Well Balanced Hybrid Schemes for Shallow Water Flows

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R. Donat[1]

V. Caselles[2], G. Haro[2], A. Martinez-Gavara [3]

[1]Dept. de Matemática Aplicada, U. Valencia

[2]Dept. de Tecnologia, U. Pompeu Fabra

[3]Dept. de E.D. y An. Numerico, U. Sevilla

Numerical schemes for Shallow Water Flows

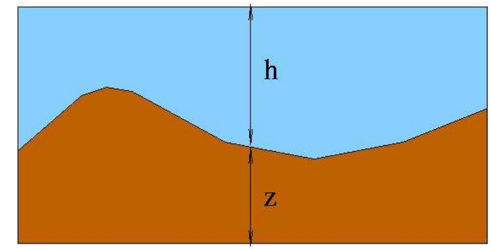
Motivation

- Engineering applications of Shallow Water Flows
  - ocean and hydraulic engineering
  - Flows in rivers, reservoirs and coastal areas
- Stationary or nearly stationary solutions need to be accurately computed.
- Numerical treatment of source terms derived from the bottom topography is a very delicate issue
  - Well Balanced Schemes [Greenberg&Leroux,96]
  - Source term Upwinding [Roe-86, Bermudez&Vazquez-Cendon-94]

Shallow Water Flow

Derived from the Navier-Stokes model after:

- depth averaging
- hydrostatic hypothesis
- neglecting viscosity and turbulence
- not considering wind effects nor Coriolis force.



System of conservation laws plus a source term:

$$U_t + F(U)_x + E(U)_y = S$$

$$\begin{pmatrix} h \\ q_1 \\ q_2 \end{pmatrix}_t + \begin{pmatrix} q_1 \\ \dfrac{q_1^2}{h} + \dfrac{1}{2} g h^2 \\ \dfrac{q_1 q_2}{h} \end{pmatrix}_x + \begin{pmatrix} q_2 \\ \dfrac{q_1 q_2}{h} \\ \dfrac{q_2^2}{h} + \dfrac{1}{2} g h^2 \end{pmatrix}_y = \begin{pmatrix} 0 \\ -g h z_x \\ -g h z_y \end{pmatrix}$$

$$q_1 = h v_x, \quad q_2 = h v_y$$

Numerical Methods - Source term upwinding

$$U_t + F(U)_x + E(U)_y = S(U)$$

[Roe, Proc. NL. Hyp. Prob., 86]	Point-wise evaluation of $S(U)$ does NOT work
[LeVeque, JCP-98]	Fractional splitting is NOT a good idea
[Bermudez, Vazquez, C&F, 94]	C-property
[Greenberg, Le Roux, SINUM 96]	Well Balanced schemes

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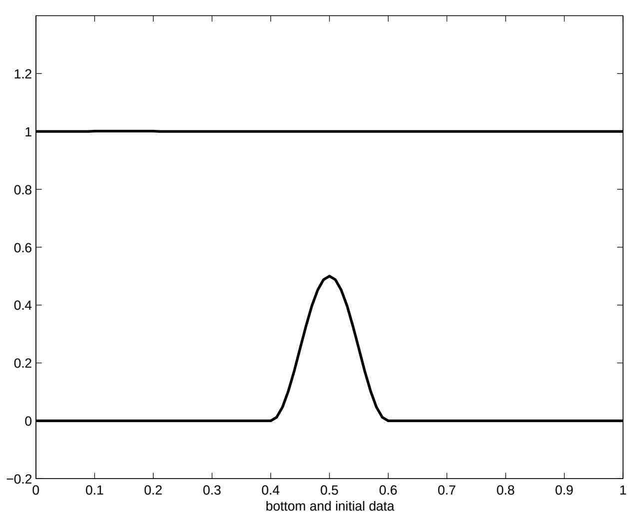
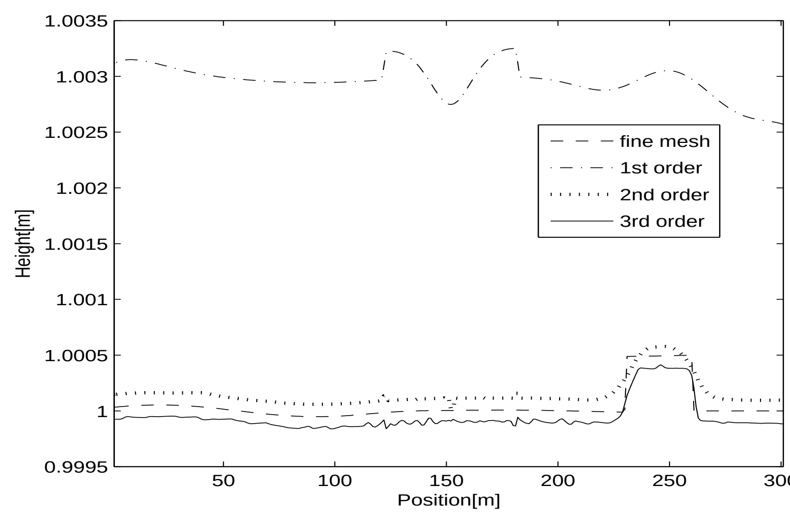
LeVeque's test, JCP98: Quasi-stationary flow.





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The spurious waves generated by schemes that are not well-balanced can distort completely the numerical solution.

General Strategy: Combine

- conservative scheme for homogeneous conservation laws
- adequate, upwind discretization of the source term

Things to address:

- Well-Balanced, High order schemes
- Wet/Dry fronts, existing and forming.

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2. The scheme of Gascon and Corberan JCP 2001
3. The 1J-2J scheme for Balance Laws, Caselles, Haro, RD, C&F 2008
4. A second order WB hybrid scheme, Martinez-Gavara, RD, (in preparation)

SC schemes for Convection Dominated Problems

$$\partial_t \vec{U} + \vec{\nabla}\vec{\mathcal{F}}(\vec{U}) = \mathcal{B}(\vec{U}, \nabla\vec{U})$$

$\vec{U} = \vec{U}(x, y, t) \qquad (x, y, t) \in \Omega\times]0, T[+$ appropriate initial and boundary conditions

- Discretization on Cartesian meshes.

$$\frac{\vec{U}_{ij}^{n+1} - \vec{U}_{ij}^{n}}{\Delta t} + \mathcal{D}_{ij} = \mathcal{B}_{ij}$$

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- $\vec{U}_{ij} \approx \frac{1}{|c_{ij}|}\int_{c_{ij}} \vec{U}(x,y,t)dxdy$ Finite Volume Schemes
- $\vec{U}_{ij} \approx \vec{U}(x_i, y_j, t)$. Point-Value Schemes, [Shu & Osher, JCP-86].

Shu-Osher SC schemes [Shu, Osher JCP 86]

- Dimension by dimension discretization in Multi-dimensions.

$$\mathcal{D}_{ij} = \frac{\vec{F}_{i+\frac{1}{2},j} - \vec{F}_{i-\frac{1}{2},j}}{\Delta x} + \frac{\vec{G}_{i,j+\frac{1}{2}} - \vec{G}_{i,j-\frac{1}{2}}}{\Delta y}$$

  - Relies on construction of Robust 1-D numerical fluxes.
    - Design first for scalar conservation laws
    - High order nonlinear reconstruction of numerical fluxes for high order accuracy in space (ENO, WENO, PHM ...).
    - Extend to systems via a local characteristic approach: Need to compute a spectral decomposition (eigenvalues, λ^p, and eigenvectors L^p, R^p) of the Jacobian matrices of the flux vectors at each interface.

