Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

UNIFORM STABILIZATION OF ANISOTROPIC MAXWELL'S EQUATIONS WITH BOUNDARY DISSIPATION

Cleverson R. da Luz

Institute of Mathematics
Federal University of Rio de Janeiro
P.O. Box 68530, Rio de Janeiro, RJ, Brazil

Gustavo P. Menzala

National Laboratory of Scientific Computation (LNCC/MCT)
Av. Getulio Vargas, 333, Quitandinha
Petrópolis, RJ, 25651-070, Brazil

(Communicated by the associate editor name)

Abstract. We consider the Maxwell system with variable anisotropic coefficients in a bounded domain Ω of $\mathbb{R}^3$. The boundary conditions are of Silver-Muller's type. We proved that the total energy decays exponentially fast to zero as time approaches infinity. This result is well known in the case of isotropic coefficients. We make use of modified multipliers with the help of an elliptic problem and some technical assumptions on the permittivity and permeability matrices.

1. **Introduction.** Maxwell's equations provide mathematical foundations for the analysis of a broad range of devices which include wave guides, energy conversion devices, electronic components, electro-optic devices, among many others. In many situations it is enough to consider the isotropic Maxwell's equations, that is, when the permittivity $\mathcal{E}(x)$ and the permeability $\mu(x)$ are scalar real-valued functions, strictly positive. In recent years, industrial applications of the so-called "smart materials" [1] and the research on electromagnetic waves in crystal optics [12] have stimulated the interest in the so-called anisotropic Maxwell's equations, which we describe next: Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set with smooth boundary and let $E(x,t)$ and $H(x,t)$ be vector-valued functions defined in $\Omega \times (0,T) \mapsto \mathbb{R}^3$ denoting the electric field intensity and magnetic field intensity, respectively. The electric permittivity and the magnetic permeability are denoted by $\mathcal{E}(x)$ and $\mu(x)$ and are 3×3 matrices, symmetric and uniformly positive definite. The entries of $\mathcal{E}(x)$ and $\mu(x)$ are real-valued functions. We assume that Ω is filled with a medium with zero conductivity and there are no electrical charges in Ω.

2000 *Mathematics Subject Classification.* Primary: 35Q99, 35Q60, 35L99.

Key words and phrases. Anisotropic Maxwell system, boundary dissipation, uniform stabilization.

The second author was partially supported by a Research Grant from CNPq (Proc. 306282/2003-8) from the Brazilian Governament.

The evolution of the electromagnetic field is governed by Maxwell's equations

$$\begin{cases} \mathcal{E}(x)E_t = \operatorname{curl} H \\ \mu(x)H_t = -\operatorname{curl} E \\ \operatorname{div}(\mathcal{E}(x)E) = 0 \\ \operatorname{div}(\mu(x)H) = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{1}$$

with boundary conditions

$$H \times \eta = \eta \times (E \times \eta) \qquad \text{on } \partial\Omega \times (0, +\infty) \tag{2}$$

and initial conditions at $t = 0$ given by

$$E(x, 0) = E_0(x), \quad H(x, 0) = H_0(x) \quad \text{in } \Omega. \tag{3}$$

Here $\eta = \eta(x)$ is the exterior unit normal at $x \in \partial\Omega$, $\times$ is the usual vector product, curl denotes the rotational operator and div the divergence operator. Observe that the boundary conditions 2 have a dissipative character for the above model. In fact, formally, by taking the inner product of the first equation 1 by E and the second by H, integrating over Ω and using the divergence theorem give us

$$\begin{aligned} \int_\Omega \big\{\mathcal{E}(x)E_t \cdot E + \mu(x)H_t \cdot H\big\}\, dx &= \int_\Omega \big\{\operatorname{curl} H \cdot E - \operatorname{curl} E \cdot H\big\}\, dx \\ &= -\int_\Omega \operatorname{div}(E \times H)\, dx = -\int_{\partial\Omega} (E \times H) \cdot \eta \, d\Gamma \\ &= -\int_{\partial\Omega} (H \times \eta) \cdot E \, d\Gamma \\ &= -\int_{\partial\Omega} [\eta \times (E \times \eta)] \cdot E \, d\Gamma \\ &= -\int_{\partial\Omega} |E \times \eta|^2 \, d\Gamma \leq 0. \end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big\{\mathcal{E}(x)E \cdot E + \mu(x)H \cdot H\big\}\, dx = -\int_{\partial\Omega} |E \times \eta|^2 \, d\Gamma \tag{4}$$

2 is known as the classical Silver-Muller condition which arises as a first order approximation to the so-called "transparent" boundary condition. Silver-Muller condition allows for reflections back into the region Ω.

