Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed

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Abstract

The general theory on exact boundary controllability for general first order quasilinear hyperbolic systems requires that the characteristic speeds of system do not vanish. This paper deals with exact boundary controllability, when this is not the case. Some important models are also shown as applications of the main result. The strategy uses the return method, which allows in certain situations to recover non zero characteristic speeds.

Keywords: Quasilinear hyperbolic system, vanishing characteristic speed, exact boundary controllability, return method

1 Introduction and main results

The general theory on exact boundary controllability for general first order quasilinear hyperbolic systems requires that the system has non vanishing characteristic speeds [21, 22]. Several papers have dealt with hyperbolic systems having a vanishing or an identically zero characteristic speed, under various assumptions. For systems with identically zero characteristic speeds, general results on exact controllability have been obtained by using internal controls [23, 24]. It is also possible to get in this case partial controllability by boundary controls, if some eigenvalue of the system is equal to zero identically [26]. A steady state

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controllability holds for some special hyperbolic models with vanishing characteristic speed as Saint-Venant equations (or shallow water equations), see Gugat [13]. For what concerns the system of isentropic gas dynamics (which contains the Saint-Venant model), a general boundary controllability result for (non constant) BV solutions was obtained by the second author in [12].

In this paper, we will discuss exact boundary controllability for a general hyperbolic system which admits a vanishing characteristic speed. Typically, the result applies to the control of physical systems admitting a critical speed: this contains the case of the Saint-Venant equation (one can travel between sub-critical, super-critical and critical states), the isentropic and full Euler equation for one-dimensional gas dynamics (one can travel between subsonic, supersonic and sonic states); see Section 4.

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0,\ (t,x) \in [0,T] \times [0,L], \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{tr}(t,x)$ is the state of the system in some nonempty open set $\Omega \subset \mathbb{R}^n$ and the $n \times n$ matrix A belongs to $C^2(\Omega; \mathbb{R}^{n\times n})$.

Let $u^* \in \Omega$ be fixed. Assume that $A(u^*)$ has n real distinct eigenvalues:

$$\lambda_1(u^*) < \cdots < \lambda_{m-1}(u^*) < \lambda_m(u^*) = 0 < \lambda_{m+1}(u^*) < \cdots < \lambda_n(u^*), \tag{1.2}$$

for some $m \in \{1, \cdots, n\}$, which are the characteristic speeds at which the system propagates. Thus in a neighborhood of the equilibrium $u = u^*$, the system is strictly hyperbolic and $A(u)$ has a complete set of left (resp. right) eigenvectors $l_1(u), \cdots, l_n(u)$ (resp. $r_1(u), \cdots, r_n(u)$):

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)), \quad i = 1, \cdots, n. \tag{1.3}$$

Without loss of generality, let us assume that

$$l_i(u)r_j(u) = \delta_{ij}, \quad i,j = 1, \cdots, n, \tag{1.4}$$

where δ_{ij} is Kronecker's symbol. Reducing Ω if necessary, we assume that

$$\forall j \in \{1, \ldots, m-1\},\ \lambda_j(u) < 0 \text{ and } \forall j \in \{m+1, \ldots, n\},\ \lambda_j(u) > 0, \quad \forall u \in \Omega. \tag{1.5}$$

Now the question is: is it possible to realize the local exact controllability near the equilibrium $u = u^*$ only by using boundary controls?

In order to overcome the difficulty of a characteristic speed vanishing at u^*, we assume the following hypothesis:

(H): for all $\varepsilon > 0$, there exists $\alpha = (\alpha_1, \cdots, \alpha_{m-1}, \alpha_{m+1}, \cdots, \alpha_n) \in L^\infty(0,1; \mathbb{R}^{n-1})$ with

$$\|\alpha\|_{L^\infty(0,1;\mathbb{R}^{n-1})} \leq \varepsilon, \tag{1.6}$$

such that the solution $z \in C^0([0,1];\mathbb{R}^n)$ of the ordinary differential equation

$$\frac{dz}{ds} = \sum_{j\neq m} \alpha_j(s) r_j(z), \quad z(0) = u^*, \tag{1.7}$$

satisfies

$$\lambda_m(z(1)) \neq 0. \tag{1.8}$$

An additional difficulty when considering hyperbolic systems with characteristic speeds whose sign may change is that it is difficult to describe the exact distribution of boundary controls, since the number of boundary data to be imposed depends on the state of the solution itself. To overcome this difficulty, we consider the system without boundary conditions (which is consequently under-determined), and aim at finding the solution u itself. Obviously, one can recover the boundary controls as the traces of u afterwards.

The main result of this paper is the following theorem:

Theorem 1.1. *Let* (1.2) *and* (H) *be true. Then, for any $\delta > 0$, there exist $T > 0$ and $\nu > 0$ such that, for all $\varphi, \psi \in C^1([0,L];\mathbb{R}^n)$ satisfying*

$$\|\varphi(\cdot) - u^*\|_{C^1([0,L])} \leq \nu, \quad \|\psi(\cdot) - u^*\|_{C^1([0,L])} \leq \nu, \tag{1.9}$$

there exists $u \in C^1([0,T]\times[0,L];\mathbb{R}^n)$ such that

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad \forall (t,x) \in [0,T]\times[0,L], \tag{1.10}$$

$$u(0,x) = \varphi(x), \quad \forall x \in [0,L], \tag{1.11}$$

$$u(T,x) = \psi(x), \quad \forall x \in [0,L], \tag{1.12}$$

$$\|u(t,\cdot) - u^*\|_{C^1([0,L])} \leq \delta, \quad \forall t \in [0,T]. \tag{1.13}$$

The hypothesis (H) seems quite difficult to check. However, we have some sufficient conditions of (H) relying on Lie brackets.

Proposition 1.1. *The following properties are sufficient conditions for (H) to hold:*

(H1): *there exists $j \in \{1,\cdots,n\}\setminus\{m\}$ such that $\nabla\lambda_m(u^*)\cdot r_j(u^*) \neq 0$,*

(H2): *there exist $j,k \in \{1,\cdots,n\}\setminus\{m\}$ such that $\nabla\lambda_m(u^*)\cdot [r_j, r_k](u^*) \neq 0$,*

(H3): *$A \in C^\infty(\Omega;\mathbb{R}^{n\times n})$ and there exists $h \in Lie\{r_1,\cdots,r_{m-1},r_{m+1},\cdots,r_n\}$, such that $\nabla\lambda_m(u^*)\cdot h(u^*) \neq 0$,*

(H4): *$A \in C^\infty(\Omega;\mathbb{R}^{n\times n})$ and $\{h(u^*),\ h \in Lie\{r_1,\cdots,r_{m-1},r_{m+1},\cdots,r_n\}\} = \mathbb{R}^n$ and u^* is in the closure of $\{u \in \Omega:\ \lambda_m(u) \neq 0\}$.*

Here $Lie\{r_1, \cdots, r_{m-1}, r_{m+1}, \cdots, r_n\}$ denotes the Lie algebra generated by the smooth vector fields $r_1,\ldots,r_{m-1},r_{m+1},\ldots,r_n$.