- Only Uniform grids [Merriman, J. Sci. Comput, 03]

- Robust, but expensive. (Relativistic Flows [RD, Font, Ibañez, Marquina JCP 99, [Marquina, Mulet JCP 2003], [Chiavassa, RD, SISC 01], [Rault, Chiavassa, RD JSC 03], Kinematic flow problems [RD,Mulet Benasque07/JsC 08])

HRSC Numerical Flux Functions

- [Shu-Osher JCP-86]

$$F_{i+\frac{1}{2}} = \sum_p F^p_{i+\frac{1}{2}} R^p_{i+\frac{1}{2}}$$

- [RD, Marquina JCP-96] Marquina Flux-Splitting $\equiv$ 2-Jacobian Shu-Osher framework [RD, Mulet AMFM-06]

$$F^M_{i+\frac{1}{2}} = \sum_p F^{p,+}_{i+\frac{1}{2}} R^p(U_L) + F^{p,-}_{i+\frac{1}{2}} R^p(U_R)$$

- Other options ....

$F^p_{i+\frac{1}{2}}$ characteristic-fluxes (constructed 'as in the scalar-case')
use characteristic information (L^p, R^p, λ^p) obtained from the Jacobian matrix $\frac{\partial F}{\partial U}$.

Numerical treatment of Source terms

Well Balanced Schemes [Greenberg & LeRoux, SINUM-96]: Schemes that preserve steady states at the discrete level.

The C-property [Bermudez & Vazquez-Cendon, C&F-94]: A scheme satisfies the *exact* C-property if it preserves exactly stationary steady states (non-moving water). If it is not exact, but it is accurate of order $O(\triangle x^2)$, it satisfies the approximate C-property.

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Source Term upwinding, Roe-86 $\quad w_t + a w_x = s(x,w) = k(x) w$ $(a > 0)$

Characteristic construction: $\dfrac{dw}{dt} = k \cdot w$ along $\dfrac{dx}{dt} = a$

$$w(x,t) = w(x-at,0) + \int_0^t k(x-a(t-s))w(x-a(t-s),s)ds$$

There is an upwind domain of dependence determined by a.
'Upwinding' in the source term discretization is essential for Well Balancing (**WB**)

The scheme of Gascon & Corberan: Automatic WB

Consider the scalar equation $\qquad w_t + f(w)_x = s(x, w),$

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When $w = w(x) \quad \longrightarrow \quad f(w)_x = s(x, w) \quad \longrightarrow \quad f(w) = K + \int_{\bar{x}}^{x} s(y, w(y))dy$

Construct schemes that treat the flux and any primitive of the source term in an analogous fashion: Flux upwinding $\rightarrow$ source term upwinding

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Construct schemes that treat the flux and any primitive of the source term in an analogous fashion: Flux upwinding $\to$ source term upwinding

Define $\qquad b(x, w) = -\int_{\bar{x}}^{x} s(y, w(y,t))dy, \quad \to \quad b_x = -s$

$$w_t + f_x = -b_x \quad \to \quad w_t + (f + b)_x = 0 \quad \to \quad w_t + g_x = 0$$

$$g(x, w) = f(w) + b(x, w)$$

[G&C JCP-2001] Construction of a second order TVD scheme for $w_t + g_x = 0$,

$$w_j^{n+1} = w_j^n - \lambda \left\{ \bar{g}_{j+1/2}^n - \bar{g}_{j-1/2}^n \right\}$$

G & C JCP-01

Gascon and Corberan start from an explicit three point scheme of the form

$$w_j^{n+1} = w_j^n - \lambda \left(\hat{g}_{j+1/2}^{n+1/2} - \hat{g}_{j-1/2}^{n+1/2} \right)$$

with a Lax-Wendroff type flux

$$\hat{g}_{j+1/2}^{n+1/2} = \frac{1}{2} \left(g_j^n + g_{j+1}^n - \lambda \left. \frac{\partial g}{\partial w} \right|_{j+1/2}^{n} (g_{j+1}^n - g_j^n) \right)$$

$$g_i^n = f_i^n + b_i^n; \quad b_i^n = \int_{\bar{x}}^{x_i} s(y, w(y, t_n)) dy, \qquad \frac{\partial g}{\partial w} = \frac{\partial f}{\partial w} + \frac{\partial b}{\partial w}$$

$$b_{i+1} = b_i + b_{i,i+1}, \qquad b_{i,i+1} = \int_{x_i}^{x_{i+1}} s(y, w(y, t_n)) dy$$

$$g_{i+1} - g_i = f_{i+1} - f_i + b_{i,i+1}$$

and the scheme can be rewriten so that only $b_{j,j+1}$ need to be computed.

Automatic Well Balancing

Steady (smooth) flow: $\equiv u_t = 0, \quad \Rightarrow \quad f_x = s = -b_x \quad \Leftrightarrow \quad g_x = 0$

$$f_{i+1} - f_i = \int_{x_i}^{x_{i+1}} f_x = \int_{x_i}^{x_{i+1}} s = \int_{x_i}^{x_{i+1}} -b_x = -(b_{i+1} - b_i) = -b_{i,i+1}$$

,

$$\boxed{\Rightarrow \quad g_{i+1} - g_i = 0 \quad \equiv \quad f_{i+1} - f_i + b_{i,i+1} = 0}$$

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Let $u(x,t) = u(x)$ be a stationary solution of $u_t + f_x = s$.

Let $u_i^0 = u(x_i)$. Then for $n = 0$, $\forall j$

$$g_{j+1}^n = g_j^n, \quad \forall j, \quad \Rightarrow \quad \hat{g}_{j+1/2}^{n+1/2} = \frac{1}{2}\left(g_j^n + g_{j+1}^n - \lambda \left.\frac{\partial g}{\partial w}\right|_{j+1/2}^{n} (g_{j+1}^n - g_j^n) \right) = g_j^n$$

Hence $\boxed{g_{j+1}^n = g_j^n, \quad \forall j, \forall n > 0, \forall j \quad \Rightarrow \quad \hat{g}_{j+1/2}^{n+1/2} - \hat{g}_{j-1/2}^{n+1/2} = 0}$

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The scheme

- is exactly well balanced (Greenberg& Leroux)
- satisfies the exact C-property (Bermudez& Vazquez-Cendon)

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Incorporating the source term into the flux divergence $\Rightarrow$
Flux-Gradient and Source-term Balancing (FSB) $\Rightarrow$ Automatic WB

FSB and Automatic WB

FSB : $\boxed{f_{i+1} - f_i + b_{i,i+1} = 0, \quad \forall i}$, $b_{i,i+1} = \int_{x_i}^{x_{i+1}} s(y, w(y,t))dy$. Key to WB

OBS! The computation of $b_{i,i+1}$ is carried out by numerical integration,

$$\begin{cases} b_{i,i+1} = \hat{b}_{i,i+1} + O(\triangle x^{p+1}), \\ b_i = \hat{b}_i + O(\triangle x^p) \end{cases} \qquad g_i = f(u_i) + b_i \approx \hat{g}_i = f(u_i) + \hat{b}_i,$$

$$\begin{aligned} \hat{g}_{j+1} - \hat{g}_j &= f_{j+1} - f_j + \hat{b}_{i,i+1} = f_{j+1} - f_j + b_{i,i+1} + O(\triangle x^{p+1}) \\ &= g_{j+1} - g_j + O(\triangle x^{p+1}) \end{aligned}$$

In steady smooth flow, $\hat{g}^n_{j+1} - \hat{g}^n_j = O(\triangle x^{p+1})$, hence

$$\hat{g}^{n+1/2}_{j+1/2} - \hat{g}^{n+1/2}_{j-1/2} = O(\triangle x^{p+1}) \quad \rightarrow \quad \frac{1}{\triangle x}(\hat{g}^{n+1/2}_{j+1/2} - \hat{g}^{n+1/2}_{j-1/2}) = O(\triangle x^p)$$

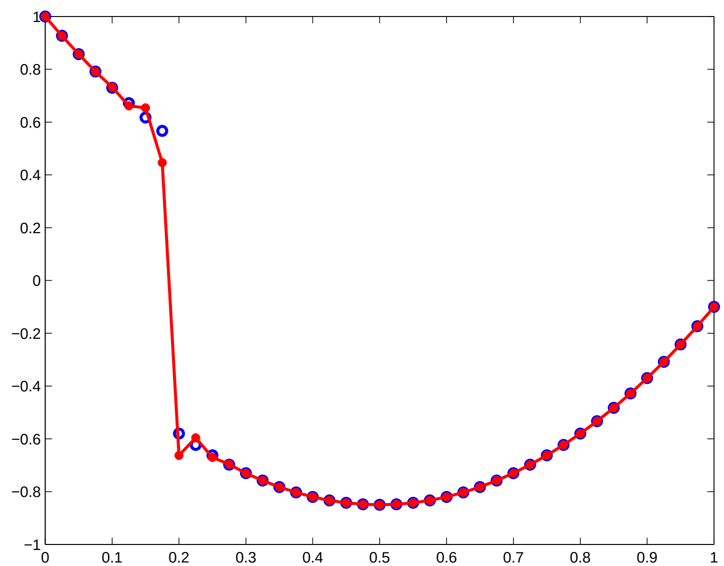
The scheme satisfies the approximate C-property (B&V-C) when $p \geq 2$.

