Optimal transport techniques for the one-dimensional sticky particle system

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Outline

Starting point: motion of a finite number of particles.

Discrete particle model

N particles $P_i := (m_i, x_i, v_i)$, $i = 1, \ldots, N$,
with positive mass m_i satisfying $\sum_{i=1}^{N} m_i = 1$
ordered positions $x_1 < x_2 < \ldots < x_{N-1} < x_N$,
and velocities v_i.

At the initial time $t = 0$ the particles are disjoint and start to move freely with constant velocity:

$$x_i(t) := x_i(0) + v_i(0)t, \quad v_i(t) := v_i.$$

The **first collision time** $t = t^1$ correspond to

$$x_j(t^1) = x_{j+1}(t^1) = \ldots = x_k(t^1) \quad \text{for some indices } j < k.$$

The particles $P_j, P_{j+1}, \ldots, P_k$ **collapse in a new particle** P with **mass** $m := m_j + \ldots + m_k$ and

"barycentric" velocity $v := \dfrac{m_j v_j(t^1) + m_{j+1} v_{j+1}(t^1) + \ldots + m_k v_k(t^1)}{m}$

After the collision the particles $P_j, \ldots, P_k$ **stick** in the same big particle P which freely moves with fixed velocity v up to the next collision t^2.

Measure-theoretic description

We thus have:
a **(finite) sequence of collision times** $0 < t^1 < t^2 < \dots$
in each interval $[t^h, t^{h+1})$ a finite number N^h of (suitably relabelled) particles $P_1(t), \cdots, P_{N^h}(t)$, $P_i(t) := (m_i, x_i(t), v_i(t))$.

We can introduce the measures

$$\rho_t := \sum_{i=1}^{N^h} m_i \delta_{x_i(t)} \in \mathcal{P}(\mathbb{R}) \quad (\rho\, v)_t := \sum_{i=1}^{N^h} m_i\, v_i\, \delta_{x_i(t)} \in \mathcal{M}(\mathbb{R}) \quad \text{if } t \in [t^h, t^{h+1}).$$

They satisfy the **1-dimensional pressureless Euler system** in the sense of distributions

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = 0, \end{cases} \quad \text{in } \mathbb{R} \times (0, +\infty); \quad \rho\big|_{t=0} = \rho_0, \quad v\big|_{t=0} = v_0,$$

and the Oleinik entropy condition

$$v_t(x_2) - v_t(x_1) \le \frac{1}{t}(x_2 - x_1) \quad \text{for } \rho_t\text{-a.e. } x_1, x_2 \in \mathbb{R},\ x_1 \le x_2.$$

Main problem: continuous limit

Consider a sequence of discrete initial data $\mu_0^n := (\rho_0^n, \rho_0^n v_0^n)$ converging to $\mu_0 = (\rho_0, \rho_0 v_0)$ in a suitable measure-theoretic sense and let $\mu_t^n = (\rho_t^n, \rho_t^n v_t^n)$ be the (discrete) solution of SPS.

Problem

- *Prove that the limit* $\mu_t = (\rho_t, \rho_t v_t)$ *of the SPS* $\mu_t^n = (\rho_t^n, \rho_t^n v_t^n)$ *as* $n \uparrow +\infty$ *exists.*
- *Find a suitable characterization of* μ_t
- *Show that* $(\rho_t, \rho_t v_t)$ *solves the pressureless Euler system*

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = 0, \end{cases} \qquad \text{in } \mathbb{R}\times(0,+\infty); \quad \rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0,$$

and satisfy Oleinik entropy condition.

Main contributions

- Existence and convergence:
  - GRENIER '95, E-RYKOV-SINAI '96: first existence and convergence result.
  - BRENIER-GRENIER '96: Characterization of the limit in terms of a suitable scalar conservation law, uniqueness.
  - HUANG-WANG '01, NGUYEN-TUDORASCU '08, MOUTSINGA '08: further refinements.