Proof of Proposition 1.1. It is a consequence of Chow and Rashevski's connectivity Theorem (see for instance [9, Theorem 3.19, p. 135]) that (H4) implies (H). Next we notice that both (H1) and (H2) clearly imply (H3). So we have left to prove that (H3) implies (H). From (H3) we deduce that the exists a direction $b \in \mathbb{R}^n$ obtained by p successive Lie brackets and such that $\nabla\lambda_m(u^*) \cdot b \neq 0$. We use [16, Lemma 1, p. 456] to deduce that there are controls α which are arbitrarily small in L^∞ norm such that the corresponding solution of (1.7) satisfies $z(4^p t^{1/(1+p)}) = u^* + tb + o(t)$ as $t \to 0$. The conclusion follows. $\square$

Remark 1.1. Theorem 1.1 can be regarded as a local boundary controllability result because one can drive any initial data φ to any desired data ψ near $u = u^*$ without using any internal controls. In the conservative case (where $A(u)$ is a Jacobian matrix $Df(u)$), the solution that we determine can enter the general theory of initial-boundary problems for systems of conservation laws in the context of entropy solutions, as it was introduced by Dubois and LeFloch [10]. The solution is not required to be exactly equal to the boundary condition on $x = 0$ or $x = L$, but one requires that the solution of the Riemann problem between the boundary condition and the value of the solution at $x = 0$ (resp. $x = L$) has only waves of negative (resp. positive) speed. For the study of the initial-boundary value problem in this framework, we refer in particular to Amadori [1] and Amadori and Colombo [2]. Actually these papers are concerned with the initial boundary problem on the half line, but due to the finite speed of propagation of hyperbolic systems, the local in time theory can be transferred to a bounded interval without additional effort.

Remark 1.2. One could ask whether the solution that we obtain Theorem 1.1 requires a finite number of switches on the boundary, that is, of change of sign of the characteristic speeds on the boundary, since this determines the number of boundary conditions to be imposed. In particular in [13], one can pass from a sub-critical to a super-critical steady state by using a single switch (hence one can move between two super-critical states of opposite sign by using two switches). In the general case this is not clear and depends on the geometry of the function A. But in particular cases such as the one described in Section 4 (which all satisfy assumption **(H1)**), this will be the case as will be clear by following the lines of the proof.

Remark 1.3. Due to the hypothesis (H), one can drive the possible vanishing characteristic speed λ_m to be nonzero after sufficiently long time by only using boundary controls. However, if some characteristic speeds of the system are identically zero, the approach of this paper is not valid anymore. Is boundary controllability possible in such cases, even for some special models? Up to our knowledge, this question remains open.

Remark 1.4. We could treat the case where $A \in C^1(\Omega; \mathbb{R}^{n\times n})$, see in particular Remark 3.1 below.

The main idea to prove Theorem 1.1 is to use a constructive approach and the return method [7]. In our framework the method consists in constructing a trajectory $\overline{u} \in C^2([0,T]\times[0,L];\mathbb{R}^n)$ of the system (1.1), close to u^* such that

$$\overline{u}(0,x) = \overline{u}(T,x) = u^*, \quad \forall x \in [0,L], \tag{1.14}$$

and that the linearized equation around $\overline{u}$ is controllable. Note indeed that the linearized equation around u^* is not controllable. Based on this, we can construct a solution $u \in C^1([0,T]\times[0,L];\mathbb{R}^n)$ to the system (1.1) which connects the initial and final data (which have to be sufficiently close to u^*).

As a matter of fact, we will not use the linearized equation. Instead, we use an argument of perturbation of the trajectory $\overline{u}$ and then reduce the original control problem to a boundary control problem without vanishing characteristic speeds, which has been solved by Li and Rao [22]. In the framework of systems of conservation laws, the return method has also been used in [6, 8, 12, 18], see also [3]. For other applications of the return method, see [9] and the references therein.

Without loss of generality, we may assume the equilibrium u^* to be 0, replacing u by $u-u^*$ as the unknown in the system (1.1) if necessary. For the convenience of statement, we denote by C various positive constants in the whole paper which may change from one line to another.

The organization of this paper is as follows: in Section 2 we construct the special trajectory $\overline{u} \in C^2([0,T]\times[0,L];\mathbb{R}^n)$ of the system (1.1) which starts at 0 and returns to 0, and such that the equation linearized around $\overline{u}$ is controllable. Then we prove the main result, Theorem 1.1, in Section 3. Some important applications are shown in Section 4, including Saint-Venant equations (shallow water equations), 1-D isentropic gas dynamics equations, 1-D full gas dynamics equations and Aw-Rascle model on traffic flow and its generalization. Finally in Appendix A, we establish a technical result.

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2 Construction of the trajectory $\overline{u}$

Definition 2.1. Let $j \in \{1,\cdots,n\}$ and $u^0 \in \Omega$. Let $s \in [-\varepsilon_0,\varepsilon_0] \mapsto U_j(s) \in \Omega$ be the orbit of the eigenvector field r_j starting at u^0 (or rarefaction curves):

$$\frac{dU_j}{ds} = r_j(U_j), \quad U_j(0) = u^0, \tag{2.1}$$

where $\varepsilon_0 > 0$ is a small constant. Let $\Phi_j(s,\cdot)$ be the corresponding flow map when s varies, i.e.,

$$\Phi_j(s,u^0) := U_j(s), \quad \forall s \in [\varepsilon_0,\varepsilon_0]. \tag{2.2}$$

Remark 2.1. For all $s \in [-\varepsilon_0, \varepsilon_0]$ one has $u^+ = \Phi_j(s, u^-) \iff u^- = \Phi_j(-s, u^+)$.

Our first proposition concerns simple waves which one can use to modify the state in $[0, L]$.

Proposition 2.1. *Let $j \in \{1, \cdots, n\} \setminus \{m\}$ and*

$$T > \frac{L}{|\lambda_j(0)|}. \tag{2.3}$$

There exist $C > 0$ and $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0]$, all $u^-, u^+ \in \Omega$ satisfying

$$|u^-|, |u^+| \leq \varepsilon \text{ and } u^+ = \Phi_j(\overline{s}, u^-) \text{ for some } \overline{s} \text{ such that } |\overline{s}| \leq \varepsilon, \tag{2.4}$$

there exists $u \in C^2([0, T] \times \mathbb{R}; \mathbb{R}^n)$ such that

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \tag{2.5}$$

$$u(0, x) = u^-, \quad \forall x \in [0, L], \tag{2.6}$$

$$u(T, x) = u^+, \quad \forall x \in [0, L], \tag{2.7}$$

$$\|u(t, \cdot)\|_{C^1(\mathbb{R})} \leq C\varepsilon, \quad \forall t \in [0, T]. \tag{2.8}$$

Proof: Without loss of generality, we may assume that $j \in \{1, \cdots, m-1\}$ (the case where $j \in \{m+1, \cdots, n\}$ can be treated similarly by symmetry in x, that is, replacing x by $L - x$ if necessary).