Embid's Problem

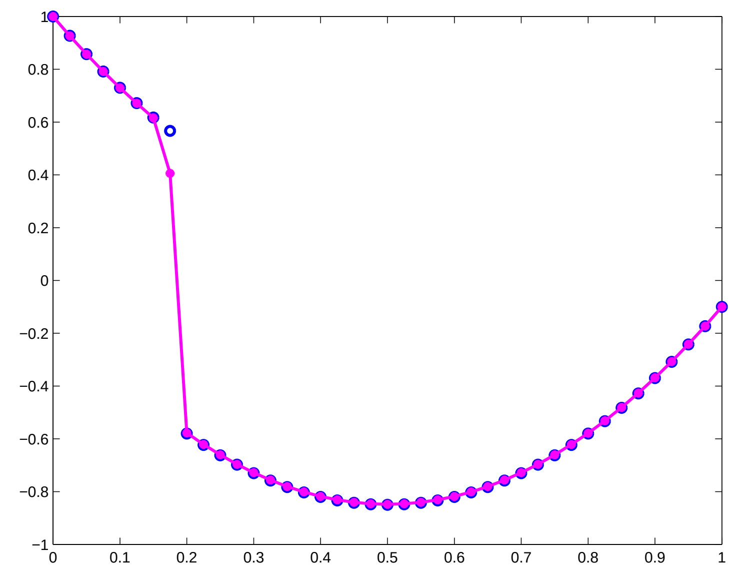
Scalar model for gas flow through a duct of variable cross section.

$$\begin{cases} w_t + \left(w^2/2\right)_x = (6x-3)w, & 0 < x < 1 \\ w(0,t) = 1; w(1,t) = -.1 & \end{cases}$$

Steady solution: $w(x) = \begin{cases} 1 + 3x^2 - 3x, & x < x_j \\ -0.1 + 3x^2 - 3x, & x > x_j \end{cases}$



WB2-scheme



Curbing the oscillations

Curbing the oscillations: G&C TVD scheme

$$\hat{g}^{n+1/2}_{j+1/2} = \frac{1}{2}\left(g^n_j + g^n_{j+1} - \lambda \left.\frac{\partial g}{\partial w}\right|^n_{j+1/2} (g^n_{j+1} - g^n_j) \right)$$

$$\frac{\partial g}{\partial w} = \frac{\partial f}{\partial w} - \frac{\partial b}{\partial w}, \qquad \alpha \approx \lambda \frac{\partial f}{\partial w}, \quad \beta \approx \frac{\partial b}{\partial w}$$

$$\alpha_{j+1/2} = \lambda \begin{cases} \frac{f_{j+1} - f_j}{w_{j+1} - w_j}, \\ \left.\frac{\partial f}{\partial w}\right|_{j+1/2}, \end{cases} \qquad \beta_{j+1/2} = \lambda \begin{cases} \frac{b_{j+1} - b_j}{w_{j+1} - w_j}, & \text{if } w_{j+1} - w_j \neq 0 \\ 0, & \text{if } w_{j+1} - w_j = 0 \end{cases}$$

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Search for a TVD scheme of the form $\boxed{\quad w^{n+1}_j = w^n_j - \lambda(\bar{g}_{j+1/2} - \bar{g}_{j-1/2})}$

$$\bar{g}_{j+1/2} = \frac{1}{2}\left(g_j + g_{j+1} - h(\alpha_{j+1/2} + \beta_{j+1/2})(g_{j+1} - g_j)\right)$$

Adjust $h(x)$ so that scheme is TVD $\rightarrow$ CFL-like restrictions on $\alpha_{i+1/2} + \beta_{i+1/2}$

First order scheme!. Upgrade to second order following Harten JCP-97

Curbing the oscillations: G&C TVD scheme

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Search for a TVD scheme of the form $\boxed{w_j^{n+1} = w_j^n - \lambda(\bar{g}_{j+1/2} - \bar{g}_{j-1/2})}$

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Applications: Nozze-flows

Revisiting Gascon and Corberan TVD scheme

Anna Martinez Gavara PhD Thesis.

BUT!!

- Solutions to scalar balance laws might not be TVD! TVD restrictions might be too stringent on numerical schemes for general balance laws.
- CFL-like restrictions based on $\alpha_{i+1/2} + \beta_{i+1/2}$ might be too restrictive

FSB for Shallow Water Flows

Incorporate the source term in the flux divergence for shallow water flows

Split source term, $S = S_1 + S_2 = \begin{pmatrix} 0 \\ -ghz_x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -ghz_y \end{pmatrix}$

Define functions:

$$B(x,y,t) = -\int_0^x S_1(s,y,t)ds; \qquad \partial_x B = -S_1$$

$$C(x,y,t) = -\int_0^y S_2(x,s,t)ds; \qquad \partial_y C = -S_2$$

Formally Rewrite the original system in 'conservation form' as follows,

$$\mathbf{U_t + (F + B)_x + (E + C)_y = 0}$$

$$\mathbf{U_t + (G)_x + (H)_y = 0}$$

$\rightarrow$ combined fluxes: physical fluxes + primitive of source term

Shu-Osher HRSC approach for FSB schemes

- Semi-discrete formulation on Cartesian meshes (separate spatial and temporal accuracy).

$$\partial_t \vec{U}_{ij} + \frac{\vec{G}_{i+\frac{1}{2},j} - \vec{G}_{i-\frac{1}{2},j}}{\Delta x} + \frac{\vec{H}_{i,j+\frac{1}{2}} - \vec{H}_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

- Accuracy in time: Runge-Kutta schemes $\vec{U}^n_{ij} \quad \Rightarrow \quad \vec{U}^{n+1}_{ij}$.
- Relies on construction of Robust 1-D numerical fluxes.
  - Design first for scalar conservation laws
  - High order nonlinear reconstruction of numerical fluxes for high order accuracy in space (ENO, WENO, PHM ...).
  - Extend to systems via a local characteristic approach: Need to compute a spectral decomposition (eigenvalues, λ^p, and eigenvectors L^p, R^p) of the Jacobian matrices of the flux vectors at each interface.

A HRSC-FSB scheme for the scalar CL

[Caselles, Haro, RD C& F 09]

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Higher order: Use nonlinear reconstruction process on $g_i = f_i + b_i$

OBS!: $b_i = \int^{x_i} s(w)dx$ has to be computed by numerical integration!

- seek to involve only $b_{l,l+1} = \int_{x_l}^{x_{l+1}} s(w)dx$

[Caselles, Haro, RD C & F 09]

Shu-Osher RF algorithm:

$$g_{i+1/2} - g_{i-1/2} = [\,\mathcal{G}_{i+1/2} + \textbf{HOT}_{i+1/2}\,] - [\,\mathcal{G}_{i-1/2} + \textbf{HOT}_{i-1/2}\,]$$

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$\mathcal{G}_{i\pm 1/2}$ first order contributions

$$\mathcal{G}_{i+1/2} = \begin{cases} g_i = f_i + b_i & \text{if } f' > 0 \text{ in } [w_i, w_{i+1}] \\ g_{i+1} = f_{i+1} + b_{i+1} & \text{if } f' < 0 \text{ in } [w_i, w_{i+1}] \\ \frac{1}{2}(g_i^+ + g_{i+1}^-) & \text{else} \end{cases}$$

$$\alpha_{i+1/2} = \max\{|f'(w)|, w \in [w_i, w_{i+1}]\}, \qquad \begin{cases} g_i^+ = (g_i + \alpha_{i+1/2} w_i) \\ g_{i+1}^- = (g_{i+1} - \alpha_{i+1/2} w_{i+1}) \end{cases}$$

HOT$_{i\pm 1/2}$ Higher order terms, based on divided diferences of g.
Divided diferences of g only depend on $b_{i,i+1}$.

