Asymptotics for nonlocal evolution equations

Liviu Ignat

Institute of Mathematics of the Romanian Academy

Benasque, August 2009

Joint work with Julio Rossi

Few words about local diffusion problems

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$ is given by:

$$u(t,x) = (H(t,\cdot) * u_0)(x)$$

where

$$H(t,x) = (4\pi t)^{d/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

Smoothing effect

$$u \in C^\infty((0,\infty), \mathbb{R}^d)$$

Decay of solutions, $1 \le p \le q \le \infty$:

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$$

Few words about local diffusion problems

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$ is given by:

$$u(t,x) = (H(t,\cdot) * u_0)(x)$$

where

$$H(t,x) = (4\pi t)^{d/2} \exp(-\frac{|x|^2}{4t})$$

Smoothing effect

$$u \in C^\infty((0,\infty), \mathbb{R}^d)$$

Decay of solutions, $1 \leq p \leq q \leq \infty$:

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$$

Refined asymptotics

Zuazua& Duoandikoetxea, CRAS '92
For all $\varphi \in L^p(\mathbb{R}^d, 1+|x|^k)$

$$u(t,\cdot) \sim \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha dx \right) D^\alpha H(t,\cdot) \qquad \text{in } L^q(\mathbb{R}^d)$$

for some p, q, k

A linear nonlocal problem

E. Chasseigne, M. Chaves and J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl., 86, 271–291, (2006).

$$\begin{cases} u_t(x,t) &= J*u-u(x,t)=\int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy-u(x,t),\\ &= \int_{\mathbb{R}^d} J(x-y)(u(y,t)-u(x,t))dy\\ u(x,0) &= u_0(x),\end{cases}$$

where $J:\mathbb{R}^N\to\mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)dr=1$

Some applications

P. Fife. *Some nonclassical trends in parabolic and parabolic-like evolutions.* Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

• $u(x,t)$ - the density of a single population at the point x at time t
• $J(x-y)$ - the probability distribution of jumping from y to x
Then
• $(J*u)(x,t) = \int_{\mathbb{R}^d} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to x from all other places and $-u(x,t) = -\int_{\mathbb{R}^d} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving x to travel to all other sites.
Thus in the absence of external or internal sources, the density u satisfies the nonlocal equation (4).

Some applications

P. Fife. *Some nonclassical trends in parabolic and parabolic-like evolutions.* Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

• $u(x,t)$ - the density of a single population at the point x at time t

• $J(x-y)$ - the probability distribution of jumping from y to x

Then

• $(J*u)(x,t) = \int_{\mathbb{R}^d} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to x from all other places and $-u(x,t) = -\int_{\mathbb{R}^d} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving x to travel to all other sites.

Thus in the absence of external or internal sources, the density u satisfies the nonlocal equation (4).

Some applications

P. Fife. *Some nonclassical trends in parabolic and parabolic-like evolutions.* Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

• $u(x,t)$ - the density of a single population at the point x at time t
• $J(x-y)$ - the probability distribution of jumping from y to x
Then
• $(J*u)(x,t)=\int_{\mathbb{R}^d} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to x from all other places and $-u(x,t)=-\int_{\mathbb{R}^d} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving x to travel to all other sites.
Thus in the absence of external or internal sources, the density u satisfies the nonlocal equation (4).

Heat equation and nonlocal diffusion

- Similarities
  - bounded stationary solutions are constant
  - a maximum principle holds for both of them

- Difference
  - there is no regularizing effect in general

  The fundamental solution can be decomposed as

$$w(x,t) = e^{-t}\delta_0(x) + v(x,t), \tag{1}$$

with $v(x,t)$ smooth

$$S(t)\varphi = e^{-t}\varphi + v * \varphi = \text{smooth as initial data} + \text{smooth part}$$
$$= \text{no smoothing effect}$$

Heat equation and nonlocal diffusion

- Similarities
  • bounded stationary solutions are constant
  • a maximum principle holds for both of them

- Difference
  • there is no regularizing effect in general
  The fundamental solution can be decomposed as

$$w(x,t) = e^{-t}\delta_0(x) + v(x,t), \tag{1}$$

with $v(x,t)$ smooth

$$\begin{aligned} S(t)\varphi = e^{-t}\varphi + v * \varphi &= \text{smooth as initial data} + \text{smooth part} \\ &= \text{no smoothing effect} \end{aligned}$$

Asymptotic Behaviour

• If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \xi \sim 0$, $1 < \alpha \leq 2$, the asymptotic behavior is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the laplacian:

$$\lim_{t\to+\infty} t^{d/\alpha} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -A(-\Delta)^{\alpha/2} v(x,t)$ with initial condition $v(x,0) = u_0(x)$.

