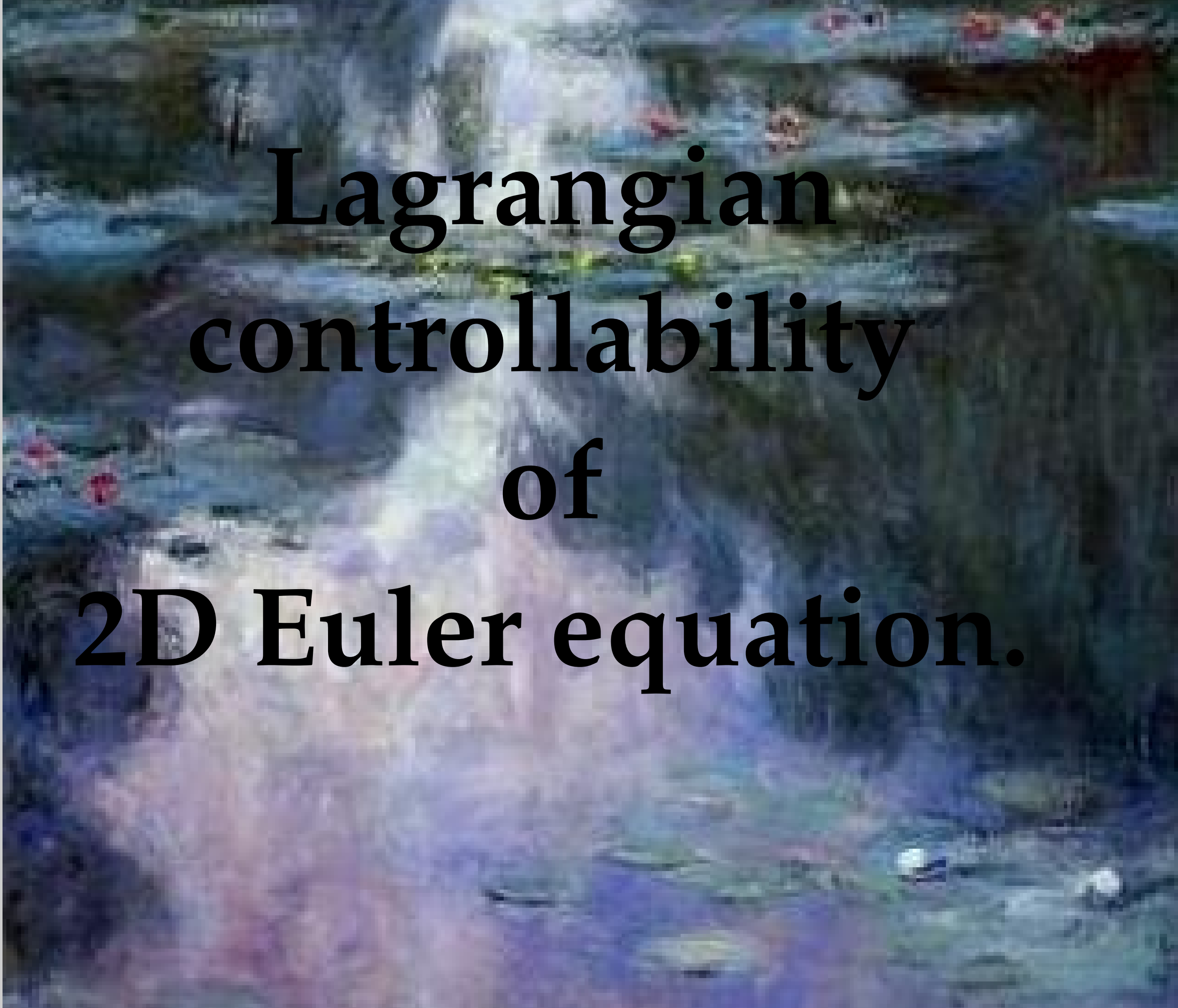
Lagrangian controllability of 2D Euler equation.



Thierry Horsin

Université de Versailles

Benasque, August 2009

- Work with Olivier Glass (university paris 6 transiting to 9).