[Caselles, Haro, RD C & F 09]

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- First order scheme, applied to $w_t + a w_x = s(x, w) = a w$ ($a > 0$)

  $$w_t = \frac{1}{\Delta x}\left(\mathcal{G}_{i+1/2} - \mathcal{G}_{i-1/2}\right) = \frac{1}{\Delta x}\left(f_i - f_{i-1} + \int_{x_{i-1}}^{x_i} s(w(z,t))dz\right)$$

  [Roe, 86]

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Notice that $\mathcal{G}_{i+1/2} - \mathcal{G}_{i-1/2}$ can be written in terms of

$$b_{l+1} - b_l = \int^{x_{l+1}} s - \int^{x_l} s = \int_{x_l}^{x_{l+1}} s(x, w(x))dx = b_{l,l+1}, \quad l = i-1, i$$

[Caselles, Haro, RD C & F 09] $\mathcal{G}_{i+1/2} - \mathcal{G}_{i-1/2} = \mathcal{G}^+_{i+1/2} - \mathcal{G}^-_{i-1/2}$

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$$\mathcal{G}^-_{i-1/2} = \begin{cases} f_{i-1} - b_{i-1,i} & \text{if } f' > 0 \text{ in } [w_{i-1}, w_i] \\ f_i & \text{if } f' < 0 \text{ in } [w_{i-1}, w_i] \\ \frac{1}{2}(f_i^+ + f_{i-1}^-) - \frac{1}{2} b_{i-1,i} & \text{else} \end{cases}$$

$$\begin{cases} f_i^+ &= f_i + \alpha_{i+1/2} w_i, \\ f_i^- &= f_i - \alpha_{i-1/2} w_i \end{cases}$$

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$$\mathcal{G}^-_{i-1/2} = \begin{cases} f_{i-1} - b_{i-1,i} & \text{if } f' > 0 \text{ in } [w_{i-1}, w_i] \\ f_i & \text{if } f' < 0 \text{ in } [w_{i-1}, w_i] \\ \frac{1}{2}(f^+_i + f^-_{i-1}) - \frac{1}{2} b_{i-1,i} & \text{else} \end{cases}$$

- $\mathcal{G}^+_{i+1/2} - \mathcal{G}^-_{i+1/2} = b_{i,i+1} = \int_{x_i}^{x_{i+1}} -s(w(x), x)dx$

  $\mathcal{G}^\pm_{i+1/2}$ split the source term contribution at the $i + 1/2$ interface according to the wind, as specified by $f'(u)$.

[Caselles, Haro, RD C & F 09]

$$
\begin{aligned}
g_{i+1/2} - g_{i-1/2} &= [\mathcal{G}^{+}_{i+1/2} + \textbf{HOT}_{i+1/2}] - [\mathcal{G}^{-}_{i-1/2} + \textbf{HOT}_{i-1/2}] \\
&= G^{+}_{i+1/2} - G^{-}_{i-1/2}
\end{aligned}
$$

$$
\mathcal{G}^{+}_{i+1/2} = \begin{cases} f_i & \text{if } f' > 0 \text{ in } [w_i, w_{i+1}] \\ f_{i+1} + b_{i,i+1} & \text{if } f' < 0 \text{ in } [w_i, w_{i+1}] \\ \frac{1}{2}(f_i^{+} + f_{i+1}^{-}) + \frac{1}{2} b_{i,i+1} & \text{else} \end{cases}
$$

$$
\mathcal{G}^{-}_{i-1/2} = \begin{cases} f_{i-1} - b_{i-1,i} & \text{if } f' > 0 \text{ in } [w_i, w_{i+1}] \\ f_i & \text{if } f' < 0 \text{ in } [w_l, w_r] \\ \frac{1}{2}(f_i^{+} + f_{i-1}^{-}) - \frac{1}{2} b_{i-1,i} & \text{else} \end{cases}
$$

HOT$_{i-1/2}$ Higher order terms, based on divided diferences of g.

only $b_{l,l+1} = \int_{x_l}^{x_{l+1}} s(w)dx$ are involved !!

Extension to Systems

- For $f(w)$ nonlinear, $G^{\pm}_{i+1/2}$ lead to an automatic upwind splitting of the source term contribution at the $i+1/2$ cell.

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Extension to 1D-systems:

$$U_t + G(U)_x = 0 \quad \Rightarrow \quad \partial_t U_i + \frac{1}{\Delta x}\left(G^{+}_{i+1/2} - G^{-}_{i-1/2}\right) = 0$$

Construction of $G^{\pm}_{i+1/2}$: Use spectral decomposition of $\frac{\partial F}{\partial U}$

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- Follow the basic design structure of Marquina Flux-Splitting technique:

  The 2J scheme: $G^{\pm}_{i+1/2} = \sum_p (G^{p,\pm}_{i+1/2})^L R^p(U^L) + (G^{p,\pm}_{i+1/2})^R R^p(U^R)$

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- If $U^L = U^R = U^*$ (Plain Extension of Shu-Osher to HCL+source term)
  The 1J Scheme: $G^{\pm}_{i+1/2} = \sum_p G^{p,\pm}_{i+1/2} R^p(U^*)$
  $U^* \equiv U_{i+1/2}$ interface state.

$G^{p,\pm}$ characteristic fluxes, computed 'as in the scalar case' for the pth field.

[Caselles, Haro, RD C & F 09]

- Numerical integration crucial for Well Balancing

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghz_x \end{pmatrix}$$

$$B^n_{i,i+1} = \begin{pmatrix} 0 \\ \int_{x_i}^{x_{i+1}} ghz_x dx \end{pmatrix}, \quad \int_{x_i}^{x_{i+1}} ghz_x dx \approx \frac{g}{2}(z_{i+1}-z_i)(h_i+h_{i+1})+O(\triangle x^3)$$

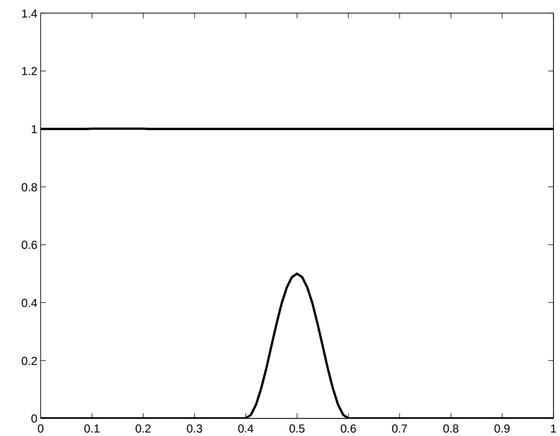
for quiescent steady flows: $q=0$, $h+z=$constant,

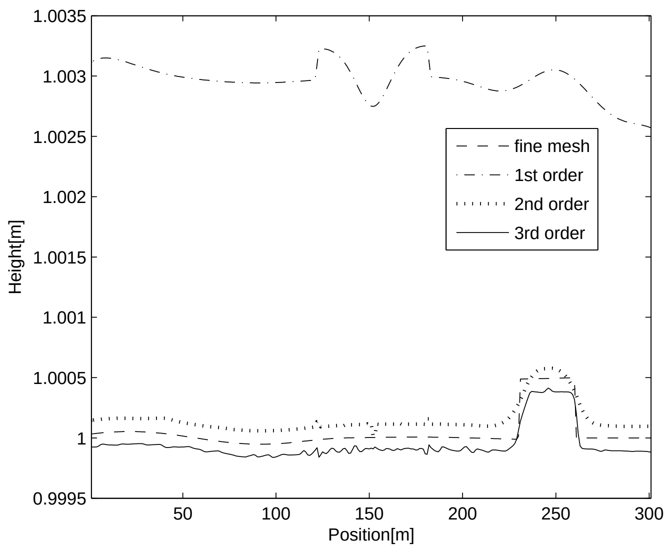
$$\int_{x_i}^{x_{i+1}} ghz_x dx = \int_{x_i}^{x_{i+1}} gh(-h)_x dx = -\frac{g}{2}(h_i^2-h_{i+1}^2) = \frac{g}{2}(z_{i+1}-z_i)(h_i+h_{i+1})$$

- 1J-scheme exactly well balanced for stationary flows.
- 2J-scheme only approximately well balanced, order ≥ 2.