Basic assumptions:

$\rho_0^n \to \rho_0$ in the L^2-Wasserstein distance,

$v_0^n = v_0$ is given by a continuous function with (at most) linear growth.

In particular the result cover the case when ρ_0^n, ρ_0 have a common compact support and $\rho_0^n \to \rho_0$ weakly in the sense of distribution (or, equivalently, in the duality with continuous functions).

- Pioneering ideas which lies (more or less explicitly) at the core of the papers by E-RYKOV-SINAI and BRENIER-GRENIER have been introduced by
  - SHNIRELMAN '86 and further clarified by
  - ANDRIEVWSKY-GURBATOV-SOBOELVSKIĬ '07 in a formal way.
- Different approaches and models:
  - BOUCHUT-JAMES '95, POUPAUD-RASCLE '97
  - SOBOLEVSKIĬ '97, BOUDIN '00: viscous regularization.
  - WOLANSKY '07: particles with finite size.

The Brenier-Grenier formulation

For every probability measure $\rho \in \mathcal{P}(\mathbb{R})$ we introduce the **cumulative distribution function**

$$M_\rho(x) := \rho\big((-\infty, x]\big), \quad x \in \mathbb{R}, \quad \text{so that } \rho = \partial_x M_\rho \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Main idea: study the evolution of $M_t := M_{\rho_t}$, where ρ_t is the solution of the SPS.

Theorem (Brenier-Grenier '96)

M is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \textit{in } \mathbb{R} \times (0, +\infty)$$

where $A : [0,1] \to \mathbb{R}$ is a continuous flux function depending only on the initial data ρ_0 and v_0. It is characterized by

$$A'(M_0(x)) = v_0(x).$$

Monotone rearrangement

Point of view of 1-dimensional optimal transport: instead of using the cumulative distribution function $M_\rho(x) = \rho\big((-\infty, x]\big)$, we

represent each probability measure ρ by its monotone rearrangement $X_\rho : (0,1) \to \mathbb{R}$

$$X_\rho(w) := \inf\Big\{x \in \mathbb{R} : M_\rho(x) > w\Big\} \quad w \in (0,1)$$

which is the so-called pseudo-inverse of M_ρ.

The map X_ρ is **nondecreasing and right-continuous**.
It pushes the Lebesgue measure $\lambda := \mathscr{L}^1|_{(0,1)}$ on $(0,1)$ onto ρ, i.e.

$$(X_\rho)_\# \mathscr{L}^1|_{(0,1)} = \rho, \qquad \mathscr{L}^1(X_\rho^{-1}(B)) = \rho(B) \quad \text{for every Borel set } B \subset \mathbb{R}$$

It satisfies the change of variable formula

$$\int_{\mathbb{R}} \phi(x)\,\mathrm{d}\rho(x) = \int_0^1 \phi(X_\rho(w))\,\mathrm{d}w$$

for every nonnegative/bounded Borel function $\phi : \mathbb{R} \to \mathbb{R}$. In particular,

$$\mathfrak{m}_2(\rho) := \int_{\mathbb{R}} |x|^2\,\mathrm{d}\rho(x) = \int_0^1 \big|X_\rho(w)\big|^2\,\mathrm{d}w = \big\|X_\rho\big\|^2_{L^2(0,1)}$$

Wasserstein distance and the L^2 isometry

The map $\rho \mapsto X_\rho$ is a **one-to-one correspondence** between

the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moment
$\mathfrak{m}_2(\rho) = \int_{\mathbb{R}} |x|^2 \,\mathrm{d}\rho(x) < +\infty$

and

the closed convex cone $\mathcal{K}$ of all the nondecreasing function in $L^2(0,1)$ (among which we can always choose the right-continuous representative).