Motivation

- The question arose from talks with G. Leugering and J.P. Puel:
  Can the optimal transportation theory and the controllability be mixed ?

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- Can one prescribe the motion of a set of fluid particles ?

Motivation

- Possible applications:
  Treatment of pollution: when a pollutant can be considered as a fluid.
  Displacement of species (animal, plant, alga, mermaids).

Formulation

- Given an open bounded set $\Omega \subset \mathbb{R}^N$, $T > 0$.
  Two Jordan domains γ_0 and γ_1 included in Ω.
  The two domains surrounded by γ_0 and γ_1 satisfy:
  $$|\mathrm{int}(\gamma_0)| = |\mathrm{int}(\gamma_1)|.$$
  Let $\Gamma \subset \partial\Omega$.

Formulation

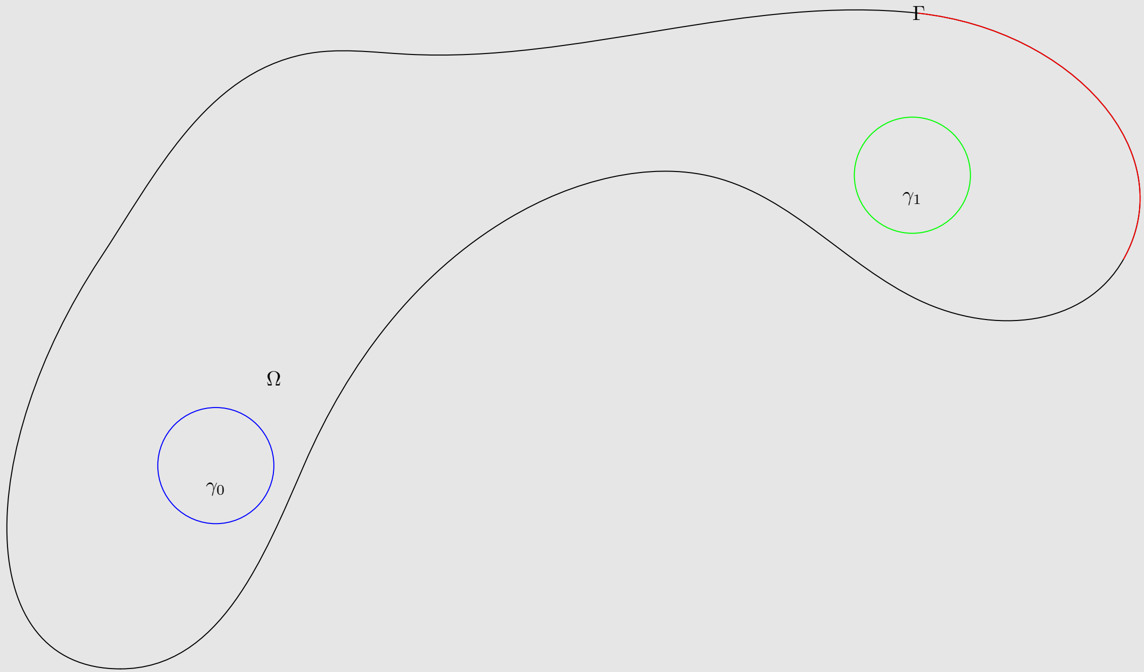

Figure 1 One possible configuration

Formulation

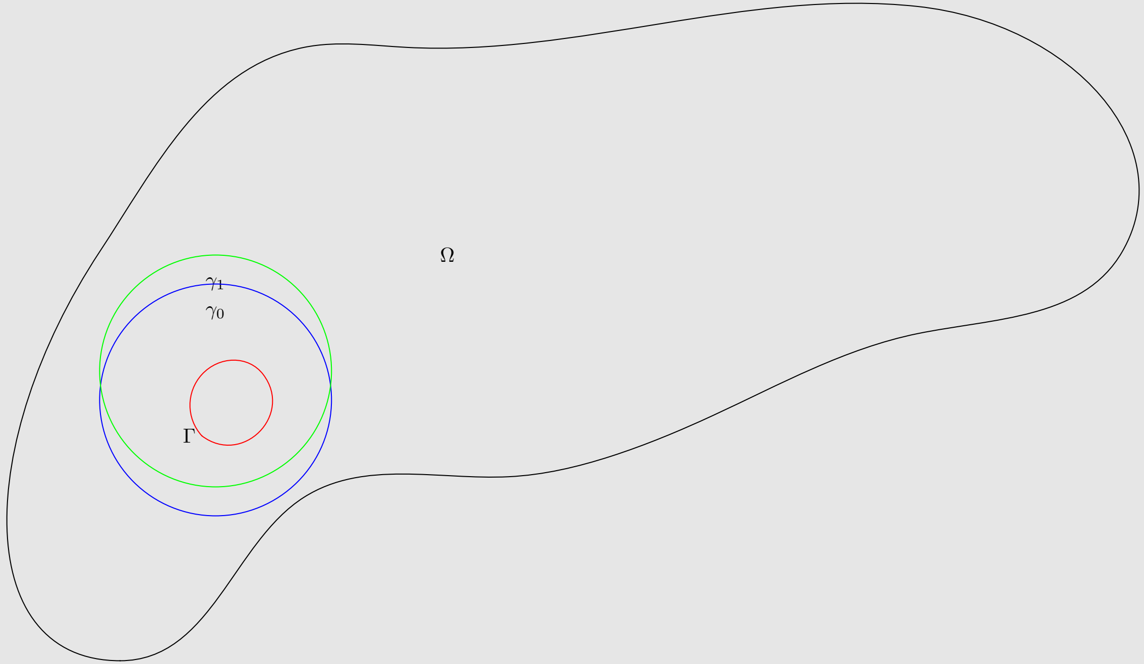

Figure 2 Another possible configuration

Formulation

- Let $u_0 : \overline{\Omega} \to \mathbb{R}^2$.
  Consider $u : (0,T) \times \Omega \to \mathbb{R}^2$ and $p : (0,T) \times \Omega \to \mathbb{R}$ satisfying the Euler system (E)

$$(E) \quad \begin{cases} (1)\ \dfrac{\partial u}{\partial t} + (u.\nabla)u + \nabla p = 0,\ \ t > 0\ x \in \Omega \\ (2)\ u(t=0,x) = u_0(x) \\ (3)\ u.n = 0,\ \ \text{on } (0,T) \times \partial\Omega \setminus \Gamma \\ (4)\ \operatorname{div}(u) = 0 \end{cases}$$