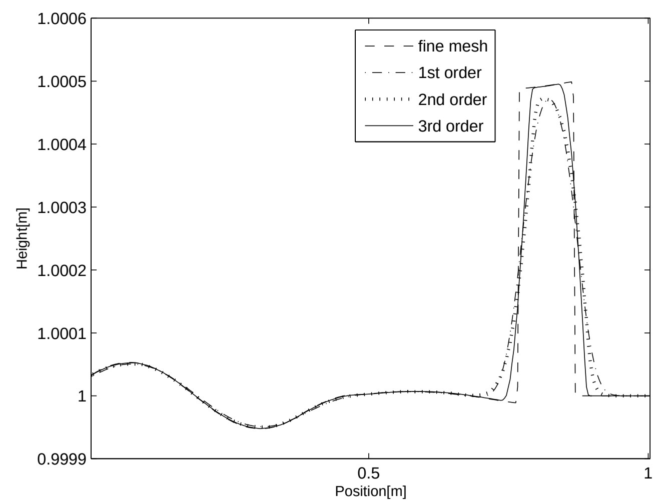
[Caselles, Haro, RD C & F 09]

LeVeque's test, JCP98: Quasi-stationary flow.





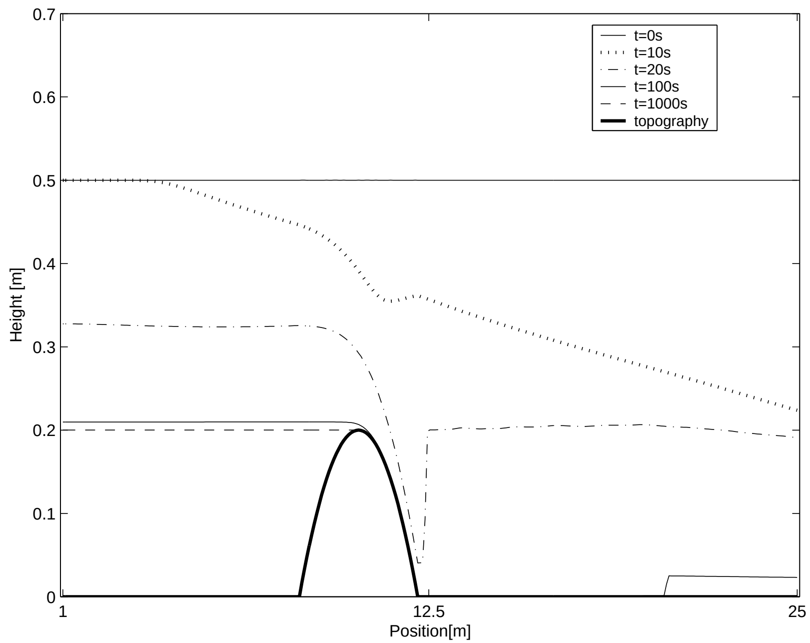
The 2J-scheme



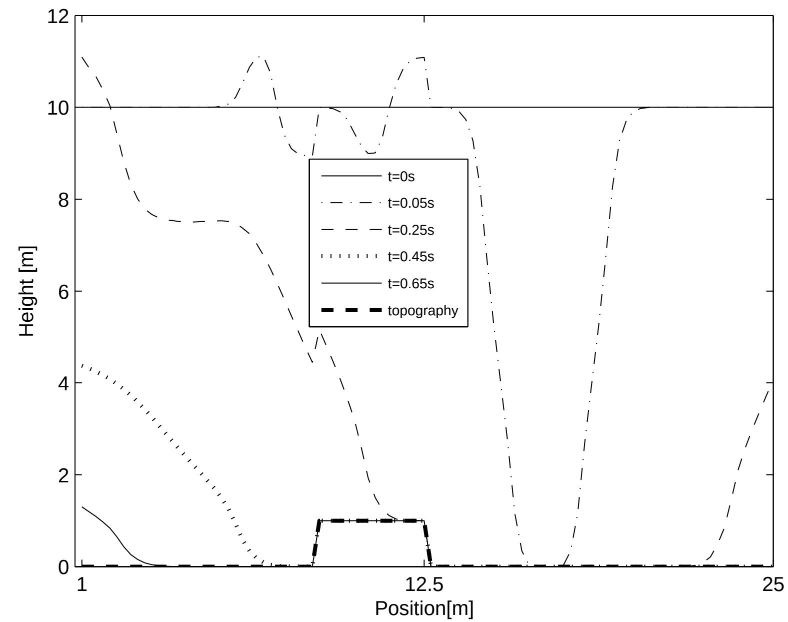
The 1J-2J scheme

[Caselles, Haro, RD C & F 09]

- The 1J-2J scheme can manage wet/dry fronts.



Drain on a non-flat bottom



Dry bed generation + non-flat bottom

- Slight loss of conservation in the 2J extension to systems.

A WB one-step, second-order FSB scheme

A. Martinez-Gavara PhD Thesis

Start with the scalar equation $w_t + f(w)_x = s(w),$

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Look for a Lax-Wendroff type second order scheme:

$$U_i^{n+1} = U_i^n - \lambda(\tilde{g}_{i+\frac{1}{2}}^{n+\frac{1}{2}} - \tilde{g}_{i-\frac{1}{2}}^{n+\frac{1}{2}}) \qquad \text{with} \qquad \tilde{g}_{i+\frac{1}{2}}^{n+\frac{1}{2}} := \hat{g}_{i+\frac{1}{2}}^{n} + \frac{\triangle t}{2} \frac{\partial \hat{g}}{\partial t}\bigg|_{i+\frac{1}{2}}^{n}.$$

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Second order accuracy:
$$\begin{cases} \hat{g}_{i+\frac{1}{2}}^n := \dfrac{1}{2}(g_{i+1}^n + g_i^n), \\ \hat{g}_t\big|_{i+\frac{1}{2}}^n := -f_w\big|_{i+\frac{1}{2}}^n \dfrac{g_{i+1}^n - g_i^n}{\triangle x} + b_t\big|_{i+\frac{1}{2}}^n, \\ b_t\big|_{i+\frac{1}{2}}^n = \displaystyle\int_0^{x_{i+\frac{1}{2}}} \frac{\partial s}{\partial u}(y, u(y,t_n))g_y(y,t_n)dy, \end{cases}$$

- Using TRAPEZOIDAL RULE

$$
\begin{aligned}
b_t\Big|_{i+\frac{1}{2}}^{n} - b_t\Big|_{i-\frac{1}{2}}^{n} &= \int_{x_{i-1/2}}^{x_{i+1/2}} s_u(y,u(y,t_n))g_x(y,t_n)dy \\
&= \left(s_u\Big|_{i+\frac{1}{2}}^{n} \frac{g_{i+1}^n - g_i^n}{\triangle x} + s_u\Big|_{i-\frac{1}{2}}^{n} \frac{g_i^n - g_{i-1}^n}{\triangle x} \right) \frac{\triangle x}{2} + \mathcal{O}(\triangle x^3)
\end{aligned}
$$

