$\mathcal{PT}$-symmetry versus pseudo-Hermiticity

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$\mathcal{PT}$-symmetry

Origins of $\mathcal{PT}$-symmetry

- Hamiltonian $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ has real, positive, discrete spectrum

  Bender, Boettcher 1998

- original hypothesis - the reality of spectrum due to $\mathcal{PT}$-symmetry
  - $[\mathcal{PT}, H] = 0$
  - parity $\mathcal{P}$, $(\mathcal{P}\psi)(x) = \psi(-x)$
  - complex conjugation $\mathcal{T}$, $(\mathcal{T}\psi)(x) = \overline{\psi}(x)$
- $\mathcal{PT}$-symmetry is not sufficient for reality of the spectrum
- some $\mathcal{PT}$-symmetric operators are similar to the self-adjoint ones

  $h = \varrho^{-1} H \varrho = h^*$

Antilinear symmetry without pseudo-Hermiticity

Example

- $\{e_n\}_{n=1}^{\infty}$ standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, $e_n(m) = \delta_{mn}$
- $Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0$
- $T^* e_n := e_{n+1}, \quad n \in \mathbb{N}$
- $T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots & \ddots \end{pmatrix}$

Pseudo-Hermiticity without antilinear symmetry

Example

- $\{e_i\}_{-\infty}^{\infty}$ orthonormal basis of $\mathcal{H} = l^2(\mathbb{Z})$, $e_n(m) = \delta_{mn}$
- $$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \overline{\lambda}_0 e_{-1}, & i = -1, \\ \overline{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases}$$
- $\lambda_0 \in \mathbb{C}, \ \operatorname{Im} \lambda_0 > \frac{1}{2}$

Pseudo-Hermiticity without antilinear symmetry

Example

- $T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \overline{\lambda_0} & 0 & 0 & 0 & 0 & \\ & 1 & \overline{\lambda_0} & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \lambda_0 & 0 & \\ & 0 & 0 & 0 & 1 & \lambda_0 & \\ & & & & & \ddots & \ddots \end{pmatrix}$