• The asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{d/\alpha} u(yt^{1/\alpha}, t) - (\int_{\mathbb{R}^d} u_0) G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G_A}(\xi) = e^{-A|\xi|^\alpha}$.

Other results on the linear problem

- I.L. Ignat and J.D. Rossi, *Refined asymptotic expansions for nonlocal diffusion equations*, Journal of Evolution Equations 2008.
- I.L. Ignat and J.D. Rossi, *Asymptotic behaviour for a nonlocal diffusion equation on a lattice*, ZAMP 2008.

The classical convection-diffusion equation

For $a \in \mathbb{R}^d$ and $q \geq 1$

$$\begin{cases} u_t - \Delta u = a \cdot \nabla(|u|^{q-1}u) \text{ in } (0,\infty) \times \mathbb{R}^d \\ u(0) = u_0 \end{cases}$$

- Asymptotic Behaviour by using

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^p dx = -\frac{4(p-1)}{p}\int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2 dx.$$

M. Schonbek, *Uniform decay rates for parabolic conservation laws*, Nonlinear Anal., 10(9), 943–956, (1986).

M. Escobedo and E. Zuazua, *Large time behavior for convection-diffusion equations in $\mathbb{R}^N$*, J. Funct. Anal., 100(1), 119–161, (1991).

Methods to obtain asymptotics

- M. Schonbek $\rightarrow$ Fourier Splitting Method
- M. Escobedo and E. Zuazua $\rightarrow$ Energy method

For $d \geq 3$ using the Sobolev inequality $\|v\|_{2d/(d-2)} \leq C(d)\|\nabla v\|_2$ with $v = |u|^{p/2}$ and the contraction of the L^1-norm of the solutions we get

$$\frac{d}{dt}\|u(t)\|_p^p + \frac{C}{\|u_0\|_1^{2p/d(p-1)}}\|u(t)\|_p^{p[d(p-1)+2]/d(p-1)} \leq 0$$

.... etc...

Nonlocal Convection-Diffusion

L.I. Ignat and J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal., 251, 399–437, (2007).

$$\begin{cases} u_t(t,x) = (J * u - u)(t,x) + (G * (f(u)) - f(u))(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

- J and G are nonnegatives and verify $\int_{\mathbb{R}^d} J(x)dx = \int_{\mathbb{R}^d} G(x)dx = 1$.
- J radially symmetric
- G not symmetric, then the individuals have greater probability of jumping in one direction than in others, provoking a convective effect
- $f(u) = |u|^{q-1}u$ with $q > 1$

Well-posedness

Theorem

For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *there exists a unique global solution*

$$u \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty); \mathbb{R}^d).$$

If u *and* v *are solutions of* (11) *corresponding to initial data* $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *respectively, then the following contraction property*

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$$

holds for any $t \ge 0$. *In addition,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

Why not $u_0 \in L^1(\mathbb{R}^d)$?

- For the local problem $\|v(t)\|_\infty \leq C\|v_0\|_1 t^{-d/2}$ for any $v_0 \in L^1 \cap L^\infty$, so for any $u_0 \in L^1$ we can choose $u_{0,\epsilon} \in L^1 \cap L^\infty$ with $u_{0,\epsilon} \to u_0$ in L^1 and we can pass to the limit.
- In the nonlocal model, we cannot prove such type of inequality independently of the L^∞-norm of the initial data.
- In the one-dimensional case with $f(u) = |u|^{q-1}u$, $1 \leq q < 2$ we have

$$\sup_{u_0 \in L^1(\mathbb{R})} \sup_{t \in [0,1]} \frac{t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = \infty.$$

- The $L^1 - L^\infty$ regularizing effect is not available for the linear equation $w_t = J * w - w$

Long time behaviour of the solutions

Theorem

Let $f = |u|^{q-1}u$ with $q > 1$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for every $p \in [1, \infty)$ the solution u of equation (11) verifies

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}) \langle t \rangle^{-\frac{d}{2}(1-\frac{1}{p})}. \tag{2}$$

Proof 1: Adapted Fourier Splitting method - JFA 2007

Why not an energy method ?

