Variational models for cavitation and fracture in nonlinear elasticity

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(joint work with Duvan Henao)

Introduction to Nonlinear Elasticity

A deformation of a body $\Omega \subset \mathbb{R}^3$ is described by a map $\mathbf{u} : \Omega \to \mathbb{R}^3$.

$$\text{Elastic energy} = \int_\Omega W(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

where $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is the stored energy function of the material.

An equilibrium solution $\mathbf{u}$ (Statics) is a solution of

$$\min_{\mathbf{u}\in W^{1,p}_{bc}} \int_\Omega W(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \; + \; \text{forces.}$$

Apart from solving a minimization problem, every physically realistic solution $\mathbf{u}$ must:

- preserve the orientation: $\det D\mathbf{u} > 0$,
- be one-to-one (no interpenetration of matter).

Typically, $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ satisfies $W(\mathbf{F}) = \infty$ whenever $\det \mathbf{F} \leq 0$.

$\implies$ W cannot be convex.

Existence in Nonlinear Elasticity (J.M. Ball 1977)

Theorem.

W polyconvex, i.e., $W(\mathbf{F}) = W_1(\mathbf{F}, \operatorname{cof} \mathbf{F}, \det \mathbf{F})$ with W_1 convex,

$$W(\mathbf{F}) \geq c_1 \left(|\mathbf{F}|^p + |\operatorname{cof} \mathbf{F}|^q + |\det \mathbf{F}|^r \right) - c_2$$

with $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$. Dirichlet boundary conditions. Then there exists a minimizer of $\int_\Omega W(D\mathbf{u}) \, \mathrm{d}\mathbf{x}$.

Key of the proof:

$$\mathbf{u}_j \overset{W^{1,p}}{\rightharpoonup} \mathbf{u} \implies \operatorname{cof} D\mathbf{u}_j \overset{L^q}{\rightharpoonup} \operatorname{cof} D\mathbf{u} \quad \text{and} \quad \det D\mathbf{u}_j \overset{L^r}{\rightharpoonup} \det D\mathbf{u}.$$

Refinements by Ball & Murat 84, Ciarlet & Nečas 87, Šverák 88, Giaquinta, Modica & Souček 89, Müller 89, 90, Müller, Tang & Yan 94...

All in function spaces that don't allow cavitation.

What is cavitation?

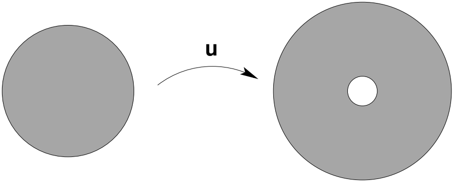
It is the process of sudden formation of voids in solids subject to sufficiently large tension. It is typical in rubbers, but some metals undergo cavitation too.

Cavitation in rubber (A.N. Gent & P.B. Lindley 1959)

Experiments in metals suggest that ductile fracture is preceded by cavitation.

Cavitation and fracture in a Titanium alloy (N. Petrinic et al. 06)

The easiest cavitation is the radial: $\mathbf{u}(\mathbf{x}) := r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$.



This deformation opens a hole at $\mathbf{0}$ of radius $r(0)$.

It satisfies $\mathbf{u} \in W^{1,p}$ for all $p < 3$, and $\operatorname{cof} D\mathbf{u} \in L^q$ for all $q < \frac{3}{2}$.

$\rightsquigarrow$ not covered by the existence theorems in Elasticity.

J.M. Ball & F. Murat 84 showed that if cavitation is allowed then

$$\mathbf{u}_j \overset{W^{1,p}}{\rightharpoonup} \mathbf{u} \;\not\Longrightarrow\; \det D\mathbf{u}_j \overset{L^1}{\rightharpoonup} \det D\mathbf{u}.$$

The model of S. Müller & S.J. Spector 1995 for cavitation

They added to the elastic energy a *surface energy* (= energy due to cavitation):

$$\text{Energy } = \int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathrm{Per}(\mathbf{u}(\Omega)).$$

This can be minimized in the class

$$\left\{\mathbf{u} \in W^{1,p} : \ \det D\mathbf{u} > 0, \ \mathbf{u} \text{ is one-to-one a.e. and satisfies (INV)}\right\}.$$

(INV) is a topological condition related to injectivity.

We would like to improve Müller-Spector's model:

- Definition of created surface. The surface energy should measure only the created surface, excluding the existing one.
- Allow for fracture too. Condition (INV) is incompatible with fracture.

Model for brittle fracture

(L. Ambrosio & A. Braides 95, G.A. Francfort & J.J. Marigo 98)

$$\int_\Omega W(\nabla \mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{H}^2(J_{\mathbf{u}})$$

$J_{\mathbf{u}}$ = set of jumps of $\mathbf{u}$.

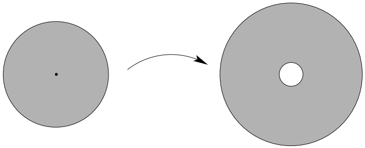
Existence in SBV guaranteed by Ambrosio 90 compactness theorem.

Condition (INV) is incompatible with SBV.

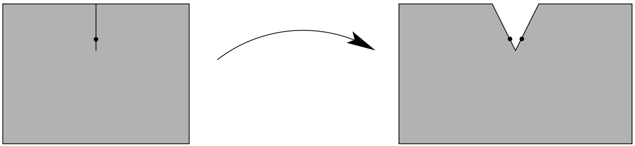
Need a new functional space and a new surface energy that detects both cavitation and fracture.