L^2-Wasserstein distance

$W_2(\rho^1, \rho^2)$ between $\rho^1, \rho^2 \in \mathcal{P}_2(\mathbb{R})$:

$$W_2^2(\rho^1, \rho^2) := \int_0^1 \big|X_{\rho^1}(w) - X_{\rho^2}(w)\big|^2 \,\mathrm{d}w = \big\|X_{\rho^1} - X_{\rho^2}\big\|^2_{L^2(0,1)}$$

In this way $\rho \leftrightarrow X_\rho$ is an **isometry** between $(\mathcal{P}_2(\mathbb{R}), W_2)$ and $(\mathcal{K}, \|\cdot\|_{L^2(0,1)})$.

A description of the evolution by the L^2-projection on $\mathcal{K}$

We denote by $\mathsf{P}_{\mathcal{K}}(Y)$ **the L^2 projection of $Y \in L^2(0,1)$ on $\mathcal{K}$**, i.e.

$$X = \mathsf{P}_{\mathcal{K}}(Y) \quad \Leftrightarrow \quad \|Y - X\| = \min\Big\{\|Y - Z\| : Z \in \mathcal{K}\Big\}$$

To the (discrete) data $\mu_t = (\rho_t, \rho_t v_t)$ we associate the functions $(X_t, V_t) \in \mathcal{K} \times L^2(0,1)$ by

$$X_t := X_{\rho_t}, \quad V_t := v_t \circ X_t.$$

Notice that the second component of (X_t, V_t) do not span the whole space $L^2(\mathbb{R})$ in general, but it is contained in the closed subspace

$$\mathcal{H}_{X_t} := \Big\{V = v \circ X_t \text{ for some Borel map } v \in L^2_{\rho_t}(\mathbb{R}).\Big\}$$

Theorem (First Lagrangian representation)

A family $\mu_t = (\rho_t, \rho_t v_t)$ is a solution of the (discrete) SPS if and only if

$$X_t = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0).$$

Stability properties (I)

Let $\tilde{X}_t := X_0 + tV_0$ be associated to the measures

$$\tilde{\rho}_t = (\tilde{X}_t)\#\mathscr{L}^1|_{(0,1)} = \sum_{i=1}^{N} m_i \delta_{x_i + tv_i}$$

corresponding to the **"collision free" motion, when the particles do not see each other.**

Thus the solution X_t can be computed by "projecting" the Lagrangian free motion $\tilde{X}_t$ on $\mathcal{K}$.

Since the projection operator $\mathsf{P}_{\mathcal{K}}$ is a contraction in $L^2(0,1)$, the representation formula

$$X_t = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0) = \mathsf{P}_{\mathcal{K}}(\tilde{X}_t)$$

easily yields

Corollary

If X_t^1, X_t^2 are the Lagrangian representation of two (discrete) solutions ρ_t^1, ρ_t^2 of the SPS we have

$$\|X_t^1 - X_t^2\|_{L^2(0,1)} \le \|X_0^1 - X_0^2\|_{L^2(0,1)} + t\|V_0^1 - V_0^2\|_{L^2(0,1)}.$$

The polar cone and the subdifferential of the indicator function of $\mathcal{K}$

The polar cone $\mathcal{K}^\circ$ is defined by

$$Y \in \mathcal{K}^\circ \quad \Leftrightarrow \quad (Y|Z) \le 0 \quad \text{for every } Z \in \mathcal{K}$$

$\mathrm{I}_\mathcal{K}$ is the indicator (convex, lower semicontinuous) function of $\mathcal{K}$ in $L^2(0,1)$

$$\mathrm{I}_\mathcal{K}(X) = \begin{cases} 0 & \text{if } X \in \mathcal{K}, \\ +\infty & \text{otherwise,} \end{cases}$$

with subdifferential $\partial \mathrm{I}_\mathcal{K} : L^2(0,1) \to 2^{L^2(0,1)}$.