If we want to use energy estimates to get decay rates (for example in $L^2(\mathbb{R}^d)$), we arrive easily to

$$\frac{d}{dt}\int_{\mathbb{R}^d}|w(t,x)|^2\,dx = (\leq) - \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(w(t,x)-w(t,y))^2\,dx\,dy$$

However, we can not go further since an inequality of the form

$$\left(\int_{\mathbb{R}^d}|u(x)|^p\,dx\right)^{\frac{2}{p}} \leq C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))^2\,dx\,dy$$

is not available for $p>2$.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))^2\,dx\,dy = \int_{\mathbb{R}^d}(1-\hat{J}(\xi))|\hat{u}(\xi)|^2$$

A $1/2$-energy method :)

Rossi J. & I.L., JMPA 2009

$$< Aw, w >= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(w(x)-w(y))^2 dxdy = \int_{\mathbb{R}^d} (1-\hat{J}(\xi))|\hat{w}(\xi)|^2 d\xi.$$

Lemma

Let be $d \geq 3$, $\epsilon \in (0,1)$ and $\alpha(\epsilon) \in (0,1)$ given by

$$\alpha(\epsilon) = \frac{(1-\epsilon)d}{d+2-\epsilon(d-2)}.$$

Then there exists a constant $C = C(\epsilon, \delta)$ such that

$$\|u\|^2_{L^2(\mathbb{R})} \leq C\|u\|^{2(1-\alpha(\epsilon))}_{L^{1+\epsilon}(\mathbb{R}^d)} < Au, u >^{\alpha(\epsilon)} + < Au, u > . \tag{3}$$

As a consequence

$$\|u\|^2_{L^2(\mathbb{R})} \lesssim f_\epsilon(< Au, u >)$$

and

$$f_\epsilon^{-1}(\|u\|^2_{L^2(\mathbb{R})}) \leq < Au, u > .$$

Using that

$$\frac{d}{dt}\|u(t)\|^2 + < Au, u > \leq 0$$

we find that $\psi(t) = \|u(t)\|^2_{L^2(\mathbb{R}^d)}$ satisfies the following differential inequality

$$\psi_t + \psi^{1/\alpha(\epsilon)}\chi_{\{\psi \lesssim 1\}} + \psi\chi_{\{\psi \gtrsim 1\}} \leq 0$$

and we get the right $L^{1+\epsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ decay.

As a consequence

$$\|u\|^2_{L^2(\mathbb{R})} \lesssim f_\epsilon(< Au, u >)$$

and

$$f_\epsilon^{-1}(\|u\|^2_{L^2(\mathbb{R})}) \leq < Au, u > .$$

Using that

$$\frac{d}{dt}\|u(t)\|^2 + < Au, u > \leq 0$$

we find that $\psi(t) = \|u(t)\|^2_{L^2(\mathbb{R}^d)}$ satisfies the following differential inequality

$$\psi_t + \psi^{1/\alpha(\epsilon)}\chi_{\{\psi \lesssim 1\}} + \psi\chi_{\{\psi \gtrsim 1\}} \leq 0$$

and we get the right $L^{1+\epsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ decay.

A short proof

For any function $u \in L^2(\mathbb{R}^d)$ we define its projections on the low, respectively high frequencies:

$$v := (\mathbf{1}_{\{|\xi|_\infty \leq R\}} \hat{u})^\vee, \ w := (\mathbf{1}_{\{|\xi|_\infty \geq R\}} \hat{u})^\vee.$$

Orthogonality gives us

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}.$$

Fourier multipliers

$$\|v\|_{L^p(\mathbb{R}^d)} \leq C(p,d)\|u\|_{L^p(\mathbb{R}^d)} \tag{4}$$

for any $1 < p < \infty$.