In view of (1.2) and (2.3), there exist $\varepsilon_1 > 0$ and $\eta > 0$ small enough such that

$$T > \max_{|u| \leq \varepsilon_1} \frac{L + \eta}{|\lambda_j(u)|}. \tag{2.9}$$

Let $\varepsilon \in (0, \varepsilon_1]$ and $u^-, u^+ \in \Omega$ be such that (2.4) holds. By Definition 2.1, it is easy to see that

$$|\Phi_j(s, u^-)| \leq C\varepsilon, \quad \forall s \in [-|\overline{s}|, |\overline{s}|]. \tag{2.10}$$

Let $\beta \in C_0^\infty((0, 1); \mathbb{R})$ be such that

$$\int_0^1 \beta(\theta) d\theta = 1. \tag{2.11}$$

Then we let

$$\overline{\beta}(\theta) := \frac{\overline{s}}{\eta} \beta(\frac{\theta}{\eta}), \tag{2.12}$$

which gives that $\overline{\beta} \in C_0^\infty((0, \eta); \mathbb{R})$ and

$$\int_0^\eta \overline{\beta}(\theta) d\theta = \overline{s}. \tag{2.13}$$

From the above, the ordinary differential equation

$$\frac{dy}{d\theta} = \overline{\beta}(\theta) r_j(y), \quad y(0) = u^-, \tag{2.14}$$

admits a unique solution $y(\cdot) = \Phi_j(\sigma(\cdot), u^-) \in C^2([0,\eta];\mathbb{R}^n)$, where

$$\sigma(s) := \int_0^s \overline{\beta}(\theta)d\theta, \quad \forall s \in [0,\eta]. \tag{2.15}$$

Let

$$\varphi(x) := \begin{cases} u^-, & x \leq L, \\ y(x-L), & L < x < L+\eta, \\ u^+, & x \geq L+\eta. \end{cases} \tag{2.16}$$

In the following, we will denote by $C^k(\mathbb{R})$ the space of functions of class C^k whose derivatives up to order k are bounded on $\mathbb{R}$ (and the norm $\|\cdot\|_{C^k(\mathbb{R})}$ is in fact the norm $\|\cdot\|_{W^{k,\infty}(\mathbb{R})}$).

Then by (2.10), (2.12), (2.14) and (2.16), we obtain that

$$\|\varphi\|_{C^0(\mathbb{R})} := \sup_{x\in\mathbb{R}} |\varphi(x)| \leq C\varepsilon, \tag{2.17}$$

$$\|\varphi'\|_{C^0(\mathbb{R})} := \sup_{x\in\mathbb{R}} |\varphi'(x)| \leq C\frac{\overline{s}}{\eta} \leq C\varepsilon. \tag{2.18}$$

Now we focus on the Cauchy problem of (2.5) on $\mathbb{R}$ with the initial condition

$$u(0,x) = \varphi(x), \quad \forall x \in \mathbb{R}. \tag{2.19}$$

It is classical that there exists a unique C^2 solution to the Cauchy problem (2.5) and (2.19) in small time; see for instance [17, p. 55]. Let us prove that: for the fixed time $T > 0$, if ε is sufficiently small, the Cauchy problem (2.5), (2.19) admits a unique solution $u \in C^2([0,T]\times\mathbb{R};\mathbb{R}^n)$ such that (2.6) to (2.8) hold.

To show that, it suffices to obtain a uniform a priori estimate of the solution in C^1 (see [17, Theorem 4.2.5, p. 55]). In order to obtain such an a priori estimate, we assume that the Cauchy problem (2.5), (2.19) admits already a solution $u \in C^2([0,T_0]\times\mathbb{R};\mathbb{R}^n)$ for some $T_0 \in (0,T)$.

For any $i \in \{1,\cdots,n\}$ and any point $(t,x) \in [0,T_0]\times\mathbb{R}$, we can define the $i-th$ characteristic curve $\xi = \xi_i(\tau)$ passing through (t,x) by

$$\frac{d\xi}{d\tau} = \lambda_i(u(\tau,\xi)), \quad \xi(t) = x. \tag{2.20}$$

Introducing

$$v_i := l_i(u)u, \;\; w_i := l_i(u)\frac{\partial u}{\partial x}, \quad i = 1,\cdots,n, \tag{2.21}$$

i.e.,

$$u = \sum_i v_i r_i(u), \;\; \frac{\partial u}{\partial x} = \sum_i w_i r_i(u), \tag{2.22}$$

we know that v_i, w_i $(i = 1,\cdots,n)$ satisfy the following (see [17, p. 47ff] and [19]):

$$\frac{dv_i}{d_i t} = \sum_{j,k} \beta_{ikl}(u)v_k w_l, \quad i = 1,\cdots,n, \tag{2.23}$$

$$\frac{dw_i}{d_i t} = \sum_{k,l} \gamma_{ikl}(u)w_k w_l, \quad i = 1,\cdots,n, \tag{2.24}$$

where

$$\frac{d}{d_i t} := \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.25}$$

denotes the derivative along the i-th characteristic, and where $\beta_{ikl}, \gamma_{ikl} \in C^1(\Omega; \mathbb{R}^n)$ satisfy in particular

$$\gamma_{ikk}(u) = 0, \quad \forall i, k \in \{1, \cdots, n\}, k \neq i, \tag{2.26}$$

$$\gamma_{iii}(u) = -\nabla\lambda_i(u) r_i(u), \quad i = 1, \cdots, n. \tag{2.27}$$

By (2.26)-(2.27), (2.24) can be written as

$$\frac{dw_i}{d_i t} = \sum_{k\neq l} \gamma_{ikl}(u) w_k w_l - (\nabla\lambda_i(u) r_i(u)) w_i^2, \quad i = 1, \cdots, n. \tag{2.28}$$

Combining (2.16)-(2.18) and (2.22), noticing (1.4), we have

$$|v_i(0,x)| = |l_i(\varphi(x))\varphi(x)| \leq C\varepsilon, \quad \forall x \in \mathbb{R}, \forall i \in \{1, \cdots, n\}, \tag{2.29}$$

$$w_i(0,x) = l_i(\varphi(x))\varphi'(x) = 0, \quad \forall x \in \mathbb{R}, \forall i \in \{1, \cdots, n\} \setminus \{j\}, \tag{2.30}$$

$$|w_j(0,x)| = |l_j(\varphi(x))\varphi'(x)| \leq C\varepsilon, \quad \forall x \in \mathbb{R}. \tag{2.31}$$

As in the proof of [17, Theorem 4.2.5, p. 55], we first assume that

$$|v_i(t,x)| \leq 1, \ |w_i(t,x)| \leq 1, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}, \forall i \in \{1, \cdots, n\}. \tag{2.32}$$

By (2.28) when $i \neq j$ and (2.30) , we deduce that

$$w_i(t,x) = 0, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}, \ \forall i \in \{1, \cdots, n\} \setminus \{j\}, \tag{2.33}$$

which then reduces (2.28) when $i = j$ to

$$\frac{dw_j}{d_j t} = -(\nabla\lambda_j(u) r_j(u)) w_j^2. \tag{2.34}$$

By comparing the norm of the solution (2.23) and (2.34) to the solution of the ordinary differential equation $\dot{x} = \|\nabla\lambda_j.r_j\| x^2$, noticing also (2.29) and (2.31), we deduce that the following estimates hold

$$|v_i(t,x)| \leq C\varepsilon, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}, \ \forall i \in \{1, \cdots, n\}, \tag{2.35}$$

$$|w_j(t,x)| \leq C\varepsilon, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}. \tag{2.36}$$

Combining (2.33) and (2.35)-(2.36), there exists $\varepsilon_0 \in (0, \varepsilon_1]$ small enough such that for any $\varepsilon \in (0, \varepsilon_0]$ the assumption (2.32) is indeed satisfied, and the uniform a priori estimate

$$\|u(t,\cdot)\|_{C^1(\mathbb{R})} \leq C\varepsilon, \quad \forall t \in [0, T_0] \tag{2.37}$$

holds for all $T_0 \in (0, T)$. This proves the existence of the solution $u \in C^2([0,T] \times \mathbb{R}; \mathbb{R}^n)$ (see again [17, Theorem 4.2.5, p. 55]). Moreover, since (2.9) implies

$$T > \frac{L + \eta}{|\lambda_j(u^+)|}, \tag{2.38}$$

we derive from the fact

$$w_i(0,x)=0, \quad \forall x\in[L+\eta,\infty), \forall i\in\{1,\cdots,n\} \tag{2.39}$$

that

$$w_i(T,x)=0, \quad \forall x\in[0,\infty), \forall i\in\{1,\cdots,n\}, \tag{2.40}$$

which in turn implies that

$$u(T,x)=const.=\lim_{x\to\infty}u(T,x)=u^{+}, \quad \forall x\in[0,\infty). \tag{2.41}$$

This concludes the proof of Proposition 2.1. □

Remark 2.2. In the proof above, the information travels from right to left through the boundary $x=L$. For $i\in\{1,\ldots,m-1\}$, the information would travel from left to right through the boundary $x=0$.