Cavitation and fracture are singularities of a different kind:

- Cavitation: the image of a point is a surface.

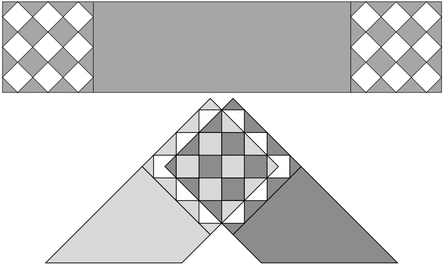


- Fracture: the image of a surface is a surface in a one-to-two way.



But they have something in common: both **create surface**.

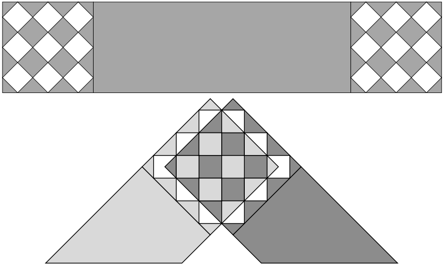
Müller & Spector 95: example of 'invisible' created surface.



Definition:

Invisible surface $\Gamma_I(\mathbf{u})$: the set of jump discontinuity points of $\mathbf{u}^{-1} : \mathbf{u}(\Omega) \to \Omega$ with density 1 in $\mathbf{u}(\Omega)$.

Visible surface $\Gamma_V(\mathbf{u})$: the set of jump discontinuity points of $\mathbf{u}^{-1}\chi_{\mathbf{u}(\Omega)} : \mathbb{R}^3 \to \Omega \cup \{\mathbf{0}\}$ with density $\frac{1}{2}$ in $\mathbf{u}(\Omega)$ and preimage in Ω.



$$\mathrm{Per}(\mathbf{u}(\Omega)) = \sup_{\|\mathbf{g}\|_\infty \le 1} \int_{\mathbf{u}(\Omega)} \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

New surface energy: $$\mathcal{E}(\mathbf{u}) := \sup_{\|\mathbf{f}\|_\infty \le 1} \int_{\mathbf{u}(\Omega)} \operatorname{div} \mathbf{f}(\mathbf{u}^{-1}(\mathbf{y}), \mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Theorem: $\mathcal{E}(\mathbf{u}) = \mathcal{H}^2(\Gamma_V(\mathbf{u})) + 2\mathcal{H}^2(\Gamma_I(\mathbf{u})).$

Weak continuity of the determinant.

Preservation of injectivity under limit.

Theorem. If

$$\mathbf{u}_j \to \mathbf{u} \ \text{ a.e.}, \qquad \operatorname{cof} \nabla \mathbf{u}_j \rightharpoonup \operatorname{cof} \nabla \mathbf{u} \ \text{ in } \ L^1, \qquad \sup_j \mathcal{E}(\mathbf{u}_j) < \infty,$$

then $\det \nabla \mathbf{u}_j \rightharpoonup \det \nabla \mathbf{u}$ in L^1.

(Improvement of M. Giaquinta, G. Modica & J. Souček 98.)

If, in addition, $\mathbf{u}_j$ is one-to-one a.e. and $\det \nabla \mathbf{u}_j > 0$ a.e.

then $\mathbf{u}$ is one-to-one a.e.

(Improvement of Ph.G. Ciarlet & J. Nečas 87,

A. Giacomini & M. Ponsiglione 08).

Regularity of inverses.

Using ideas of Ambrosio 95 we can prove that

$$\mathbf{u} \in BV,\ \mathcal{E}(\mathbf{u}) < \infty \implies \mathbf{u}^{-1}\chi_{\mathbf{u}(B)} \in SBV_{\mathrm{loc}} \quad \text{for a.e. ball} \quad B$$

Related regularity results by S. Hencl, P. Koskela, J. Onninen, J. Malý, M. Csörnyei…2006–08.

The functional $\mathcal{E}$ is a known object in the theory of **Cartesian currents** (M. Giaquinta, G. Modica & J. Souček 98):

$$\mathcal{E}(\mathbf{u}) = \mathbb{M}((G_{\mathbf{u}})_2).$$

Existence theory for cavitation and fracture.

Theorem (only cavitation).

$p \geq 2, \;\; r > 1, \;\; W$ polyconvex,

$$W(\mathbf{F}) \geq c_1 \left(|\mathbf{F}|^p + |\operatorname{cof} \mathbf{F}|^r + |\det \mathbf{F}|^r \right) - c_2 .$$

Then $\displaystyle\int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{E}(\mathbf{u})$ has a minimizer in

$$\left\{ \mathbf{u} \in W^{1,p} : \; \det D\mathbf{u} > 0, \; \mathbf{u} \text{ one-to-one a.e.}, \;\; \text{Dirichlet} \right\} .$$

Theorem (cavitation and fracture).

p, r, W as above.

Then $\displaystyle\int_\Omega W(\nabla\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{E}(\mathbf{u}) + \mathcal{H}^2(J_{\mathbf{u}})$ has a minimizer in

$$\{ \mathbf{u} \in SBV : \; \nabla\mathbf{u} \in L^p, \; \det \nabla\mathbf{u} > 0, \; \mathbf{u} \text{ one-to-one a.e.}, \;\; \text{Dirichlet} \} .$$