The assumptions on $\hat{J}$ give us

$$< Au, u > = \int_{\mathbb{R}^d} (1 - \hat{J}(\xi))\hat{u}^2(\xi)d\xi \geq \int_{|\xi|\leq R} \frac{\xi^2}{2}\hat{u}^2(\xi)d\xi + \delta \int_{|\xi|\geq R} \hat{u}^2(\xi)d\xi$$
$$\geq c(\delta)\Big(\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\Big)$$
$$\geq c(\delta)(\|v\|^2_{L^{2^*}(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\Big).$$

The low frequency projection, v, satisfies:

$$\|v\|^2_{L^2(\mathbb{R}^d)} \leq \left(\|v\|^{1-\alpha(\epsilon)}_{L^{1+\epsilon}(\mathbb{R}^d)}\|v\|^{\alpha(\epsilon)}_{L^{2^*}(\mathbb{R}^d)}\right)^2 \leq \left(\|u\|^{1-\alpha(\epsilon)}_{L^{1+\epsilon}(\mathbb{R}^d)}\|v\|^{\alpha(\epsilon)}_{L^{2^*}(\mathbb{R}^d)}\right)^2 \leq \|u\|^{2(1-\alpha(\epsilon))}_{L^{1+\epsilon}(\mathbb{R}^d)} < Au, u >^{\alpha(\epsilon)} . \tag{5}$$

Thus

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^d)} &= \|v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \|u\|^{2(1-\alpha(\epsilon))}_{L^{1+\epsilon}(\mathbb{R}^d)} < Au, u >^{\alpha(\epsilon)} + < Au, u >\end{aligned}$$

Using the differential inequality

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^d)} + < Au, u > \leq 0$$

we now can get the $L^{1+\epsilon} - L^2$ decay property.

Variants of the energy method

- For the linear part we can remove the convolution property of the function J

$$u_t(t,x) = \int_{\mathbb{R}} J(x,y)(u(t,y) - u(t,x))dy$$

under suitable properties of J: symmetric, etc...

- An energy method for the nonlocal p-Laplacian: $< A_p u, u >= \int_{\mathbb{R}^d} J(x,y)|u(x) - u(y)|^p dxdy$.

A theorem without Fourier analysis

Let $p \in [1, \infty)$ and $J(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d$ be a symmetric function satisfying
HJ1) There exist positive constant $0 < C_1, C_2 < \infty$ such that

$$C_1 \leq \int_{\mathbb{R}^d} J(x,y)\, dy \leq C_2. \tag{6}$$

HJ2) There exist positive constants c_1, c_2 and a function $a \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfying

$$\sup_{x \in \mathbb{R}^d} |\nabla a(x)| < \infty \tag{7}$$

such that the set

$$B_x = \{y \in \mathbb{R}^d : |y - a(x)| \leq c_2\} \tag{8}$$

verifies

$$B_x \subset \{y \in \mathbb{R}^d : \ J(x,y) > c_1\}.$$

For any function $u \in L^p(\mathbb{R}^d)$ there exist two functions v and w such that $u = v + w$ and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)|u(x) - u(y)|^p \, dx \, dy.$$

Moreover, if $u \in L^q(\mathbb{R}^d)$ with $q \in [1, \infty]$ then the function v satisfies

$$\|v\|_{L^q(\mathbb{R}^d)} \leq C(J,p) \|u\|_{L^q(\mathbb{R}^d)}$$

Proof of the case $p = 2$.

Choose ψ is a smooth function supported in $B_{c_2}(0)$ with mass one and

$$v(x) = \int_{\mathbb{R}^d} \psi(a(x) - y)u(y)dy.$$

Observe that

$$w(x) = u(x) - v(x) = \int_{\mathbb{R}^d} \psi(a(x) - y)(u(y) - u(x))dy$$

and

$$\partial_{x_i} v(x) = \partial_{x_i} a(x) \int_{\mathbb{R}^d} \nabla\psi(a(x) - y)(u(y) - u(x))dy$$

Using Hölder inequality we obtain the desired result:

$$|w(x)|^2 = (\int_{|y-a(x)|\le c_2} \psi(a(x)-y)|u(y)-u(x)|dy)^2$$
$$\le \|\psi\|^2_{L^2(\mathbb{R}^d)} \int_{|y-a(x)|\le c_2} |u(y)-u(x)|^2 dy.$$

and

$$\int_{\mathbb{R}^d} |w(x)|^2 \le \|\psi\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{|y-a(x)|\le c_2} |u(y)-u(x)|^2 dydx$$
$$\lesssim \|\psi\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)|u(y)-u(x)|^2 dydx.$$

Similar results hold for v provided $\nabla a \in L^\infty(\mathbb{R}^d)$.