$\Xi \in \partial \mathrm{I}_\mathcal{K}(X)$ if and only if $X \in \mathcal{K}$ and one of the following equivalent conditions holds

$$(\Xi|Z - X) \le 0 \quad \text{for every } Z \in \mathcal{K},$$
$$\Xi \in \mathcal{K}^\circ \quad \text{and} \quad (\Xi|X) = 0,$$
$$P_\mathcal{K}(X + \Xi) = X.$$

Differential inclusion (I)

Recall that to the (discrete) data $\mu_t = (\rho_t, \rho_t v_t)$ we associate the functions $(X_t, V_t) \in \mathcal{K} \times L^2(0,1)$ by

$$X_t := X_{\rho_t}, \quad V_t := v_t \circ X_t.$$

Theorem (Second Lagrangian representation)

A family $\mu_t = (\rho_t, \rho_t v_t)$ is a solution of the (discrete) SPS if and only if X is the unique strong solution of the differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t \in -\partial \mathrm{I}_{\mathcal{K}}(X_t) + V_0, \quad \lim_{t \downarrow 0} X_t = X_0.$$

Stability properties (II)

By general results on solution of differential inclusion of the type $X_t' \in -\partial\phi(X_t) + F_t$, ϕ convex,

X is right-differentiable in each point and the velocity field v_t can be recovered by the formula

$$V_t = \frac{\mathrm{d}^+}{\mathrm{d}t} X_t = v_t \circ X_t \in \mathcal{H}_{X_t}.$$

One gets [S. '96] the following integral estimate for the velocity component:

Corollary

If X_t^1, X_t^2 are the Lagrangian representation of two (discrete) solutions ρ_t^1, ρ_t^2 of the SPS, their velocities $V_t^\ell = \frac{\mathrm{d}}{\mathrm{d}t} X_t^\ell$ satisfy

$$\int_0^t \|V_r^1 - V_r^2\|_{L^2(0,1)}^2 \,\mathrm{d}r \le C(1+t)\Big(\sum_\ell \|X_0^\ell\| + \|V_0^\ell\|\Big)\Big(\|X_0^1 - X_0^2\| + \|V_0^1 - V_0^2\|\Big).$$

Moreover, if $\psi : \mathbb{R} \to \mathbb{R}$ is convex then the map

$$t \mapsto \Psi(V_t) = \int_0^1 \psi(V_t(w))\,\mathrm{d}w \quad \textit{is non increasing in } [0, +\infty).$$

Further references: scalar conservation laws, L^2-theory and Wasserstein distance

- The link between the
  BRENIER-GRENIER formulation based on the **scalar conservation law**

  $$\partial_t M + \partial_x A(M) = 0$$

  and the
  Hilbertian theory of gradient flows like $\frac{\mathrm{d}}{\mathrm{d}t} X_t \in -\partial \mathrm{I}_{\mathcal{K}}(X_t) + V_0$
  is not at all surprising, after the illuminating recent paper

  > BRENIER '09*: L^2–formulation of multidimensional scalar conservation laws*

  whose ideas, in particular concerning the SPS, could be traced back to other papers by BRENIER in 2004-2005.
- Wasserstein contraction properties of solutions of one–dimensional scalar conservation laws have also been recently obtained by
  BOLLEY-BRENIER-LOEPER '05
  and further developed by CARRILLO-DI FRANCESCO-LATTANZIO '06

Differential inclusion (II)

Recall that to the (discrete) data $\mu_t = (\rho_t, \rho_t v_t)$ we associate the functions $(X_t, V_t) \in \mathcal{K} \times L^2(0,1)$ by

$$X_t := X_{\rho_t}, \quad V_t := v_t \circ X_t.$$

Theorem (Third Lagrangian representation)

A family $\mu_t = (\rho_t, \rho_t v_t)$ is a solution of the (discrete) SPS if and only if X is the unique strong solution of the differential inclusion

$$t \frac{\mathrm{d}}{\mathrm{d}t} X_t \in -\partial \mathrm{I}_{\mathcal{K}}(X_t) + X_t - X_0, \quad \lim_{t \downarrow 0} \frac{X_t - X_0}{t} = V_0.$$

Up to the rescaling $\tau = \log t$, $\hat{X}_\tau = X_t$, the equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \hat{X}_\tau \in -\partial \mathrm{I}_{\mathcal{K}}(\hat{X}_\tau) + \hat{X}_\tau - X_0;$$

it is the gradient flow in $L^2(0,1)$ of the functional

$$\Phi(X) := \mathrm{I}_{\mathcal{K}}(X) - \frac{1}{2} \|X - X_0\|^2$$

which is (-1)-convex.