Remark 2.3. Because (2.5) is an autonomous system, the conclusion of Proposition 2.1 on $[0,T]$ can be achieved on $[t_0,t_0+T]$ for any $t_0\in\mathbb{R}$ by translation in time.

The next proposition proves that one can approximate the trajectory given by (1.7) by a trajectory composed of simple waves.

Proposition 2.2. *There exist $C>0$ and $\varepsilon_0>0$ such that the following holds. For any $\varepsilon\in(0,\varepsilon_0]$, any $\alpha=(\alpha_1,\cdots,\alpha_{m-1},\alpha_{m+1},\cdots,\alpha_n)\in L^\infty(0,1;\mathbb{R}^{n-1})$ satisfying*

$$\|\alpha\|_{L^\infty(0,1;\mathbb{R}^{n-1})}\leq\varepsilon, \tag{2.42}$$

we consider $z\in C^0([0,1];\mathbb{R}^n)$ the solution to the ordinary differential equation

$$\frac{dz}{ds}=\sum_{j\neq m}\alpha_j(s)r_j(z), \quad z(0)=0, \tag{2.43}$$

Then, for any $\eta>0$, there exist $p\in\mathbb{N}$, $i_1,\cdots,i_p\in\{1,\cdots,n\}\setminus\{m\}$ and $t_1,\cdots,t_p\in\mathbb{R}$ such that

$$\sum_{l=1}^{p}|t_l|\leq C\varepsilon, \tag{2.44}$$

$$|z(1)-\Phi_{i_p}(t_p,\cdot)\circ\cdots\circ\Phi_{i_2}(t_2,\cdot)\left(\Phi_{i_1}(t_1,0)\right)|\leq\eta. \tag{2.45}$$

Proposition 2.2 will be established in Appendix A. The next proposition, which establishes the existence of the special trajectory $\overline{u}$, is the principal result of this section.

Proposition 2.3. *Let K be a compact subset of Ω. There exist $C>0$ and $\varepsilon_0>0$ such that the following holds. For any $\varepsilon\in(0,\varepsilon_0]$, there exist $T>0$ and a state $\overline{u}^*\in K$ satisfying*

$$\lambda_i(\overline{u}^*)\neq 0, \quad \forall i\in\{1,\cdots,n\}, \tag{2.46}$$

there exist $p \in \mathbb{N}$ *and times* $0 = \tau_0 < \tau_1 < \cdots < \tau_{2p+1} = T$ *with*

$$\tau_{p+1} - \tau_p > \max_{i=1\dots n} \frac{L}{|\lambda_i(\overline{u}^*)|}, \tag{2.47}$$

and a function $\overline{u} \in L^\infty((0,T) \times \mathbb{R}; \mathbb{R}^n)$ *such that*

$$\overline{u}_{|[\tau_{l-1},\tau_l]\times\mathbb{R}} \in C^2([\tau_{l-1},\tau_l] \times \mathbb{R}; \mathbb{R}^n),\ \forall l \in \{1,\dots,2p+1\}, \tag{2.48}$$

$$\overline{u}_{|[0,T]\times[0,L]} \in C^2([0,T] \times [0,L]; \mathbb{R}^n), \tag{2.49}$$

$$\frac{\partial \overline{u}}{\partial t} + A(\overline{u})\frac{\partial \overline{u}}{\partial x} = 0,\ \text{for } (t,x) \text{ in each } [\tau_{l-1},\tau_l] \times \mathbb{R},\ \forall l \in \{1,\dots,2p+1\}, \tag{2.50}$$

$$\overline{u}(0,x) = \overline{u}(T,x) = 0, \quad \forall x \in [0,L],$$

$$\overline{u}(t,x) = \overline{u}^*, \quad \forall t \in [\tau_p,\tau_{p+1}], \forall x \in [0,L],$$

$$\|\overline{u}(t,\cdot)\|_{C^1(\mathbb{R})} \le C\varepsilon, \quad \text{for } t \text{ in each } [\tau_{l-1},\tau_l],\ \forall l \in \{1,\dots,2p+1\}. \tag{2.51}$$

Proof: By Proposition 2.2 and the hypothesis (H), we can deduce that there exist $C > 0$ and $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0,\varepsilon_1]$, one can find $p \in \mathbb{N}$ and $i_1,\cdots,i_p \in \{1,\cdots,n\} \setminus \{m\}$, $t_1,\cdots,t_p \in \mathbb{R}$ such that (2.44) applies and

$$\lambda_m(\Phi_{i_p}(t_p,\cdot) \circ \cdots \circ \Phi_{i_2}(t_2,\cdot)(\Phi_{i_1}(t_1,0))) \neq 0. \tag{2.52}$$

And thus

$$\lambda_i(\Phi_{i_p}(t_p,\cdot) \circ \cdots \circ \Phi_{i_2}(t_2,\cdot)(\Phi_{i_1}(t_1,0))) \neq 0, \quad \forall i \in \{1,\cdots,n\}. \tag{2.53}$$

We let

$$\overline{u}^* := \Phi_{i_p}(t_p,\cdot) \circ \cdots \circ \Phi_{i_2}(t_2,\cdot)(\Phi_{i_1}(t_1,0)).$$

Now for every $l \in \{1,\cdots,p\}$, let

$$T_l := \frac{L}{|\lambda_{i_l}(0)|} + 1 \tag{2.54}$$

and in addition

$$\begin{cases} \tau_l := \sum_{k=1}^{l} T_k \text{ for } l = 1,\dots,p, \\ \tau_{p+1} := \tau_p + \max_{i=1\dots n} \frac{L}{|\lambda_i(\overline{u}^*)|} + 1 \\ \tau_l := \tau_{p+1} + \sum_{k=2p+2-l}^{p} T_k \text{ for } l = p+2,\dots,2p+1. \end{cases} \tag{2.55}$$

Observe that there is a symmetry with respect to the central time interval $[\tau_p,\tau_{p+1}]$, that is, $[\tau_{p-1},\tau_p]$ is symmetric of $[\tau_{p+1},\tau_{p+2}]$, etc.