Our main result in this article is the uniform decay of the total energy associated $I(t)$ with problem 1–3 where

$$I(t) = \frac{1}{2}\int_\Omega \big\{\mathcal{E}(x)E \cdot E + \mu(x)H \cdot H\big\}\, dx.$$

The behavior of $I(t)$ as $t \to +\infty$ has been previously considered only in the isotropic case with internal or boundary dissipation ([6], [9], [10], [11], [13] and the references therein). In this isotropic case, Maxwell's equations 1 can be reduced (by differentiating once more in time) to a vector wave equation which can be treated using well-known standard results in literature. This argument does not work in the anisotropic case. Consequently, in order to analyze $I(t)$, we have to deal with a first order hyperbolic system with variable coefficients at all times.

There are not so many articles in the mathematical literature giving rigorous results on the anisotropic case for Maxwell's equations. Let us mention some of them: V. Vogelsang [16] and T. Okaji [14] proved strong unique continuation for time-harmonic anisotropic Maxwell's equations, V. G. Yakhno [17] gave a construction of a Green's function for the time-dependent anisotropic Maxwell's system, M. Eller [4, 5] proved a unique continuation of the system across non-characteristic surfaces and also a boundary observability inequality for the anisotropic case. This result implies the exact boundary controllability of an electromagnetic field in Ω, M. Eller and M. Yamamoto [8] obtained a Carleman estimate for the time-harmonic anisotropic Maxwell system.

A classical reference describing the importance of the anisotropic Maxwell's equations in crystal optics is the book [12] by M. Kline and D. Kay.

This paper is organized as follows: global solvability of problem 1–3 in the required class is briefly indicated in Section 2. Uniform exponential decay of the total energy $I(t)$ is established in Section 3.

2. **Well-posedness of 1–3.** Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set with smooth boundary $\partial\Omega$ (class C^2 is enough) and $\mathcal{E}(x)$, $\mu(x)$ be symmetric 3×3 matrices whose entries are real-valued functions belonging to $W^{1,\infty}(\Omega)$. Furthermore the condition (uniformly positive definite) is assumed:

There exists positive constants A_1, A_2 in such a way that

$$\begin{cases} v^t\mathcal{E}(x)v \geq A_1|v|^2 & \text{for any } v \in \mathbb{R}^3 \text{ and } x \in \Omega \text{ a.e.} \\ v^t\mu(x)v \geq A_2|v|^2 & \text{for any } v \in \mathbb{R}^3 \text{ and } x \in \Omega \text{ a.e.} \end{cases} \tag{5}$$

Here if $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ then, we denote $v^t = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $|v|^2 = \sum_{j=1}^{3} v_j^2$. Let us describe the function spaces where we will consider the solution of problem 1-3: If $\alpha(x) = [\alpha_{ij}(x)]_{3\times 3}$ is symmetric, $\alpha_{ij} \in L^\infty(\Omega)$ and there exists $A_0 > 0$ such that $\xi^t\alpha(x)\xi \geq A_0|\xi|^2$, for any $\xi \in \mathbb{R}^3$ and all $x \in \Omega$ a.e., then we consider the weighted space:

$$\begin{aligned} L^2(\Omega;\alpha) =& \Big\{ z(x) = (z_1(x), z_2(x), z_3(x)) \ \text{ in such a way} \\ & \int_\Omega z^t(x)\,\alpha(x)\,z(x)\,dx = \sum_{i,j=1}^{3} \int_\Omega \alpha_{ij}(x)\,z_i(x)\,z_j(x)\,dx < +\infty \Big\} \end{aligned}$$

with inner product

$$(v,u)_{L^2(\Omega;\alpha)} = \sum_{i,j=1}^{3} \int_\Omega \alpha_{ij}(x)\,v_i(x)\,u_j(x)\,dx = \int_\Omega u^t(x)\,\alpha(x)\,v(x)\,dx$$

and norm

$$||z||^2_{L^2(\Omega;\alpha)} = \int_\Omega z^t(x)\,\alpha(x)\,z(x)\,dx.$$

Clearly $L^2(\Omega;\alpha) = [L^2(\Omega)]^3$ and the norms $||\cdot||_{L^2(\Omega;\alpha)}$ and $||\cdot||_{[L^2(\Omega)]^3}$ are equivalent.