A corollary

Let $J(\cdot,\cdot):\mathbb{R}^d\times\mathbb{R}^d$ satisfying HJ and $d\geq 3$. There exist two positive constants $C_1=C_1(J)$ and $C_2=C_2(J)$ such that for any $u\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ the following holds:

$$\|u\|^2_{L^2(\mathbb{R}^d)}\leq C_1\|u\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}\langle A_2u,u\rangle^{\alpha(2)}+C_2\langle A_2u,u\rangle, \tag{9}$$

where $\alpha(2)$ satisfies:

$$\frac{1}{2}=\frac{\alpha(2)}{2^*}+\frac{1-\alpha(2)}{1}. \tag{10}$$

Using now that

$$\frac{d}{dt}\int_{\mathbb{R}^d}u^2+<A_2u,u>\leq 0$$

we obtan the decay of u in the L^2-norm.

A corollary

Let $J(\cdot,\cdot):\mathbb{R}^d\times\mathbb{R}^d$ satisfying HJ and $d\geq 3$. There exist two positive constants $C_1=C_1(J)$ and $C_2=C_2(J)$ such that for any $u\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ the following holds:

$$\|u\|^2_{L^2(\mathbb{R}^d)}\leq C_1\|u\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}\langle A_2u,u\rangle^{\alpha(2)}+C_2\langle A_2u,u\rangle, \tag{9}$$

where $\alpha(2)$ satisfies:

$$\frac{1}{2}=\frac{\alpha(2)}{2^*}+\frac{1-\alpha(2)}{1}. \tag{10}$$

Using now that

$$\frac{d}{dt}\int_{\mathbb{R}^d}u^2+<A_2u,u>\leq 0$$

we obtan the decay of u in the L^2-norm.

Another question: optimality of the decay?

There are cases where the decay is exponential.

Lemma

Let $a:\mathbb{R}\to\mathbb{R}$ be a diffeomorfism. Assume that

$$J(x,y)>\frac{1}{2} \quad \text{on} \quad |y-a(x)|\leq 1,$$

where the function a satisfies

$$\sup_{\mathbb{R}}|a_x|<1 \quad \text{or} \quad \inf_{\mathbb{R}}|a_x|>1$$

then there exists a positive constant C such that

$$\langle A_2 u, u\rangle \geq C\|u\|^2_{L^2(\mathbb{R})}.$$

Another question: optimality of the decay?

There are cases where the decay is exponential.

Lemma

Let $a:\mathbb{R}\to\mathbb{R}$ be a diffeomorfism. Assume that

$$J(x,y)>\frac{1}{2} \quad \textit{on} \quad |y-a(x)|\le 1,$$

where the function a satisfies

$$\sup_{\mathbb{R}}|a_x|<1 \quad \textit{or} \quad \inf_{\mathbb{R}}|a_x|>1$$

then there exists a positive constant C such that

$$\langle A_2 u,u\rangle \ge C\|u\|^2_{L^2(\mathbb{R})}.$$

Let us consider $\psi : \mathbb{R} \to \mathbb{R}$ a smooth positive function, supported in $(-1,1)$. Then

$$
\begin{aligned}
2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2 u, u\rangle &\geq \iint_{\mathbb{R}^2} \psi(y-a(x))(u(x)-u(y))^2 dxdy \\
&\geq (1-\theta)\iint_{\mathbb{R}^2} \psi(y-a(x))\Big(u^2(x)-\frac{u^2(y)}{\theta}\Big)dxdy \\
&= (1-\theta)\int_{\mathbb{R}} u^2(x)\Big(\int_{\mathbb{R}}\psi(y)dy - \frac{1}{\theta}\int_{\mathbb{R}}\psi(x-a(y))dy\Big)dx \\
&= \frac{1-\theta}{\theta}\int_{\mathbb{R}}\psi(y)dy\int_{\mathbb{R}} u^2(x)\Big(\theta - \frac{(\psi * |(a^{-1})_x|)(x)}{\int_{\mathbb{R}}\psi(y)dy}\Big)dx.
\end{aligned}
$$

Some about eigenvalues

For any interval $I_r = (-r, r)$ we consider

$$\lambda_1(I_r) = \inf_{u \in L^2(I_r)} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} J(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dx dy}{\int_{I_r} u^2(x) dx}$$