Applying Proposition 2.1 and Remark 2.2 with

$$u_- = \Phi_{i_{l-1}}(t_{l-1},\cdot) \circ \cdots \circ \Phi_{i_2}(t_2,\cdot)(\Phi_{i_1}(t_1,0)) \text{ and } u_+ = \Phi_{i_l}(t_l,\cdot) \circ \cdots \circ \Phi_{i_2}(t_2,\cdot)(\Phi_{i_1}(t_1,0)),$$

for $l = 1, \dots, p$, we deduce that provided that ε_0 is small enough, for any $\varepsilon \in (0, \varepsilon_0]$, there exists $\overline{u}^l \in C^2([\tau_{l-1}, \tau_l] \times \mathbb{R}; \mathbb{R}^n)$ such that

$$\frac{\partial \overline{u}^l}{\partial t} + A(\overline{u}^l)\frac{\partial \overline{u}^l}{\partial x} = 0, \quad \forall (t,x) \in [\tau_{l-1}, \tau_l] \times \mathbb{R}, \tag{2.56}$$

$$\overline{u}^l(\tau_{l-1}, x) = u_-, \quad \forall x \in [0, L], \tag{2.57}$$

$$\overline{u}^l(\tau_l, x) = u_+, \quad \forall x \in [0, L], \tag{2.58}$$

$$\|\overline{u}^l(t, \cdot)\|_{C^1(\mathbb{R})} \le C\varepsilon, \quad \forall t \in [\tau_{l-1}, \tau_l]. \tag{2.59}$$

Then, we let

$$T := \tau_{2p+1}. \tag{2.60}$$

Finally, letting

$$\overline{u}(t,x) := \begin{cases} \overline{u}^l(t,x), & (t,x) \in [\tau_{l-1}, \tau_l] \times \mathbb{R}, l = 1, \cdots, p, \\ \overline{u}^*, & (t,x) \in [\tau_p, \tau_{p+1}] \times \mathbb{R}, \\ \overline{u}^{2p+1-l}(\tau_l - t, L - x), & (t,x) \in [\tau_{l-1}, \tau_l] \times \mathbb{R}, l = p+2, \cdots, 2p+1. \end{cases} \tag{2.61}$$

we can see that $\overline{u} \in L^\infty((0,T) \times \mathbb{R}; \mathbb{R}^n)$ satisfies the required properties. $\square$

3 Proof of Theorem 1.1

In order to conclude the proof, we will use a perturbation argument together with a result by Li and Rao [22]. First, we have the following perturbation result.

Proposition 3.1. *Consider $K \subset \Omega$ a nonempty compact subset. Let $T > 0$. For any $\widetilde{u} \in C^2([0,T] \times \mathbb{R}; K)$ satisfying*

$$\frac{\partial \widetilde{u}}{\partial t} + A(\widetilde{u})\frac{\partial \widetilde{u}}{\partial x} = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{3.1}$$

$$\widetilde{u}(0,x) = \tilde{\psi}(x) \quad \forall x \in \mathbb{R}, \tag{3.2}$$

there exist $\nu_0 > 0$ and $C > 0$ such that for any $\nu \in (0, \nu_0)$ and any $\psi \in C^1(\mathbb{R}; \Omega)$ satisfying

$$\|\psi(\cdot) - \tilde{u}(0, \cdot)\|_{C^1(\mathbb{R})} \le \nu, \tag{3.3}$$

the unique maximal solution $u \in C^1([0, T_0] \times \mathbb{R}; \Omega)$ of

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{3.4}$$

$$u(0,x) = \psi(x), \quad \forall x \in \mathbb{R}, \tag{3.5}$$

is defined on $[0,T] \times \mathbb{R}$ and satisfies

$$\|u(t, \cdot) - \widetilde{u}(t, \cdot)\|_{C^1(\mathbb{R})} \le C\nu, \quad \forall t \in [0,T]. \tag{3.6}$$

Proof: Given $\psi \in C^1(\mathbb{R};\Omega)$, there exists a local in time solution $u \in C^1([0,T_0]\times\mathbb{R})$ of (3.4)-(3.5). We show in the same time that u does not blow up before T and that (3.6) holds.

For that, let us make the difference of (3.1) and (3.4), we get

$$\frac{\partial}{\partial t}(u-\widetilde{u}) + A(u)\frac{\partial}{\partial x}(u-\widetilde{u}) = (A(\widetilde{u})-A(u))\frac{\partial \widetilde{u}}{\partial x}, \quad \forall (t,x)\in[0,T]\times\mathbb{R}, \tag{3.7}$$

$$u(0,x)-\widetilde{u}(0,x) = \psi(x)-\widetilde{\psi}(x), \quad \forall x\in\mathbb{R}. \tag{3.8}$$

By Gronwall's inequality we deduce that

$$\|u(t,\cdot)-\widetilde{u}(t,\cdot)\|_{C^0(\mathbb{R})} \leq C\|\psi-\widetilde{\psi}\|_{C^0(\mathbb{R})} \leq C\nu. \quad \forall t\in[0,T]. \tag{3.9}$$

Differentiating (3.7) with respect to x and observing that $\widetilde{u}$ is of class C^2, we can use the same Gronwall argument to infer (3.6) and that the maximal solution is defined on $[0,T]$.

□

Remark 3.1. We could use only a C^1 regularity assumption on $\tilde{u}$ provided that this $\tilde{u}$ has the particular structure given by Proposition 2.1. While the estimate (3.9) should be replaced by a weaker one (but sufficient for the proof of Theorem 1.1):

$$\|u(T,\cdot)-\widetilde{u}(T,\cdot)\|_{C^1([0,L])} \leq C\nu. \tag{3.10}$$

This can be proved by using the wave decomposition formula (2.28) in the sense of integral equations and the fact that $\tilde{u}$ satisfies the properties (2.33), (2.36) and (2.39). This technique is central in [20]. Hence $A\in C^1(\Omega,\mathbb{R}^{n\times n})$ is sufficient for Theorem 1.1.

Remark 3.2. As previously, the conclusion of Lemma 3.1 on $[0,T]$ can be achieved on $[t_0,t_0+T]$ for any $t_0\in\mathbb{R}$ by translation in time.

Proof of Theorem 1.1: Again, we may assume the equilibrium u^* to be 0, otherwise we can replace u by $u-u^*$ as the unknown in the system (1.1).

By Proposition 2.3, we can deduce that: there exist $C>0$, $\varepsilon_0>0$ and $T>0$, such that for any $\varepsilon\in(0,\varepsilon_0]$, there exists $\overline{u}\in L^\infty((0,T)\times\mathbb{R};\mathbb{R}^n)$ such that (2.46)-(2.51) hold.

For every $l\in\{1,\cdots,p\}$, let τ_l be given by (2.55). Let

$$u^0(0,x) = \varphi(x), \quad \forall x\in[0,L]. \tag{3.11}$$

The proof relies on a induction argument on l. By Proposition 3.1, we see that there exist $C>0$, $\varepsilon_l>0$ and $\nu_l>0$, for any $\varepsilon\in(0,\varepsilon_l]$ and any $\nu\in(0,\nu_l]$, if

$$\|u^{l-1}(\tau_{l-1},\cdot)-\overline{u}(\tau_{l-1},\cdot)\|_{C^1([0,L])} \leq \nu, \tag{3.12}$$

then there exists $u^l\in C^1([\tau_{l-1},\tau_l]\times\mathbb{R};\mathbb{R}^n)$ such that

$$\frac{\partial u^l}{\partial t} + A(u^l)\frac{\partial u^l}{\partial x} = 0, \quad \forall (t,x)\in[\tau_{l-1},\tau_l]\times\mathbb{R}, \tag{3.13}$$

$$u^l(\tau_{l-1},x) = u^{l-1}(\tau_{l-1},x), \quad \forall x\in[0,L], \tag{3.14}$$

$$\|u^l(t,\cdot)\|_{C^1(\mathbb{R})} \leq C\varepsilon + C\nu, \quad \forall t\in[\tau_{l-1},\tau_l], \tag{3.15}$$

$$\|u^l(\tau_l,\cdot)-\overline{u}(\tau_l,\cdot)\|_{C^1([0,L])} \leq C\nu. \tag{3.16}$$