Formulation

- The system (E) is well posed if one imposes, for example,

$$(5) \qquad \begin{cases} u.n \text{ on } (0,T)\times\Gamma,\ \int_{\partial\Omega} u.n d\sigma = 0 \\ \mathrm{curl} u \text{ on each point of } \partial\Omega \text{ such that } u.n < 0 \end{cases}$$

Formulation

- Definition: One says that there is (exact) Lagrangian controllability (resp. approx. Lagrang. controll. in norm $\|\cdot\|$) between γ_0 and γ_1 in time T if (resp. $\forall\varepsilon > 0$) one can find u solution of (E) such that the flow φ^u of u defined by

$$\frac{\partial\varphi^u}{\partial t}(s,t,x) = u(t,\varphi^u(s,t,x))$$
$$\varphi^u(s,s,x) = x$$

satisfies

$$\varphi^u(0,T,\gamma_0) = \gamma_1.$$

respectively

$$\|\varphi^u(0,T,\gamma_0) - \gamma_1\| < \varepsilon,$$

up to a reparameterization.

Results

The idea of moving fluid particles in controllability amounts to JM. Coron,. O. Glass ,.. to show the controllability in the Eulerian description.

In fact the Coron's method gives the exact lagrangian controllability but by allowing the fluid particles to be controlled to leave the domain Ω. Here we will impose that controlled particles remain in the domain.

L. Rosier studied the controllability of the surface of a 1-d fluid in Lagrangian description.

Recent works of A. Agrachev are linked to the Lagrangian controllability.