Q: When

$$\lim_{r \to \infty} \lambda_1(I_r) > 0?$$

In the case $J(x,y) = J(x-y)$, Rossi and Garcia-Melian proved that

$$\lambda_1(I_r) \simeq \frac{1}{r^2}, \; r \to \infty$$

Some about eigenvalues

For any interval $I_r = (-r, r)$ we consider

$$\lambda_1(I_r) = \inf_{u \in L^2(I_r)} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} J(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dxdy}{\int_{I_r} u^2(x) dx}$$

Q: When

$$\lim_{r\to\infty} \lambda_1(I_r) > 0?$$

In the case $J(x,y) = J(x-y)$, Rossi and Garcia-Melian proved that

$$\lambda_1(I_r) \simeq \frac{1}{r^2}, \, r \to \infty$$

Some about eigenvalues

For any interval $I_r = (-r, r)$ we consider

$$\lambda_1(I_r) = \inf_{u \in L^2(I_r)} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} J(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dx dy}{\int_{I_r} u^2(x) dx}$$

Q: When

$$\lim_{r \to \infty} \lambda_1(I_r) > 0?$$

In the case $J(x,y) = J(x-y)$, Rossi and Garcia-Melian proved that

$$\lambda_1(I_r) \simeq \frac{1}{r^2}, \, r \to \infty$$

Some about eigenvalues

In the particular case

$$J(x,y) = \psi(y - a(x)) + \psi(x - a(y)),$$

ψ compactly supported, we conjecture that

$$\lim_{r\to\infty} \lambda_1(I_r) > 0$$

iff the set $\{x : a_x(x) = 1\}$ is ...

The following two problems are related

$$\inf_u \frac{\int_{\mathbb{R}}\int_{\mathbb{R}} \psi(y-a(x))(u(x)-u(y))^2 dxdy}{\int_{\mathbb{R}} u(x)^2}$$

and

$$\inf_u \frac{\int_{\mathbb{R}}(u(x)-u(a(x)))^2}{\int_{\mathbb{R}} u(x)^2}$$

by choosing $\psi_\epsilon \to \delta_0$.

For the problem

$$\inf_u \frac{\int_{\mathbb{R}}(u(x)-u(a(x))^2}{\int_{\mathbb{R}} u(x)^2}$$

we know

1. $\inf = 0$ if $a(I) = I$ for some bounded interval.
2. $\inf = 0$ if $a(x_0) = x_0$ with $a_x(x_0) = 1$
3. $\inf > 0$ if $a(x) = mx + n$ with $m \neq 1$
4. $\inf = 0$ if $a(x) = x + 1$

The infimum could be related with the orbit (A. Galatean)

$$\{a^{(n)}(x) : n \in \mathbb{Z}\}$$

For the problem

$$\inf_u \frac{\int_{\mathbb{R}}(u(x)-u(a(x)))^2}{\int_{\mathbb{R}} u(x)^2}$$

we know

1. $\inf = 0$ if $a(I) = I$ for some bounded interval.
2. $\inf = 0$ if $a(x_0) = x_0$ with $a_x(x_0) = 1$
3. $\inf > 0$ if $a(x) = mx + n$ with $m \neq 1$
4. $\inf = 0$ if $a(x) = x + 1$

The infimum could be related with the orbit (A. Galatean)

$$\{a^{(n)}(x) : n \in \mathbb{Z}\}$$

Some Open Problems

- Convection-diffusion. Until now we treat the case $q > 1 + 1/d$, case when the diffusion play the important role. What it happens when $q < 1 + 1/d$? Is there something similar to the local case, where the convection part gives the asymptotic behaviour (Escobedo, Vazquez, Zuazua, ARMA 1993)
- The optimality of the decay in the non-symmetric case. The case $a(x) = 2x$ enters in the two lemmas, but $a(x) = x^2 \operatorname{sgn}(x)$ does not entry in any of them.
- Discrete models. Work in progress with A. Galatean (Bucharest)
- Analysis of the long time behaviour of $\lambda_1(B_R)$ when $a(x)$ is close to x

End of the talk

Julio Rossi's web page http://mate.dm.uba.ar/~jrossi/
My web page http://sites.google.com/site/liviuignat/

Supported by Romanian reintegration grant PN-II-Rp-3-2007 of CNCSIS, Romania.

THANKS!!!