Therefore, there exist $C > 0$, $\varepsilon_{\mathbf{f}} > 0$ and $\nu_{\mathbf{f}} > 0$, such that for any $\varepsilon \in (0, \varepsilon_{\mathbf{f}}]$ and for any $\nu \in (0, \nu_{\mathbf{f}}]$, if

$$\|\varphi\|_{C^1([0,L])} \leq \nu, \tag{3.17}$$

then there exists $u^{\mathbf{f}} \in C^1([0, \tau_p] \times [0, L]; \mathbb{R}^n)$ such that

$$\frac{\partial u^{\mathbf{f}}}{\partial t} + A(u^{\mathbf{f}})\frac{\partial u^{\mathbf{f}}}{\partial x} = 0, \quad \forall (t, x) \in [0, \tau_p] \times [0, L], \tag{3.18}$$

$$u^{\mathbf{f}}(0, x) = \varphi(x), \quad \forall x \in [0, L], \tag{3.19}$$

$$\|u^{\mathbf{f}}(t, \cdot)\|_{C^1([0,L])} \leq C\varepsilon + C\nu, \quad \forall t \in [0, \tau_p], \tag{3.20}$$

$$\|u^{\mathbf{f}}(\tau_p, \cdot) - \overline{u}(\tau_p, \cdot)\|_{C^1([0,L])} \leq C\nu. \tag{3.21}$$

In the same way and in view of Remark 3.2, there exist $C > 0$, $\varepsilon_{\mathbf{b}} > 0$ and $\nu_{\mathbf{b}} > 0$, such that for any $\varepsilon \in (0, \varepsilon_{\mathbf{b}}]$ and for any $\nu \in (0, \nu_{\mathbf{b}}]$, if

$$\|\psi\|_{C^1([0,L])} \leq \nu, \tag{3.22}$$

then there exists $u^{\mathbf{b}} \in C^1([\tau_{p+1}, T] \times [0, L]; \mathbb{R}^n)$ such that

$$\frac{\partial u^{\mathbf{b}}}{\partial t} + A(u^{\mathbf{b}})\frac{\partial u^{\mathbf{b}}}{\partial x} = 0, \quad \forall (t, x) \in [\tau_{p+1}, T] \times [0, L], \tag{3.23}$$

$$u^{\mathbf{b}}(T, x) = \psi(x), \quad \forall x \in [0, L], \tag{3.24}$$

$$\|u^{\mathbf{b}}(t, \cdot)\|_{C^1([0,L])} \leq C\varepsilon + C\nu, \quad \forall t \in [\tau_{p+1}, T], \tag{3.25}$$

$$\|u^{\mathbf{b}}(\tau_{p+1}, \cdot) - \overline{u}(\tau_{p+1}, \cdot)\|_{C^1([0,L])} \leq C\nu. \tag{3.26}$$

Now we can apply the result of Li and Rao [22] near the equilibrium of $\overline{u}(\tau_p, \cdot) = \overline{u}(\tau_{p+1}, \cdot) = \overline{u}^* \in \Omega$: due to (2.47) there exists $\nu_{\mathbf{m}} > 0$, such that for any $\nu \in (0, \nu_{\mathbf{m}}]$, if $\|u(\tau_p, \cdot) - \overline{u}^*\|_{C^1([0,L])}$ and $\|u(\tau_{p+1}, \cdot) - \overline{u}^*\|_{C^1([0,L])}$ are small enough, there exists $u^{\mathbf{m}} \in C^1([\tau_p, \tau_{p+1}] \times [0, L]; \mathbb{R}^n)$ such that

$$\frac{\partial u^{\mathbf{m}}}{\partial t} + A(u^{\mathbf{m}})\frac{\partial u^{\mathbf{m}}}{\partial x} = 0, \quad \forall (t, x) \in [\tau_p, \tau_{p+1}] \times [0, L], \tag{3.27}$$

$$u^{\mathbf{m}}(\tau_p, x) = u^{\mathbf{f}}(\tau_p, x), \quad \forall x \in [0, L], \tag{3.28}$$

$$u^{\mathbf{m}}(\tau_{p+1}, x) = u^{\mathbf{b}}(\tau_{p+1}, x), \quad \forall x \in [0, L], \tag{3.29}$$

$$\|u^{\mathbf{m}}(t, \cdot)\|_{C^1([0,L])} \leq C\nu, \quad \forall t \in [\tau_p, \tau_{p+1}]. \tag{3.30}$$

Combining all of the above, there exists $C > 0$ such that for any $\delta > 0$, there exist $\varepsilon > 0$ and $\nu > 0$ small enough, such that for any $\varphi, \psi \in C^1([0, L]; \mathbb{R}^n)$ satisfying

$$\|\varphi\|_{C^1([0,L])} \leq \nu, \quad \|\psi\|_{C^1([0,L])} \leq \nu, \tag{3.31}$$

one can construct $u \in C^1([0, T] \times [0, L]; \mathbb{R}^n)$ by

$$u(t, x) = \begin{cases} u^{\mathbf{f}}(t, x), & \forall (t, x) \in [0, \tau_p] \times [0, L], \\ u^{\mathbf{m}}(t, x), & \forall (t, x) \in [\tau_p, \tau_{p+1}] \times [0, L], \\ u^{\mathbf{b}}(t, x), & \forall (t, x) \in [\tau_{p+1}, T] \times [0, L]. \end{cases} \tag{3.32}$$

Now this function u clearly satisfies

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \quad \forall (t,x) \in [0,T] \times [0,L], \tag{3.33}$$

$$u(0,x) = \varphi(x), \quad \forall x \in [0,L], \tag{3.34}$$

$$u(T,x) = \psi(x), \quad \forall x \in [0,L], \tag{3.35}$$

$$\|u(t,\cdot)\|_{C^1([0,L])} \leq C\varepsilon + C\nu \leq \delta, \quad \forall t \in [0,T]. \tag{3.36}$$

This finishes the proof of Theorem 1.1.

4 Some models

In this section, we give several examples of systems to which the main result can be applied.

Model 1: Saint-Venant equations (shallow water equations) [8, 13, 14, 15]:

$$\begin{aligned} &\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(HV) = 0, \\ &\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}(\frac{V^2}{2} + gH) = 0, \end{aligned} \tag{4.1}$$

where $g > 0$ is the gravity constant. Let $U = (H,V)^{tr}$, (4.4) is reduced to

$$U_t + A(U)U_x = 0 \tag{4.2}$$

with

$$A(U) = \begin{pmatrix} V & H \\ g & V \end{pmatrix}. \tag{4.3}$$

By the study of Model 2 (see below), Theorem 1.1 can be applied to (4.1) near the equilibrium $U^* := (H^*, V^*)$ where $V^* = \sqrt{gH^*}$ with $H^* > 0$ or near the equilibrium $U^\star := (H^\star, V^\star)$ where $V^\star = -\sqrt{gH^\star}$ with $H^\star > 0$.

