies

\[ \pi^h_{\gamma}(y_h(T), y_h'(T)) = (0, 0). \]

Moreover, \((v_h) \longrightarrow v\) strongly in \(L^2(0, T; dt/\eta)\), where \(v\) is the HUM control of the continuous wave equation for \((w^0, w^1)\).
Order of convergence

Approximation of smooth data

\[ \exists C \text{ independent of } h > 0 \text{ such that} \]
\[ \forall (w^0, w^1) \in H^1_0(0, 1) \times L^2(0, 1), \text{ there exists a sequence} \]
\[ (w^0_h, w^1_h) \text{ of discrete data such that} \forall h > 0, \]
\[ \| (w^0_h, w^1_h) \|_{H^1_0 \times L^2} \leq C \| (w^0, w^1) \|_{H^1_0 \times L^2} \]
\[ \| (w^0_h, w^1_h) - (w^0, w^1) \|_{L^2 \times H^{-1}} \leq C h \| (w^0, w^1) \|_{H^1_0 \times L^2}. \]
Theorem (SE & Zuazua)

∃C independent of $h > 0$ such that for all $(w^0, w^1) \in H^1_0(0, 1) \times L^2(0, 1)$, the discrete controls $v_h$ computed for the discrete data $(w^0_h, w^1_h)$ given above satisfy:

$$\|v_h - v\|_{L^2(0,T;dt/\eta)} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}$$

First result on the order of convergence of discrete controls.
For \((w^0, w^1) \in H^1_0(0, 1) \times L^2(0, 1)\), the control is \(v = \eta(t)\partial_x Q(1, t)\) for a solution \(Q\) of the adjoint wave equation, with initial data \((Q^0, Q^1) \in (H^2 \cap H^1_0(0, 1) \times H^1_0(0, 1)) \cap C_h(\gamma)^2\).

One can approximate \((Q^0, Q^1)\) and \(Q\) by discrete data \((\tilde{Q}^0_h, \tilde{Q}^1_h)\) such that

\[
\left\| (\tilde{Q}^0_h, \tilde{Q}^1_h) \right\|_{H^2 \cap H^1_0 \times H^1_0} \leq C \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}.
\]

Set \(\tilde{v}_h = \eta(t) \frac{\tilde{Q}_{N,h}}{h}\):

\[
\left\| \tilde{v}_h - v \right\|_{L^2(0,T; dt/\eta)} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}.
\]
**The control** $\tilde{v}_h = \eta(t) \frac{\tilde{Q}_{N,h}}{h}$ **is an approximate control for the discrete equations:** if $\tilde{w}_h$ denotes the solution of the discrete equation with control $\tilde{v}_h$, we have

$$\| (\tilde{w}_h(T), \tilde{w}_h'(T)) \|_{L^2 \times H^{-1}} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}.$$

**Compute the control** $\hat{v}_h$ **of minimal $L^2(0, T; dt/\eta)$ norm such that**

$$\begin{cases}
    p_j'' - \frac{1}{h^2}(p_{j-1} + p_{j+1} - 2p_j) = 0, & j \in \{1, \cdots, N\}, t \geq 0, \\
    p_0(t) = 0, \quad p_{N+1}(t) = \hat{v}_h(t), & t \geq 0, \\
    (p_h(0), p_h'(0)) = (0, 0), \quad (p_h(T), p_h'(T)) = -(\tilde{w}_h(T), \tilde{w}_h'(T))
\end{cases}$$

$$\Rightarrow \hat{v}_h = -\eta(t) \frac{\hat{Q}_h}{h}, \hat{Q}_h \text{ solution of the discrete adjoint system:}$$

$$\| \hat{v}_h \|_{L^2(0, T; dt/\eta)} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}.$$
The function $\tilde{v}_h + \hat{v}_h$ is a discrete exact control which can be written as

$$\tilde{v}_h + \hat{v}_h = -\eta \frac{Q_{N,h}}{h},$$

where $Q_h$ is a solution of the discrete adjoint system in $C_h(\gamma)$. Uniqueness of such exact controls $\longrightarrow v_h = \tilde{v}_h + \hat{v}_h$

$$\|v_h - v\|_{L^2(0,T;dt/\eta)} \leq \|\tilde{v}_h - v\|_{L^2(0,T;dt/\eta)} + \|\hat{v}_h\|_{L^2(0,T;dt/\eta)} \leq C h^{2/3} \left\| (w^0, w^1) \right\|_{H^1_0 \times L^2}.$$
Remark that $\sqrt{\lambda^k(h)} = \frac{2}{h} \sin \left( \frac{k\pi h}{2} \right) \sim k\pi$ for $k = o(h^{-2/3})$

$\Rightarrow$ Convergence of the eigenvalues OK at scale $h^{-2/3}$.

See also Baker SIAM JNA ’76 and Rauch SIAM JNA ’85: Distance between the continuous and semi-discrete semi-groups is exactly $h^{2/3}$.

- **Optimality** of this rate of convergence?
- **Applications to other situations:**
  - Different numerical methods:
    - ★ finite element (Infante Zuazua ’99, SE ’09),
    - ★ mixed finite elements (Castro Micu ’06, SE’09),
    - ★ bi-grid techniques (Negreanu Zuazua ’04)
  - Higher dimensions

$\longrightarrow$ See Zuazua’s Survey ’05 for extensive references)
Thank you for your attention!