Controllability or inverse problems in fluid-structure interactions, Imanuvilov-Takahashi, Cumsille-Ortega-Rosier, Conca-Kavian & al.....

Burgers and heat equation in 1d, heat equation in N-d.

Results

- We have the following results: (everything is now in 2-d)

  Theorem 1: Assume that $u_0 \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$, $u_0.n = 0$ on $\partial\Omega \setminus \Gamma$, $\mathrm{div}(u_0) = 0$ in Ω, that γ_0 and γ_1 are C^∞, that γ_0 andγ_1 surround surfaces of same areas and are homotopic in Ω, and that Ω is connected. For all $k \in \mathbb{N}$ there is approximate controllability between γ_0 and γ_1 in time T and in norm $C^k(\mathbb{S}^1)$. Moreover $\forall t \in [0, T]$, $\varphi^u(0, t, \gamma_0) \subset \Omega$.

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- Remark 1: If one assumes that $\mathrm{curl} u_0 = 0$ in a neighborhood of γ_0, since the vorticity ($\mathrm{curl} u$) is moved by the flow of the given solution of (E) it remains null in a neighborhood of $\varphi^u(0,t,\gamma_0)$ for all $t \in [0,T]$. Therefore if γ_0 is a real analytic (Jordan) curve, the same holds for $\varphi^u(0,t,\gamma_0), \forall t$. So as long as one imposes $\forall t \in [0,T]$, $\varphi^u(0,t,\gamma_0) \subset \Omega$ there cannot be exact lagrangian controllability (it suffices to take γ_1 nonanalytic).
  $$(\frac{\partial \omega}{\partial t} + (u.\nabla)\omega = 0, \text{ with } \omega := \mathrm{curl} u.)$$

Results

- Remark 2: Contrary to the works on the controllability of (E) in Eulerian description (where the state is the velocity) [Coron] [Glass],...., one may control not on all connected components of $\partial\Omega$.

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- Remark 3: The solution of (E) that we build is C^∞:-> its flow is well defined. All the fluid surrounded by γ_0 is approximatively sent inside Int(γ_1). As long as one considers regular controls, the situation will be the same. If γ_0 and γ_1 are not homotopic in Ω the situation is not clear since the flow has to be a weak flow that does not preserve the topology.

Results

- **Wolibner, Yudovich, Chemin**, Depauw, Dutrifoy.... have studied vortex patches, *i.e.* solutions that satisfy

$$\mathrm{curl} u(t,x) = \chi_{\mathrm{Int}(\gamma)}(t,x)$$

$$\mathrm{div} u = 0 \text{ in } \Omega$$

$$u.n = 0 \text{ on } \partial\Omega \setminus \Gamma$$

$$(5\mathrm{bis}) \begin{cases} \text{u.n given on } \Gamma \\ \text{curlu=0 on } \Gamma \bigcap \{\text{u.n<0}\} \end{cases}$$

This problem is well-posed and if γ_0 is C^∞ then so is for $\gamma(t)$ for all t.
We then have the following result:

Results

- Theorem 2: If γ_0 and γ_1 are C^∞, and if u_0 is lipschitz on Ω and $u_0.n \in C^\infty(\partial\Omega)$ with

$$\begin{cases} \mathrm{curl}(u_0) = \chi_{\mathrm{Int}(\gamma_0)} \\ \mathrm{div} u_0 = 0 \text{ in } \Omega \\ u_0.n = 0 \text{ on } \partial\Omega \setminus \Gamma. \end{cases}$$

there exists then $\forall \varepsilon > 0$ $u \in L^\infty([0,T], \mathcal{L}ip(\overline{\Omega}))$ satisfying (1) (2) (3) (4) (5bis) whose flow φ^u satisfies $\varphi^u(0,t,\gamma_0) \subset \Omega$ and

$$\|\varphi^u(0,T,\gamma_0) - \gamma_1\|_{C^k} < \varepsilon.$$

Results

- The cornerstone for proving theorems 1 and 2 is constituted of the following result :