Model 2: 1-D isentropic gas dynamics equations in Eulerian coordinates [12]:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0, \\ &\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}(\frac{m^2}{\rho} + p) = 0, \end{aligned} \tag{4.4}$$

where

$$p = K\rho^\gamma \quad (K > 0,\ 1 < \gamma < 3). \tag{4.5}$$

We can see (4.1) is a special case of (4.4) when $p = g\rho^2/2$.

Moreover, let $U = (\rho, u)^{tr}$, (4.4) is reduced to

$$U_t + A(U)U_x = 0 \tag{4.6}$$

with

$$A(U) = \begin{pmatrix} u & \rho \\ \frac{p'(\rho)}{\rho} & u \end{pmatrix}. \tag{4.7}$$

The characteristic speeds and the corresponding eigenvectors are

$$\lambda_1(U) = u - \sqrt{p'(\rho)}, \quad \lambda_2(U) = u + \sqrt{p'(\rho)}, \tag{4.8}$$

$$r_1(U) = (\frac{\rho}{\sqrt{p'(\rho)}}, -1)^{tr}, \quad r_2(U) = (\frac{\rho}{\sqrt{p'(\rho)}}, 1)^{tr}. \tag{4.9}$$

Let $U^* := (\rho^*, u^*)$ where $u^* = \sqrt{p'(\rho^*)}$ with $\rho^* > 0$, that is, the fluid reaches the sound speed. Then it is easy to check that

$$\lambda_1(U^*) = 0 < \lambda_2(U^*) = 2\sqrt{p'(\rho^*)} \tag{4.10}$$

and the hypothesis (H1) is satisfied as:

$$\nabla\lambda_1(U^*) \cdot r_2(U^*) = \frac{3-\gamma}{2} > 0. \tag{4.11}$$

Similarly, if we let $U^\star := (\rho^\star, u^\star)$ where $u^\star = \sqrt{p'(\rho^\star)}$ with $\rho^\star > 0$ (which is the symmetric case of the latter), one can see that

$$\lambda_1(U^\star) = -2\sqrt{p'(\rho^\star)} < \lambda_2(U^\star) = 0 \tag{4.12}$$

and the hypothesis (H1) is satisfied as:

$$\nabla\lambda_2(U^\star) \cdot r_1(U^\star) = \frac{\gamma-3}{2} < 0. \tag{4.13}$$

Therefore, Theorem 1.1 can be applied to (4.4) near the equilibrium U^* or $U^\star$.

Model 3: 1-D full gas dynamics equations in Eulerian coordinates [25]:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \\ &\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0, \\ &\frac{\partial}{\partial t}\Big[\rho(\frac{u^2}{2} + e)\Big] + \frac{\partial}{\partial x}\Big[\rho u(\frac{u^2}{2} + e) + pu\Big] = 0. \end{aligned} \tag{4.14}$$

Assume the gas is polytropic, so that

$$e = c_v T = \frac{c_v R \rho}{p} \quad (c_v > 0,\ R > 0) \tag{4.15}$$

and

$$p = k e^{\frac{S}{c_v}} \rho^\gamma \quad (k > 0,\ 1 < \gamma < 3). \tag{4.16}$$

Thus, on the domain of $\rho > 0$, we have $p_\rho > 0$, $p_{\rho\rho} > 0$ and $p_S > 0$. Model 3 generalizes Model 2 if we let $m := \rho u$ and $S \equiv S_0 \in \mathbb{R}$.

Let $U = (\rho, u, S)^{tr}$, then (4.14) can be rewritten as

$$U_t + A(U)U_x = 0, \tag{4.17}$$

with

$$A(U) = \begin{pmatrix} u & \rho & 0 \\ \frac{p_\rho}{\rho} & u & \frac{p_S}{\rho} \\ 0 & 0 & u \end{pmatrix}. \tag{4.18}$$

The characteristic speeds and the corresponding eigenvectors are

$$\lambda_1(U) = u - c, \quad \lambda_2(U) = u, \quad \lambda_3(U) = u + c, \tag{4.19}$$

$$r_1(U) = (\rho, -c, 0)^{tr}, \quad r_2(U) = (p_S, 0, -p_\rho)^{tr}, \quad r_3(U) = (\rho, c, 0)^{tr}, \tag{4.20}$$

with $c = \sqrt{p_\rho}$.

Let $U^* := (\rho^*, 0, S^*)$ where $\rho^* > 0, S^* \in \mathbb{R}$, then it is easy to check that

$$\lambda_1(U^*) < \lambda_2(U^*) = 0 < \lambda_3(U^*) \tag{4.21}$$

and the hypothesis (H1) is satisfied as:

$$\nabla\lambda_2(U^*) \cdot r_1(U^*) = -c(U^*) < 0 \quad \text{or} \quad \nabla\lambda_2(U^*) \cdot r_3(U^*) = c(U^*) > 0. \tag{4.22}$$

Therefore, we can apply Theorem 1.1 to obtain boundary controllability for (4.14) near the equilibrium U^*.

Model 4: AR and MAR traffic flow system [4, 5]:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \\ &\frac{\partial}{\partial t}(\rho(u + p(\rho))) + \frac{\partial}{\partial x}(\rho u(u + p(\rho))) = 0, \end{aligned} \tag{4.23}$$

with

$$p = \rho^\gamma \quad (\gamma > 0), \tag{AR}$$

and

$$p = (\frac{1}{\rho} - \frac{1}{\rho_0})^{-\gamma} \quad (\gamma > 0,\ \rho_0 > 0). \tag{MAR}$$

We deduce system (AR) form system (MAR) by letting $\rho_0 = +\infty$.

Let $U = (\rho, u)^{tr}$, (4.23) is reduced to

$$U_t + A(U)U_x = 0, \tag{4.24}$$

with

$$A(U) = \begin{pmatrix} u & \rho \\ 0 & u - \rho p'(\rho) \end{pmatrix}. \tag{4.25}$$

The characteristic speeds and the corresponding eigenvectors are

$$\lambda_1(U) = u - \rho p'(\rho), \quad \lambda_2(U) = u, \tag{4.26}$$

$$r_1(U) = (1, -p'(\rho))^{tr}, \quad r_2(U) = (1, 0)^{tr}. \tag{4.27}$$

Let $U^* := (\rho^*, u^*)$ where $u^* = \rho^* p'(\rho^*)$ with $0 < \rho^* < \rho_0$, then we have

$$\lambda_1(U^*) = 0 < \lambda_2(U^*) = \rho^* p'(\rho^*) > 0, \tag{4.28}$$

and the hypothesis (H1) is satisfied as:

$$\nabla\lambda_1(U^*) \cdot r_2(U^*) = -p'(\rho^*) - \rho^* p''(\rho^*) = -\gamma(\rho^*)^{-3}(\frac{1}{\rho^*} - \frac{1}{\rho_0})^{-\gamma-2}(\frac{\rho^*}{\rho_0} + \gamma) < 0. \tag{4.29}$$

Similarly, if we let $U^\star := (\rho^\star, 0)$ with $0 < \rho^\star < \rho_0$, then it is easy to check that

$$\lambda_1(U^\star) = -\rho^\star p'(\rho^\star) < \lambda_2(U^\star) = 0, \tag{4.30}$$

and the hypothesis (H1) is satisfied as:

$$\nabla\lambda_2(U^\star) \cdot r_1(U^\star) = -p'(\rho^\star) = -\gamma(\rho^\star)^{-2}(\frac{1}{\rho^\star} - \frac{1}{\rho_0})^{-\gamma-1} < 0. \tag{4.31}$$

Theorem 1.1 can thus be applied to (4.23) near the equilibrium U^* or $U^\star$.