A metric space for the measure-momentum couples $(\rho, \rho v)$

We consider the space of couples $(\rho, \rho v)$, with $\rho \in \mathcal{P}_2(\mathbb{R})$ and $v \in L^2_\rho(\mathbb{R})$:

$$\mathcal{V}_2(\mathbb{R}) := \Big\{\mu = (\rho, \rho v) \subset \mathcal{P}_2(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) : v \in L^2_\rho(\mathbb{R})\Big\}.$$

thus ρ is a probability measure and $\eta = \rho v$ is a finite signed measure in $\mathcal{M}(\mathbb{R})$ with $\int_{\mathbb{R}} |v(x)|^2 \,\mathrm{d}\rho(x) < +\infty$.
We can introduce a **semi-distance** U_2 in $\mathcal{V}_2(\mathbb{R})$:

$$U_2^2(\mu^1, \mu^2) := \int_{\mathbb{R}} \left|v^1(X_{\rho^1}(w)) - v^2(X_{\rho^2}(w))\right|^2 \mathrm{d}w = \left\|v^1 \circ X_{\rho^1} - v^2 \circ X_{\rho^2}\right\|^2_{L^2(0,1)}$$

and a **distance** D_2

$$D_2^2(\mu^1, \mu^2) := W_2^2(\rho^1, \rho^2) + U_2^2(\mu^1, \mu^2).$$

Theorem (Ambrosio-Gigli-S. '05)

$(\mathcal{V}_2(\mathbb{R}), D_2)$ is a metric (but not complete) space whose topology is stronger than the one induced by the weak convergence of measures.
The collection $\mathscr{V}_{\mathrm{discr}}(\mathbb{R})$ of all the discrete measures $\mu = \big(\sum_{i=1}^N m_i \delta_{x_i}, \sum_{i=1}^N m_i v_i \delta_{x_i}\big)$ is a dense subset of $\mathcal{V}_2(\mathbb{R})$.
A sequence $\mu_n = (\rho_n, \rho_n v_n)$ converges to $\mu = (\rho, \rho v)$ in $\mathcal{V}_2(\mathbb{R})$ if and only if

$$W_2(\rho_n, \rho) \to 0, \quad \rho_n v_n \rightharpoonup \rho v \quad \text{weakly in } \mathcal{M}(\mathbb{R}), \quad \int_{\mathbb{R}} |v_n|^2 \, d\rho_n \to \int_{\mathbb{R}} |v|^2 \,\mathrm{d}\rho.$$

The fundamental estimate

Let $\mathscr{V}_{\mathrm{discr}}(\mathbb{R})$ the collection of all the discrete measures in $\mathcal{V}_2(\mathbb{R})$ and let us denote by $\mathscr{S}_t : \mathscr{V}_{\mathrm{discr}}(\mathbb{R}) \to \mathscr{V}_{\mathrm{discr}}(\mathbb{R})$ the map associating to any discrete initial datum $(\rho_0, \rho_0 v_0) \in \mathscr{V}_{\mathrm{discr}}$ the solution $(\rho_t, \rho_t v_t)$ of the (discrete) sticky-particle system. $\mathscr{S}_t$ is a **semigroup in $\mathscr{V}_{\mathrm{discr}}(\mathbb{R})$.**
For $\mu \in \mathcal{V}_2(\mathbb{R})$ we set

$$[\mu]_2^2 := \int_{\mathbb{R}} \Big(|x|^2 + |v(x)|^2\Big)\,\mathrm{d}\rho(x) = D_2^2(\mu, (\delta_0, 0)).$$

Theorem (Stability with respect to the initial data)