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- Theorem 3: Let γ_0 and γ_1 be as in theorem 1 and $\varepsilon > 0$, there exists $\theta \in C_0^\infty([0,1]; C^\infty(\overline{\Omega}, \mathbb{R}))$, such that

$$
\begin{aligned}
&-\Delta_x \theta = 0 \text{ in } \Omega \text{ for all } t \in [0,1] \\
&\frac{\partial \theta}{\partial n} = 0 \text{ on } [0,1] \times (\partial\Omega \setminus \Gamma) \\
&\|\varphi^{\nabla\theta}(0,1,\gamma_0) - \gamma_1\|_{C^k} \leq \varepsilon,
\end{aligned}
$$

that is to say we can do the job by potential flows.
This idea has been extensively used by Coron, and later by Glass in the Eulerian controllability of (E).

Results

- To prove theorem 3, we prove the following results:
  Theorem 4: Let γ_0 and γ_1 be two jordan curves of class C^∞ homotopic in Ω and which enclose surfaces of same area, there exists $v \in C_0^\infty((0,1)\times\Omega;\mathbb{R}^2)$ with $\mathrm{div}(v) = 0$ such that $\varphi^v(0,1,\gamma_0) = \gamma_1$.

  Proposition 1: If γ is a C^∞ Jordan curve and X a $C^0([0,1], C^\infty(\overline{\Omega}))$ divergence free vector field with $X.n = 0$ on $[0,1] \times \partial\Omega$. Let us define $\gamma_1 = \varphi^X(0,1,\gamma_0)$ there exists for all $\varepsilon > 0$ a $\theta \in C^\infty([0,1] \times \Omega, \mathbb{R})$ harmonic in space with null normal derivative on $(\partial\Omega \setminus \Gamma)$, $\varphi^{\nabla\theta}(0,t,\gamma_0) \subset \Omega$ and $||\gamma_1 - \varphi^{\nabla\theta}(0,1,\gamma_0)||_k \leq \varepsilon$.

Ideas of the proofs

- **Proposition 1.** One first assume that X and γ_0 are real analytic: Let $\gamma(t) := \varphi^X(0, t, \gamma_0)$. One can solve

$$\begin{cases} \Delta_x \psi(t,x) = 0 \text{ in } \mathrm{Int}(\gamma(t)) \cap \Omega \text{ (or } \Omega \setminus \mathrm{Int}(\gamma(t)) \text{ if } \Gamma \cap \mathrm{Int}(\gamma(t)) \neq \emptyset) \\ \dfrac{\partial \psi}{\partial \nu} = X.\nu \text{ on } \gamma(t), \ (\nu \text{ denotes the normal}) \\ \dfrac{\partial \psi}{\partial n} = 0 \text{ on } \mathrm{Int}(\gamma(t)) \cap \partial\Omega \end{cases}$$

  and extend ψ to a harmonic mappings on a neighborhood of $\mathrm{Int}(\gamma(t))$ (for simplicty we assume $\mathrm{Int}(\gamma(t)) \cap \partial\Omega = \emptyset$) thanks to the Cauchy-Kowalewski's theorem (precisely a version of C.B. Morrey) and by compactness one can take this neighborhood locally constant in time, and besides we get uniform bounds in time on these neighborhoods.

- By the Runge's theorem (and the correspondance between gradient of harmonic maps and holomorphic functions with zero circulation) one extends (not exactly but approximately) ψ on this neighborhood and 0 on a neighborhood of $\partial\Omega \setminus \Gamma$ to a harmonic map θ (that we slightly correct) on $\overline{\Omega}$ and we get uniform bounds in time on a neighborhood of $\gamma(t)$. One concludes with the Gronwall's lemma.