A Proof of Proposition 2.2

Proposition 2.2 belongs to the folklore of finite-dimensional control theory (see in particular Fillipov [11]). Since we have not found the exact required formulation in the literature, we give the proof in details for the sake of completeness.

We begin with a few notations.

Definition A.1. $\mathcal{P}^N_{(a,b)} \subset L^\infty(a,b;\mathbb{R}^N)$ is defined as the set consisting of all piecewise constant vector functions on (a,b). Next $\mathcal{F}^N_{(a,b)} \subset \mathcal{P}^N_{(a,b)}$ is defined as the set consisting of all piecewise constant vector functions on (a,b) with at most one nontrivial component, i.e., $f = (f_1, \cdots, f_N)^{tr} \in \mathcal{F}^N_{(a,b)}$ if and only if there exist $p \in \mathbb{N}$, indices $i_1, \cdots, i_p \in \{1, \cdots, N\}$, constants $f^1_{i_1}, \cdots, f^p_{i_p} \in \mathbb{R}$ and $a = t_0 < t_1 < \cdots < t_p = b$ such that

$$f(t) = f^l_{i_l} e_{i_l}, \quad \forall t \in (t_{l-1}, t_l), l = 1, \cdots, p, \tag{A.1}$$

where $e_1, \cdots, e_N$ denote the standard basis of $\mathbb{R}^N$.

Now we deduce the following statement.

Proposition A.1. *$\mathcal{F}^N_{(0,1)}$ is dense in $L^\infty(0,1;\mathbb{R}^N)$ with respect to the weak-$*$ topology, more precisely, for any $f \in L^\infty(0,1;\mathbb{R}^N)$, there exists a sequence $\{f^k\}_{k=1}^\infty \subset \mathcal{F}^N_{(0,1)}$ such that*

$$\lim_{k\to\infty} \int_0^1 f^k(t) \cdot h(t)dt = \int_0^1 f(t) \cdot h(t)dt, \quad \forall h \in L^1(0,1;\mathbb{R}^N), \tag{A.2}$$

$$\|f^k\|_{L^\infty(0,1;\mathbb{R}^N)} \leq C\|f\|_{L^\infty(0,1;\mathbb{R}^N)}, \quad \forall k \in \mathbb{N}. \tag{A.3}$$

Proof: It is classical that $\mathcal{P}^N_{(0,1)}$ is dense in $L^\infty(0,1;\mathbb{R}^N)$ for the weak-$*$ topology (moreover one can require (A.3) to hold on an approximating sequence). Hence it suffices to prove that $\mathcal{F}^N_{(0,1)}$ is dense in $\mathcal{P}^N_{(0,1)}$ with respect to weak-$*$ topology. To do this, we first prove (A.2) in the simpler case where f is a constant function:

$$f(t) = \overline{f} = (\overline{f}_1, \cdots, \overline{f}_N)^{tr} \in \mathbb{R}^N, \quad \forall t \in [0,1]. \tag{A.4}$$

For any $k \in \mathbb{N}$, we let $\overline{f}^k \in \mathcal{F}^N_{(0,1)}$ be defined as

$$\overline{f}^k(t) := N\overline{f}_i e_i, \ \forall t \in \Big(\frac{(j-1)N+i-1}{kN}, \frac{(j-1)N+i}{kN}\Big), \forall j \in \{1, \cdots, k\}, i = 1, \cdots N. \tag{A.5}$$

Clearly $\overline{f}^k$ converges weakly-$*$ to f in $L^\infty(0,1;\mathbb{R}^N)$ as k tends to ∞. Now we treat the general case where $f \in \mathcal{P}^N_{(0,1)}$. We introduce times $0 = t_0 < t_1 < \cdots < t_p = 1$ such that

$$f(t) = \overline{f}^l, \quad \forall t \in (t_{l-1}, t_l), l = 1, \cdots, p, \tag{A.6}$$

where the $\overline{f}^l$ are constants.

From the previous arguments, we can obtain by translation and scaling that there exists $\{\overline{f}^{l^k}\}_{k=1}^\infty \subset \mathcal{F}^N_{(t_{l-1},t_l)}$ such that $\overline{f}^{l^k}$ converges weakly-$*$ to $\overline{f}^l$ in $L^\infty(t_{l-1}, t_l; \mathbb{R}^N)$ as k tends to ∞. Finally, for any $k \in \mathbb{N}$, we let

$$f^k(t) := \overline{f}^{l^k}(t), \quad t \in (t_{l-1}, t_l), l = 1, \cdots, p. \tag{A.7}$$

It is obvious that $\{f^k\}_{k=1}^\infty \subset \mathcal{F}^N_{(0,1)}$ and f^k converges weakly-$*$ to f in $L^\infty(0,1;\mathbb{R}^N)$ as k tends to ∞, i.e., (A.2) holds. Observe that (A.3) holds. $\square$

Back to the proof of Proposition 2.2. Since $\alpha \in L^\infty(0,1;\mathbb{R}^{n-1})$, the solution $z \in C^0([0,1];\mathbb{R}^n)$ to the ordinary differential equation (2.43) is Lipschitz continuous, since

$$z(s) = \int_0^s \sum_{j \neq m} \alpha_j(\theta) r_j(z(\theta)) d\theta, \quad \forall s \in [0,1]. \tag{A.8}$$

Let α be such that (2.42) holds. By Proposition A.1, there exists a sequence $\{\alpha^k\}_{k=1}^\infty \subset \mathcal{F}^{n-1}_{(0,1)}$ with the notation $\alpha^k := (\alpha_1^k, \cdots, \alpha_{m-1}^k, \alpha_{m+1}^k, \cdots, \alpha_n^k)$, which converges weakly-$*$ to α in $L^\infty(0,1;\mathbb{R}^{n-1})$ and

$$\|\alpha^k\|_{L^\infty(0,1;\mathbb{R}^{n-1})} \leq C\|\alpha\|_{L^\infty(0,1;\mathbb{R}^{n-1})} \leq C\varepsilon, \quad \forall k \in \mathbb{N}. \tag{A.9}$$

Let $z^k \in C^0([0,S_k];\mathbb{R}^n)$ be the solution to the Cauchy problem

$$\frac{dz^k}{ds} = \sum_{j \neq m} \alpha_j^k(s) r_j(z^k), \quad z^k(0) = 0, \tag{A.10}$$

where $S_k \in (0,1]$. By (A.9), z^k is uniformly Lipschitz continuous.

If ε is small enough, then by (2.42), we can deduce that

$$S_k = 1, \tag{A.11}$$

that is, z_k is defined on the whole time interval $[0,1]$, for all $k \in \mathbb{N}$, and

$$\|z^k\|_{W^{1,\infty}(0,1;\mathbb{R}^n)} \le C\varepsilon, \quad \forall k \in \mathbb{N}. \tag{A.12}$$

By the Arzelà-Ascoli Theorem, there exists a subsequence $\{z^{k^l}\}_{l=1}^{\infty} \subset \{z^k\}_{k=1}^{\infty}$ and $z^\infty \in C^0([0,1];\mathbb{R}^n)$ such that z^{k^l} converges to z^∞ in $C^0([0,1];\mathbb{R}^n)$ as l tends to ∞. Now it is straightforward to pass to the limit in (A.8) (even, the limit is unique). The conclusion follows.

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