Let $\mu_t^\ell = (\rho_t^\ell, \rho_t^\ell v_t^\ell) = \mathscr{S}_t[\mu_0^\ell]$, $\ell = 1, 2$, be the solutions of the (discrete) sticky-particle system with initial data $\mu_0^\ell \in \mathscr{V}_{\mathrm{discr}}(\mathbb{R})$.

$$W_2(\rho_t^1, \rho_t^2) \le W_2(\rho_0^1, \rho_0^2) + tU_2(\mu_0^1, \mu_0^2),$$

$$\int_0^t U_2^2(\mu_r^1, \mu_r^2)\,\mathrm{d}r \le C(1+t)\Big([\mu^1]_2 + [\mu^2]_2\Big)\Big(W_2(\rho_0^1, \rho_0^2) + U_2(\mu_0^1, \mu_0^2)\Big),$$

for a suitable "universal" constant C independent of t and the data.

Evolution semigroup

Theorem (The evolution semigroup in $\mathscr{V}_2(\mathbb{R})$)

- *The semigroup $\mathscr{S}_t$ can be uniquely extended by density to a right-continuous semigroup (still denoted $\mathscr{S}_t$) of strongly-weakly continuous transformations in $\mathscr{V}_2(\mathbb{R})$, thus satisfying*

$$\mathscr{S}_{s+t}[\mu] = \mathscr{S}_s[\mathscr{S}_t[\mu]] \quad \forall\, s, t \geq 0, \qquad \lim_{t \downarrow 0} D_2(\mathscr{S}_t[\mu], \mu) = 0. \tag{2}$$

  $\mathscr{S}_t$ complies with the same discrete stability estimates of the previous Theorem.

- *$(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$, $\mu \in \mathscr{V}_2(\mathbb{R})$, is a distributional solution of Euler system satisfying Oleinik entropy condition.*
- *If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function such that $\psi(v_0) \in L^1_{\rho_0}(\mathbb{R})$, and $(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu_0]$, then*

$$\textit{the map} \quad t \mapsto \int_{\mathbb{R}} \psi(v_t)\, \mathrm{d}\rho_t(x) \quad \textit{is nonincreasing in } [0, +\infty). \tag{3}$$

  and its jump set is contained in an at most countable set $J(\mu) \subset (0, +\infty)$ independent of ψ.

A gradient flow formulation in $\mathcal{P}_2(\mathbb{R})$

The semigroup $\mathscr{S}_t$ can also be characterized by the **(metric) gradient flow** $\mathscr{G}_\tau$ (introduced in [Ambrosio-Gigli-S. '05]) of the (-1)-geodesically convex functional

$$\Phi(\rho) := -\frac{1}{2} W_2^2(\rho, \rho_0)$$

in $\mathcal{P}_2(\mathbb{R})$.

$\hat\rho_\tau = \mathscr{G}_\tau(\rho)$ is a **semigroup in** $\mathcal{P}_2(\mathbb{R})$ which can be characterized by the family of **Evolution Variational Inequalities**

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} W_2^2(\hat\rho_\tau, \sigma) - \frac{1}{2} W_2^2(\hat\rho_\tau, \sigma) \le \Phi(\sigma) - \Phi(\hat\rho_\tau) \quad \text{for every } \sigma \in \mathcal{P}_2(\mathbb{R}).$$

Theorem (The gradient flow of the opposite Wasserstein distance)

If $\mu_t = (\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0)$ is a solution of SPS then the rescaling $\tau = \log t$, $\hat\mu_\tau = \mu_t$, $\hat\rho_\tau = \rho_t$ satisfy

$$\hat\rho_{\tau+\delta} = \mathscr{G}_\delta(\hat\rho_\tau) \quad \textit{or, equivalently} \quad \rho_{t\, e^\delta} = \mathscr{G}_\delta(\rho_t).$$

The L^2-projection on $\mathcal{K}$

Theorem

If $X \in L^2(0,1)$ *and* $\mathscr{X}(w) = \int_0^w X(s)\,\mathrm{d}s$ *is its primitive then*

$$\mathsf{P}_{\mathcal{K}}(X) = \frac{\mathrm{d}}{\mathrm{d}w}\mathscr{X}^{**}$$

where $\mathscr{X}^{**}$ *is the convex envelope of* $\mathscr{X}$.