Ideas of the proofs

- We get estimates of the form $\|\varphi^{\nabla\theta}(0,t,\gamma_0)-\varphi^{\nabla\psi}(0,t,\gamma_0)\|_k \lesssim \|\nabla\theta-\nabla\psi\|_{C^0,C^k(V(\gamma(t)))}\exp(\|\nabla\psi\|_{L^\infty,W^{k+1,\infty}(V(\gamma(t)))})$ and

$$\|\nabla\theta(t,.)\|_{C^k(\mathrm{int}(\varphi^{\nabla\theta}(0,t,\gamma_0))} \leq \|\nabla\psi(t,.)\|_{C^k(\mathrm{int}(\varphi^X(0,t,\gamma_0))} + 1$$

Ideas of the proofs

- When γ_0 is only C^∞: the complement $\mathrm{Int}(\gamma_0)$ in the Riemann sphere is connected and simply connected:
  $T : \mathbb{S}^2 \setminus \overline{\mathrm{Int}(\gamma_0)} \to \mathbb{D}^2$ conformal and C^∞ up to the boundary: One takes $\gamma_0^\mu = T^{-1}(\mathbb{S}^1_{1-\mu})$. We proceed as before and thanks to the fact that T is smooth up to the boundary (Kellogs-Warschawski's theorem) one shows that the ψ^μ are bounded uniformly in t and μ on $\mathrm{int}(\varphi^X(t, 0, \gamma^0))$.

- Lastly when X and γ are only C^∞: One writes $X = \nabla^\perp h$ (for X is divergence free) and one approaches h by a h^μ (uniformly in t) which is analytic thanks to the Whitney's theorem and one corrects the boundary condition $X^\mu.n \neq 0$ but small on $\partial\Omega$ by the gradient of a harmonic function.

Ideas of the proofs

- Theorem 4. One starts by assuming that γ_0 and γ_1 intersect transversally: Let $P \in \gamma_0$ and $Q \in \mathrm{Int}(\gamma_1)$ one choose a curve $t \to s(t)$ in Ω such that $s(0) = P$ and $s(1) = Q$.
  One chooses $h(t,x)$ such that $\nabla^\perp h(t,x) = s'(t)$ along $s(t)$ and we extend h arbitrarily $[0,1] \times \Omega$. $\nabla^\perp h$ is divergence free and its flow maps γ_0 on a curve that intersects γ_1 which we now denote γ_0. By the Thom's transversality theorem one may choose $r \in \mathbb{R}^2$ such that $\gamma_0 + r$ intersects γ_1 transversally and is included in Ω.