- $\alpha_{i+\frac{1}{2}}^n = \lambda \dfrac{\partial f}{\partial u}\Big|_{i+\frac{1}{2}}^{n}$, $\quad \beta_{i+\frac{1}{2}}^n = \dfrac{\triangle t}{2}\dfrac{\partial s}{\partial u}\Big|_{i+\frac{1}{2}}^{n}$,

$$
U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}\left(\mathcal{G}_{i+\frac{1}{2}}^n - \mathcal{G}_{i-\frac{1}{2}}^n\right) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n
$$

$$
\begin{aligned}
\mathcal{G}_{i+\frac{1}{2}}^n &= \tfrac{1}{2}\left(g_{i+1}^n + g_i^n - \alpha_{i+\frac{1}{2}}^n (g_{i+1}^n - g_i^n) \right) \\
\mathcal{S}_i^n &= \tfrac{1}{2}\left(\beta_{i+\frac{1}{2}}^n (g_{i+1}^n - g_i^n) + \beta_{i-\frac{1}{2}}^n (g_i^n - g_{i-1}^n) \right)
\end{aligned}
$$

Automatic Well Balancing

Let $u(x)$ be a stationary solution $\Leftrightarrow u_t = 0 \Leftrightarrow g_x = 0 \Leftrightarrow g_{i+1} = g_i$

$$\begin{aligned}
\mathcal{G}^n_{i+\frac{1}{2}} &= \frac{1}{2}\left(g^n_{i+1} + g^n_i - \alpha^n_{i+\frac{1}{2}}(g^n_{i+1} - g^n_i)\right) = g^n_i \\
\mathcal{S}^n_i &= \frac{1}{2}\left(\beta^n_{i+\frac{1}{2}}(g^n_{i+1} - g^n_i) + \beta^n_{i-\frac{1}{2}}(g^n_i - g^n_{i-1})\right) = 0
\end{aligned}$$

$$U^{n+1}_i = U^n_i - \frac{\triangle t}{\triangle x}(\mathcal{G}^n_{i+\frac{1}{2}} - \mathcal{G}^n_{i-\frac{1}{2}}) - \frac{\triangle t}{\triangle x}\mathcal{S}^n_i = U^n_i \qquad \text{exactly WB}$$

$$g_{i+1} - g_i = f_{i+1} - f_i + b_{i,i+1} \qquad \text{if } b_{i,i+1} = \int_{x_i}^{x_{i+1}} s = \hat{b}_{i,i+1} + O(\triangle x^{p+1})$$

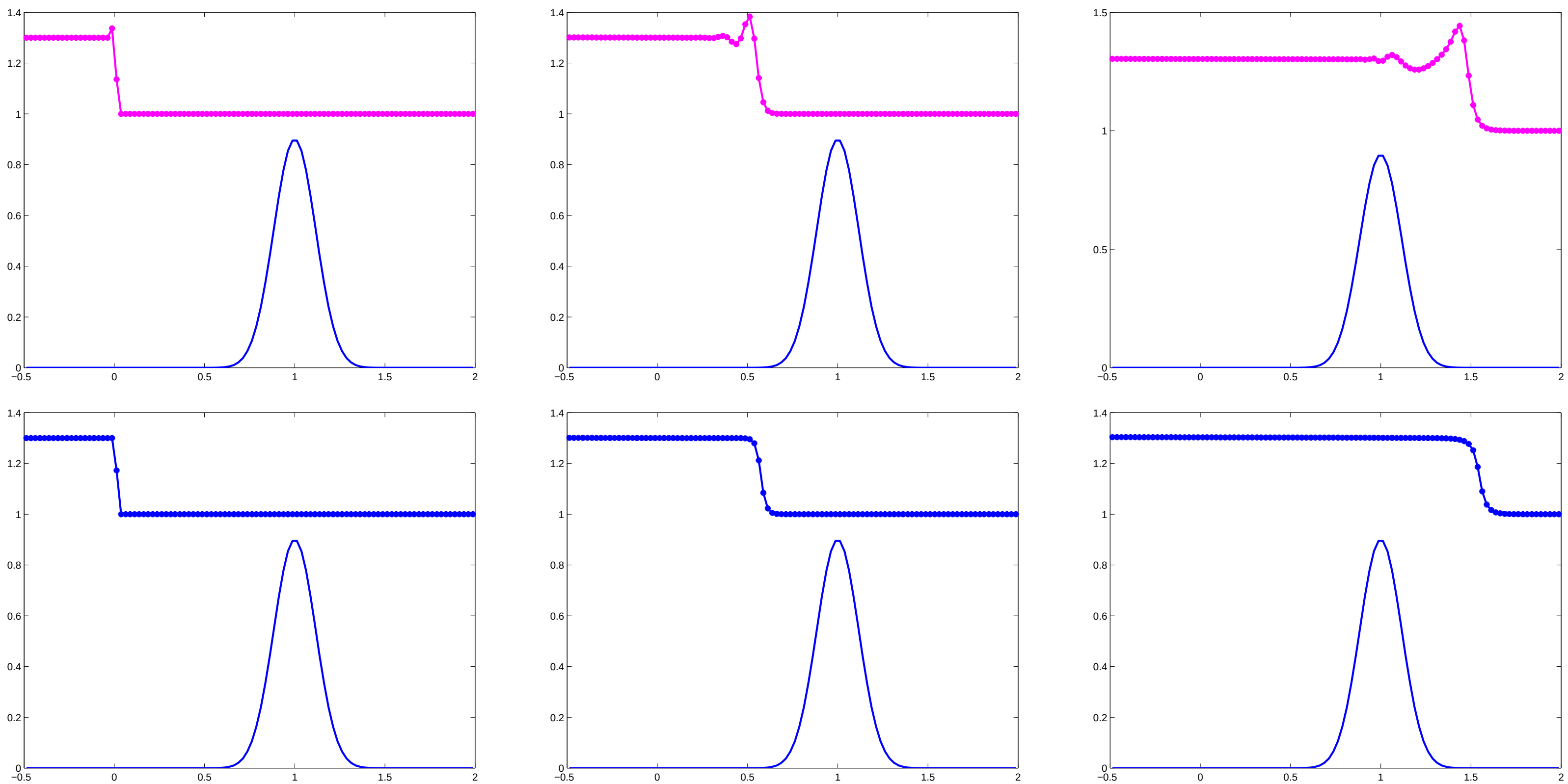
$$\mathcal{G}^n_{i+\frac{1}{2}} - \mathcal{G}^n_{i-\frac{1}{2}} = O(\triangle x^{p+1}) \qquad \mathcal{S}^n_i = O(\triangle x^{p+1}) \quad \Rightarrow \quad U^{n+1}_i = U^n_i + O(\triangle x^{p+1})$$

The scheme satisfies theapproximate C-property ($p \geq 2$).

Greenberg&Leroux tests [SINUM,96]

$$u_t + \frac{u^2}{2} = -a_x(x)u, \qquad u + a = \begin{cases} 1.3, & x < 0 \\ 1 & x > 0 \end{cases}$$

the WB second order scheme



Curbing the oscillations: A WB second order hybrid scheme

A Well Balanced second order Hybrid scheme

Second order method $$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(\mathcal{G}^n_{i+\frac{1}{2}} - \mathcal{G}^n_{i-\frac{1}{2}}) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$$

$$\mathcal{G}^n_{i+\frac{1}{2}} = \mathcal{G}^{HI}_{i+\frac{1}{2}} = \frac{1}{2}\left(g^n_{i+1} + g^n_i - \alpha^n_{i+\frac{1}{2}}(g^n_{i+1} - g^n_i)\right), \qquad \alpha^n_{i+\frac{1}{2}} = \lambda \frac{\partial f}{\partial u}\bigg|^n_{i+\frac{1}{2}}$$

A Well Balanced second order Hybrid scheme

Second order method $\quad U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(\mathcal{G}_{i+\frac{1}{2}}^n - \mathcal{G}_{i-\frac{1}{2}}^n) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$

$$\mathcal{G}_{i+\frac{1}{2}}^n = \mathcal{G}_{i+\frac{1}{2}}^{HI} = \frac{1}{2}\left(g_{i+1}^n + g_i^n - \alpha_{i+\frac{1}{2}}^n(g_{i+1}^n - g_i^n)\right), \qquad \alpha_{i+\frac{1}{2}}^n = \lambda \frac{\partial f}{\partial u}\Big|_{i+\frac{1}{2}}^n$$

Lax-Wendroff-type scheme: Oscillatory behavior at discontinuous fronts

Apply the Flux-Limiter technology to curb oscillations.