The subdifferential of $\mathrm{I}_{\mathcal{K}}$

If $X \in \mathcal{K}$ we consider the open set Ω_X where X is (essentially) constant:

$$\Omega_X := \big\{w \in (0,1) : X \text{ is essentially constant in a neighborhood of } w\big\},$$

and the cone

$$\mathcal{N}_X := \big\{\mathscr{Y} \in C^0([0,1]) : \mathscr{Y} \geq 0, \quad \mathscr{Y} = 0 \text{ in } [0,1] \setminus \Omega_X\big\}.$$

Theorem

Let $X \in \mathcal{K}$ and $Y \in L^2(0,1)$ with $\mathscr{Y}(w) := \int_0^w Y(s)\,\mathrm{d}s$. Then

$$Y \in \partial\mathrm{I}_{\mathcal{K}}(X) \quad \Leftrightarrow \quad \mathscr{Y} \in \mathcal{N}_X.$$

Notice that if $Z = f(X) \in \mathcal{H}_X$ depends on X then

$$\Omega_Z \subset \Omega_Z, \quad \mathcal{N}_Z \subset \mathcal{N}_Z$$

Corollary (Monotonicity property of $\partial\mathrm{I}_{\mathcal{K}}$)

If $Z = f(X) \in \mathcal{H}_X$ depends on X then

$$\partial\mathrm{I}_{\mathcal{K}}(X) \subset \partial\mathrm{I}_{\mathcal{K}}(Z).$$

Order reduction for differential inclusions

Differential equation for the collision free motion in Lagrangian coordinates:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t = 0$$

"Formal" differential inclusion for the SPS:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t \in -\partial \mathrm{I}_{\mathcal{K}}(X_t) \quad \text{i.e.} \quad \Big[\frac{\mathrm{d}}{\mathrm{d}t} X_t\Big]_{t=t^h} = V_{t^h+} - V_{t^h-} \in -\partial \mathrm{I}_{\mathcal{K}}(X_t). \qquad (\star)$$

Sticky condition:

$$s < t \quad \Rightarrow \quad X_t \in \mathcal{H}_{X_s} \quad \text{i.e.} \quad X_t \text{ "depends on" } X_s.$$

By the monotonicity property of $\partial \mathrm{I}_{\mathcal{K}}$ we have

$$\partial \mathrm{I}_{\mathcal{K}}(X_s) \subset \partial \mathrm{I}_{\mathcal{K}}(X_t)$$

We can then integrate $(\star)$ with respect to time from 0 to a final time t:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} X_t - V_0 \in -\partial \mathrm{I}_{\mathcal{K}}(X_t)} \quad \text{since} \quad \int_0^t \partial \mathrm{I}_{\mathcal{K}}(X_s)\,\mathrm{d}s \in \partial \mathrm{I}_{\mathcal{K}}(X_t). \qquad (\star\star)$$

Integrating again we get

$$X_t - X_0 - tV_0 \in -\partial \mathrm{I}_{\mathcal{K}}(X_t) \quad \text{i.e.} \quad \boxed{X_t = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0)} \qquad (\star\star\star)$$

Extensions and open problems

Extensions:

- $L^2 \rightsquigarrow L^p$, $p \geq 2$;
- (in collaboration with W. Gangbo and M. Westdickenberg) Adding a force induced by a potential V

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = \boxed{-\rho\, \partial_x V}. \end{cases}$$

- Adding a force induced by a smooth interaction potential

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = \boxed{-\rho\left(\rho * \partial_x W\right)} \end{cases}$$

- Adding a force induced by a non-smooth interaction potential, e.g. the **Euler-Poisson system when** $W(x) = \pm|x|$**.**

Open problems:

- The SPS in the multidimensional case.
- The displacement-extrapolation problem.