Ideas of the proofs

- We arrive to different type of situations described below

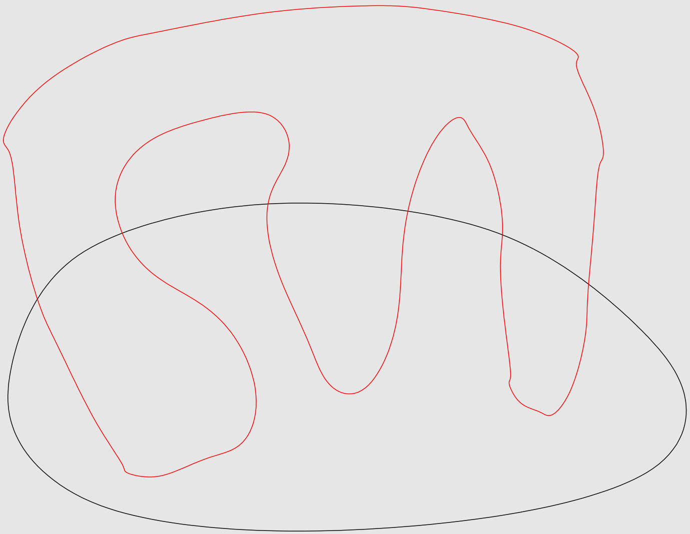

Figure 3

Ideas of the proofs

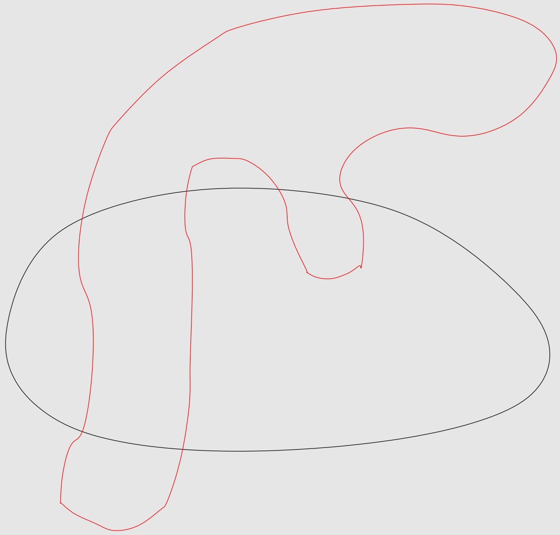

Figure 4

Ideas of the proofs

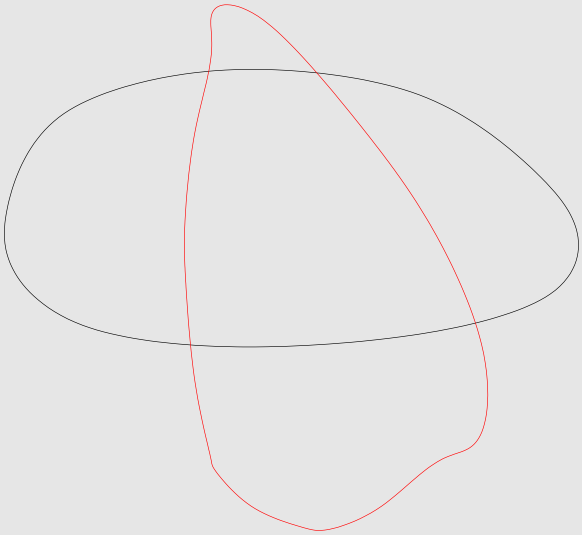

Figure 5

Ideas of the proofs

- One proves that you can go by the flow of a divergence free vector field from the first and second situation to the third. The proof is very technical but is based essentially on the following:

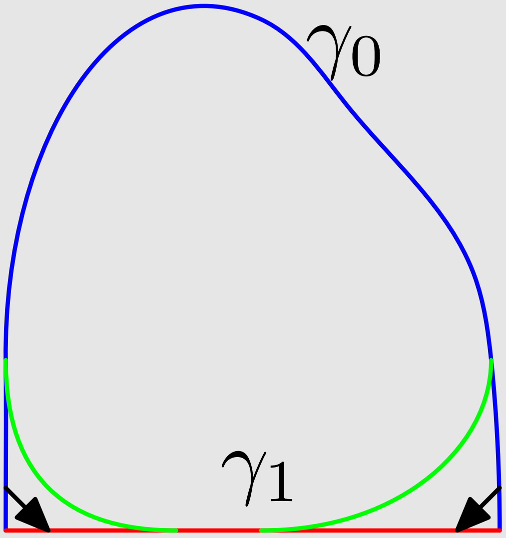

Figure 6

Ideas of the proofs

- On the left green part one imposes $b = 1$, and on the right green part one takes $b = 0$ and b is monotone.
  One then solve
  $$-\Delta b = 0$$
  in the subdomain bounded by the three colors with the b given on the boundary, by the strict maximum principle and the topoligical degree one easily sees that ∇b does not vanish on the boudary of the 3-color boundar. Inside one sets $v = \nabla c$ with $v = \nabla^{\perp} b$ and by the corners one takes v mapping by its flow the vertical edges to the horizontal ones. One glues theses maps, while taking into account the necessity to have a divergence free vector field ....

Ideas of the proofs

Theorem 3 follows from the combination of Theorem 4 and the proposition 1.