A Well Balanced second order Hybrid scheme

Second order method $$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(\mathcal{G}_{i+\frac{1}{2}}^n - \mathcal{G}_{i-\frac{1}{2}}^n) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$$

$$\mathcal{G}_{i+\frac{1}{2}}^n = \mathcal{G}_{i+\frac{1}{2}}^{HI} = \frac{1}{2}\left(g_{i+1}^n + g_i^n - \alpha_{i+\frac{1}{2}}^n (g_{i+1}^n - g_i^n)\right), \qquad \alpha_{i+\frac{1}{2}}^n = \lambda \frac{\partial f}{\partial u}\bigg|_{i+\frac{1}{2}}^n$$

$$\mathcal{G}_{i+\frac{1}{2}}^n = \mathcal{G}_{i+\frac{1}{2}}^{LO} + \phi_{i+\frac{1}{2}}(\mathcal{G}_{i+\frac{1}{2}}^{HI} - \mathcal{G}_{i+\frac{1}{2}}^{LO})$$

- $\mathcal{G}_{i+\frac{1}{2}}^{LO} = \frac{1}{2}\left(g_{i+1}^n + g_i^n - \text{sign}(\alpha_{i+\frac{1}{2}}^n)(g_{i+1}^n - g_i^n)\right)$
- $\mathcal{G}_{i+\frac{1}{2}}^{HI} = \frac{1}{2}\left(g_{i+1}^n + g_i^n - \alpha_{i+\frac{1}{2}}^n (g_{i+1}^n - g_i^n)\right)$
- $\phi_{i+\frac{1}{2}} = \phi(r_{i+\frac{1}{2}})$

Upwind smoothness indicator $$r_{i+\frac{1}{2}} = \begin{cases} \dfrac{g_i - g_{i-1}}{g_{i+1} - g_i}, & \alpha_{i+\frac{1}{2}} > 0; \\ \dfrac{g_{i+2} - g_{i+1}}{g_{i+1} - g_i}, & \alpha_{i+\frac{1}{2}} < 0. \end{cases}$$

A Well Balanced second order Hybrid scheme

Second order method $$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(\mathcal{G}^n_{i+\frac{1}{2}} - \mathcal{G}^n_{i-\frac{1}{2}}) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$$

$$\mathcal{G}^n_{i+\frac{1}{2}} = \mathcal{G}^{HI}_{i+\frac{1}{2}} = \frac{1}{2}\left(g^n_{i+1} + g^n_i - \alpha^n_{i+\frac{1}{2}}(g^n_{i+1} - g^n_i)\right), \qquad \alpha^n_{i+\frac{1}{2}} = \lambda \frac{\partial f}{\partial u}\Big|^n_{i+\frac{1}{2}}$$

$$\mathcal{G}^n_{i+\frac{1}{2}} = \mathcal{G}^{LO}_{i+\frac{1}{2}} + \phi_{i+\frac{1}{2}}(\mathcal{G}^{HI}_{i+\frac{1}{2}} - \mathcal{G}^{LO}_{i+\frac{1}{2}})$$

- $\mathcal{G}^{LO}_{i+\frac{1}{2}} = \frac{1}{2}\left(g^n_{i+1} + g^n_i - \text{sign}(\alpha^n_{i+\frac{1}{2}})(g^n_{i+1} - g^n_i)\right)$
- $\mathcal{G}^{HI}_{i+\frac{1}{2}} = \frac{1}{2}\left(g^n_{i+1} + g^n_i - \alpha^n_{i+\frac{1}{2}}(g^n_{i+1} - g^n_i)\right)$
- $\phi_{i+\frac{1}{2}} = \phi(r_{i+\frac{1}{2}})$

Upwind smoothness indicator $$r_{i+\frac{1}{2}} = \begin{cases} \dfrac{g_i - g_{i-1}}{g_{i+1} - g_i}, & \alpha_{i+\frac{1}{2}} > 0; \\ \dfrac{g_{i+2} - g_{i+1}}{g_{i+1} - g_i}, & \alpha_{i+\frac{1}{2}} < 0. \end{cases}$$

$$g_{i+1} = g_i, \quad \forall i \rightarrow U^{n+1} = U^n$$ Well Balanced

The WBH2 scheme for 1D Shallow water flows

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghz_x \end{pmatrix}$$

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}\left(\mathcal{G}_{i+1/2}^n - \mathcal{G}_{i-1/2}^n\right) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$$

$$\mathcal{G}_{i+1/2}^n = \sum_{p=1}^{2} \mathcal{G}_{i+1/2}^{n,p} R_{i+1/2}^{n,p}$$

The WBH2 scheme for 1D Shallow water flows

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2} g h^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -g h z_x \end{pmatrix}$$

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$$\mathcal{G}_{i+1/2}^n = \sum_{p=1}^{2} \mathcal{G}_{i+1/2}^{n,p} R_{i+1/2}^{n,p}$$

- Roe's average state at the interface.
- TVDB $\mathcal{G}_{i+1/2}^{n,p}$ pth characteristic flux
- $\alpha_{i+\frac{1}{2}}^n =$ p-th eigenvalue of the Jacobian matrix $\frac{\partial F}{\partial U}$.

The WBH2 scheme for 1D Shallow water flows

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghz_x \end{pmatrix}$$

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}\left(\mathcal{G}_{i+1/2}^n - \mathcal{G}_{i-1/2}^n\right) - \frac{\triangle t}{\triangle x}\mathcal{S}_i^n$$

$$\mathcal{G}_{i+1/2}^n = \sum_{p=1}^{2} \mathcal{G}_{i+1/2}^{n,p} R_{i+1/2}^{n,p}$$

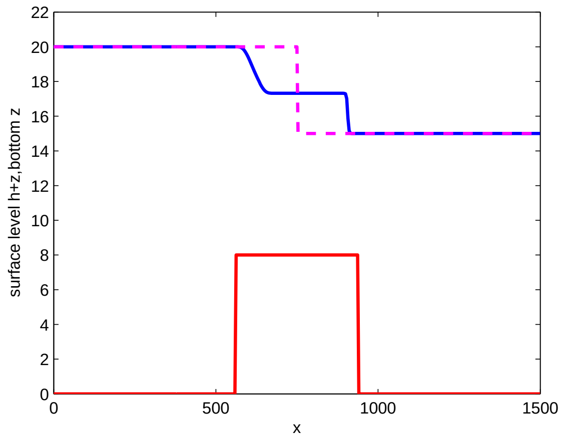
$$\mathcal{S}_i^n = \frac{1}{2}\left(\beta_{i+\frac{1}{2}}^n (F_{i+1}^n - F_i^n + B_{i,i+1}^n) + \beta_{i-\frac{1}{2}}^n (F_i^n - F_{i-1}^n + B_{i-1,i}^n)\right)$$

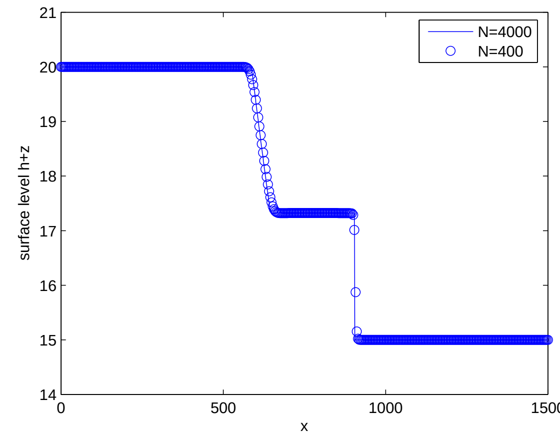
$$\beta_{i+1/2} = \frac{\partial S}{\partial U}\Big|_{i+\frac{1}{2}}^n = \begin{pmatrix} 0 & 0 \\ -gz_x\big|_{i+\frac{1}{2}}^n & 0 \end{pmatrix} \qquad B_{i,i+1}^n = \begin{pmatrix} 0 \\ \int_{x_i}^{x_{i+1}} ghz_x dx \end{pmatrix}$$