Ideas of the proofs

- Theorem1: Return method of Coron. The $\nabla\theta$ given by theorem 3 goes from 0 to 0.
  Let us fix ν small.
  On $[0, T-\nu]$ no control.
  On $[T-\nu, T]$ one puts $\nabla\theta$ fastly (with the associated pressure,) i.e the solution is thought like
$$\frac{1}{\nu}\nabla\theta(\frac{t-T+\nu}{\nu}, x)$$

Ideas of the proofs

Precisely For $\kappa \in \{0,1\}$ there exists a fixed point to : (denote $\omega_0 := \mathrm{curl} u_0$).

$$\mathcal{T} : C^0([0,T], C^{l,\alpha}(\overline{\Omega})) \to C^0([0,T], C^{l,\alpha}(\overline{\Omega}))$$

$$\mathcal{T}(\omega) = \pi\omega_0(\varphi^{\pi(y)}(t,0,x))$$

$$\mathrm{curl} y = \omega \text{ on } [0,T] \times \Omega$$

$$\mathrm{div}(y) = 0 \text{ on } [0,T] \times \Omega$$

$$y.n = \rho(t/\mu)u_0.n + \frac{\kappa}{\nu}\nabla\theta(\frac{t-T+\nu}{\nu}, x).n \text{ sur } [0,T] \times \partial\Omega$$

$$\int_{\Gamma_i} y(0,x).\tau(x)dx = \int_{\Gamma_i} u_0(x).\tau(x)dx,\ i = 1,..,d$$

$$\int_{\Gamma_i} (\frac{\partial y}{\partial t} + (y.\nabla)y).\tau(x)dx = 0, i = 1,..,d$$

where $\Gamma_0, ..., \Gamma_d$ are the connected components and you (may by restriction) assume that $\Gamma \subset \Gamma_0$.

π is an extension operator from $C^{l,\alpha}(\overline{\Omega})$ to $C^{l,\alpha}(B(0,R))$ and from $LL(\Omega)$ to $LL(B(0,R))$ where R is large. ρ ist 0 away from 0.

Ideas of the proofs

- $LL(\Omega)$ is the Log-lipschitz functions.The fixed point gives boundary data for which there exists a unique solution of (E).

Ideas of the proofs

We skip the proof of theorem 2 which is of the same flavour but the fxed point is made directly on u due to the lack of regularity, and we use an adapted contour dynamic introduced by Bertozzi-Constantin.

Possible extensions and open questions

Remark: Potential flows are also solutions of the Navier-Stokes equations, all except the return method work but we do not have the good "traditionnal" boundary conditions. What happens with these classical boundary conditions ? Or with the Navier slip boundary condition ?

Possible extensions and open questions

In 3d: Some problems may occur: for some u_0 the solution may blow-up, and we have difficulties by mapping γ_0 onto γ_1 with the flow of a divergence free vector field, that is the existence of volume conserving isotopy between γ_0 and γ_1 in Ω. Up to this moment we can prove along the same ideas

Theorem: Assume that γ_0 and γ_1 are smooth embeddings of the 2-sphere which do not intersect and are contractible in Ω, then for any u_0 (as in theorem 1), there exists a time T such that $\forall \varepsilon > 0$ there exists a solution of the 3d euler equation (1) (2) (3) (4) such that

$$\|\varphi^u(0, T, \gamma_0) - \gamma_1\|_\infty \leq \varepsilon.$$

Possible extensions and open questions

We have some ideas with O.Glass O. Kavian JP Puel to perform numerical simulations. Some of these are in progress but not very convincing for the moment being.

Possible extensions and open questions

They relies on the following theorem that in fact we prove in 2d with complex analysis, but can be proved however in any dimension (to simplify we assume that Ω has a trivial topology):

Theorem Let $H_m^{-1/2}(\Gamma) := \{h \in H^{-1/2}(\partial\Omega), < h, 1 >= 0, h = 0 \text{ in } \mathcal{D}'(\partial\Omega \setminus \Gamma)\}$.
For any $h \in H_m^{-1/2}(\Gamma)$ one defines Ψ with zero mean value such that

$$\Delta\Psi = 0 \text{ in } \Omega$$
$$\frac{\partial\Psi}{\partial n} = h \text{ on } \partial\Omega$$

then the map

$$h \mapsto \frac{\partial\Psi}{\partial n}_{/\gamma}$$

has a strict dense image in $H_m^{-1/2}(\gamma)$.

Possible extensions and open questions

In fact we have density in more regular trace space. We can compute the adjoint and then apply it to determine the h of minmal norm that gives a precise goal to order ε.