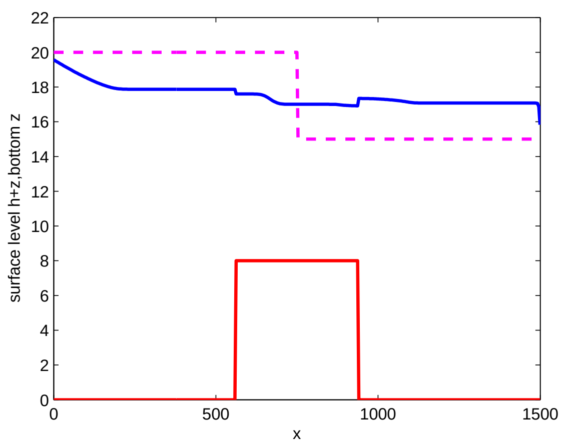
$$\int_{x_i}^{x_{i+1}} ghz_x dx \approx \frac{g}{2}(z_{i+1} - z_i)(h_i + h_{i+1}); \qquad z_x\Big|_{i+\frac{1}{2}} = \frac{z_{i+1} - z_i}{\Delta x}$$

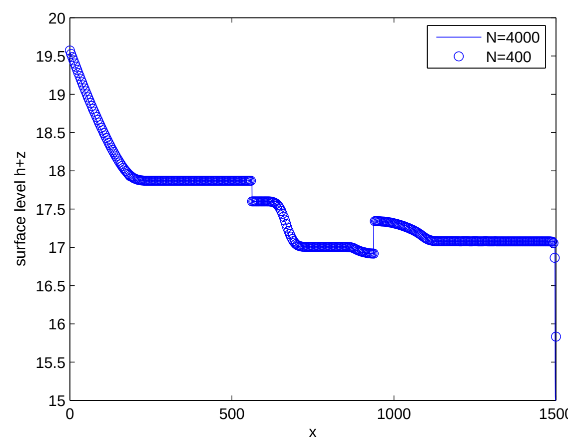
Dam Break over a discontinuous topography

Surface level $h+z$. Left: Numerical $h+z$ using 400 grid cells, and bottom topography. Right: Numerical $h+z$ using 400 and 4000 grid cells. Top: $t = 15$s, Bottom $t = 60$s



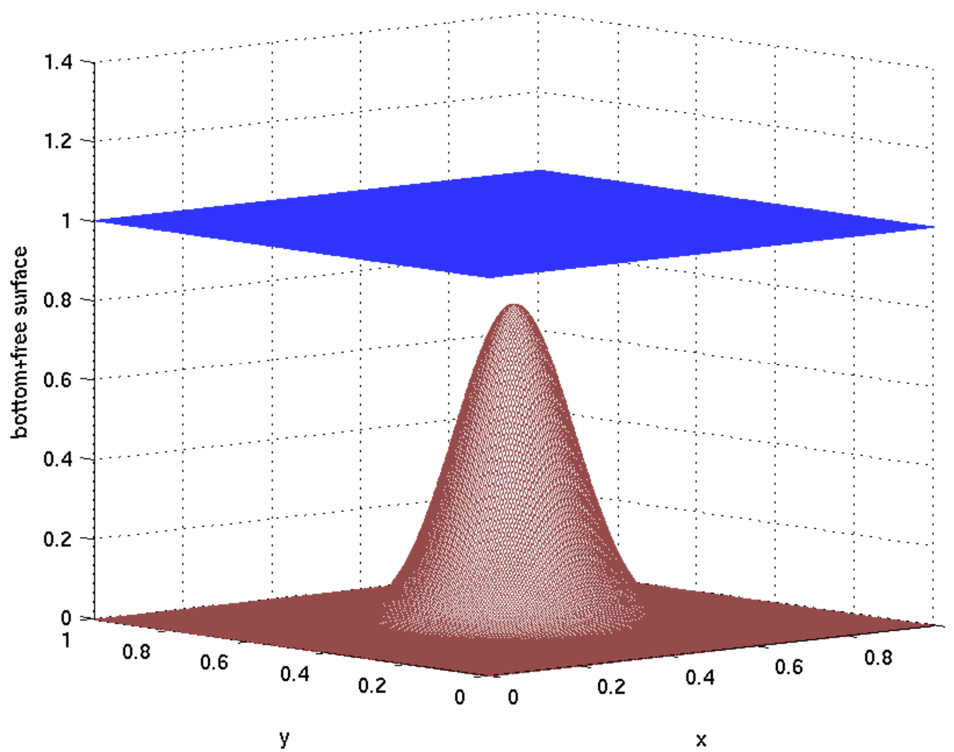






2D-extension: The C-property

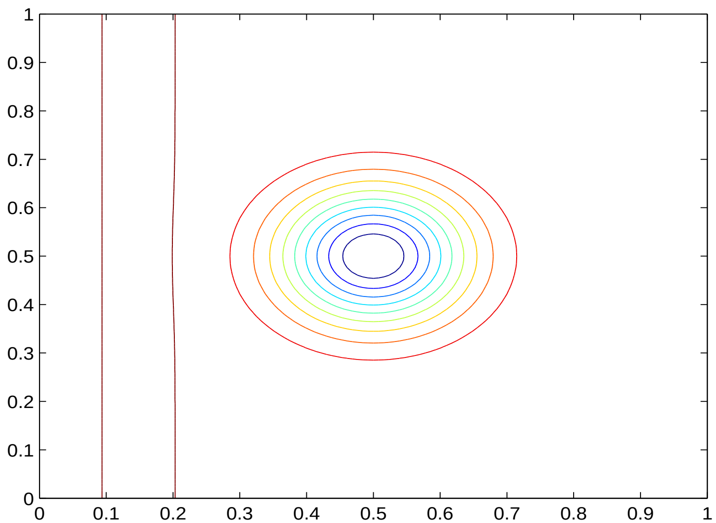
Grid size	l_1-error
128×128	$1.215 \cdot 10^{-15}$
256×256	$2.627 \cdot 10^{-15}$
512×512	$7.690 \cdot 10^{-16}$

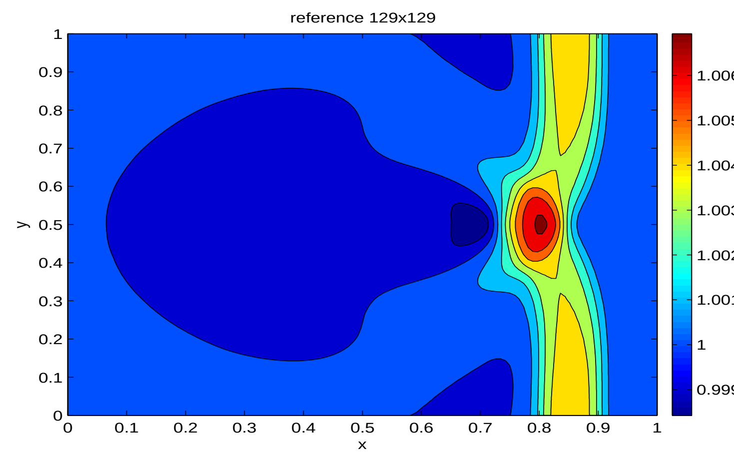


2-D Test: Quasi-stationary flow

Same topography: $z(x,y) = 0.5e^{-50((x-0.5)^2+(y-0.5)^2)}$. Initial Data

$$h(x,y) = \begin{cases} 1.01 - z(x,y), & 0.1 < x < 0.2; \\ 1 - z(x,y), & otherwise. \end{cases} \quad \begin{pmatrix} q_1(x,y) \\ q_2(x,y) \end{pmatrix} = 0$$





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Perspectives

- Treatment of Dry areas.
- Combine WBH2 with Adaptive Mesh Refinement techniques for complex situations.

Thanks for your attention!