Let $X = L^2(\Omega;\mathcal{E}) \times L^2(\Omega;\mu)$ be the Hilbert space with the inner product

$$\begin{aligned} \langle z, w\rangle_X &= (z_1, w_1)_{L^2(\Omega;\mathcal{E})} + (z_2, w_2)_{L^2(\Omega;\mu)} \\ &= \int_\Omega w_1^t(x)\,\mathcal{E}(x)\,z_1(x)\,dx + \int_\Omega w_2^t(x)\,\mu(x)\,z_2(x)\,dx. \end{aligned} \tag{6}$$

We also consider the Hilbert spaces

$$H(\mathrm{curl};\Omega) = \big\{ v \in [L^2(\Omega)]^3;\ \mathrm{curl}\, v \in [L^2(\Omega)]^3 \big\}$$

with inner product

$$\langle v, u\rangle_{H(\mathrm{curl};\Omega)} = (v, u)_{[L^2(\Omega)]^3} + (\mathrm{curl}\, v, \mathrm{curl}\, u)_{[L^2(\Omega)]^3}$$

and

$$Y = \big\{(z, w) \text{ belongs to } H(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega) \text{ such that } w \times \eta \in [L^2(\partial\Omega)]^3\big\}.$$

The following lemma is well-known (see [3] or [11]).

Lemma 2.1. *The mapping $u = (u_1, u_2) \mapsto u_1 - \eta(u_1 \cdot \eta) - (u_2 \times \eta)$ from $[C^2(\overline{\Omega})]^3 \times [C^1(\overline{\Omega})]^3$ into $[C^1(\partial\Omega)]^3$ extends by continuity to a continuous linear mapping from $Y \mapsto [H^{-1/2}(\partial\Omega)]^3$ which we also denote by*

$$u \mapsto u_1 - \eta(u_1 \cdot \eta) - (u_2 \times \eta).$$

Here $\eta = \eta(x)$ denotes the unit outward normal on $\partial\Omega$ at x and $H^{-1/2}(\partial\Omega)$ is the dual space of the Sobolev space $H^{1/2}(\partial\Omega)$.

Observe that using the results of [11] it follows that whenever $u = (u_1, u_2) \in Y$ then $u_2 \times \eta$ is well defined on $\partial\Omega$. Also, since the relation $u_1 - (\eta \times u_1) \times \eta = \eta(u_1 \cdot \eta)$ always holds then $(\eta \times u_1) \times \eta$ is well defined on $\partial\Omega$. This observation makes it possible to introduce the subspace

$$\overset{\circ}{Y} = \big\{(z, w) \in Y, \quad \eta \times (z \times \eta) = w \times \eta \text{ on } \partial\Omega\big\}$$

which is dense in X. In $\overset{\circ}{Y}$ we define the unbounded operator A with domain

$$\mathcal{D}(A) = \overset{\circ}{Y}$$

given by

$$A(z, w) = (\mathcal{E}^{-1} \mathrm{curl}\, w, -\mu^{-1} \mathrm{curl}\, z), \quad \forall\, (z, w) \in \mathcal{D}(A).$$

Here $\mathcal{E}^{-1}(x)$ and $\mu^{-1}(x)$ denote the inverse matrices of $\mathcal{E}(x)$ and $\mu(x)$ respectively.

Matrices $\mathcal{E}(x)$ and $\mu(x)$ are invertible a.e. in Ω because they are positive definite, therefore the eigenvalues of $\mathcal{E}(x)$ and $\mu(x)$ are positive (see [15]). Consequently the determinant of those matrices are positive. Hence $\mathcal{E}(x)$ and $\mu(x)$ are invertible a.e. in Ω. We can also prove that the entries of $\mathcal{E}^{-1}(x)$ and $\mu^{-1}(x)$ belong to $L^\infty(\Omega)$.

Following the same ideas of [11] we can prove the following.

Lemma 2.2. *a) The domain of the adjoint operator A^* coincides with the subspace*

$$\big\{(z, w) \in Y, \quad \eta \times (z \times \eta) = -w \times \eta \quad \text{on } \partial\Omega\big\}.$$

If $(z, w) \in \mathcal{D}(A^*)$ then

$$A^*(z, w) = -(\mathcal{E}^{-1} \mathrm{curl}\, w, -\mu^{-1} \mathrm{curl}\, z).$$

b) A is a closed operator and $\mathcal{D}(A)$ is dense in X.
c) A and A^* are dissipative, that is,

$$\langle Au, u\rangle_X \le 0 \quad \text{whenever } u \in \mathcal{D}(A)$$

and

$$\langle A^* v, v\rangle_X \le 0 \quad \text{whenever } v \in \mathcal{D}(A^*).$$

Using Lumer-Phillips' Theorem, Lemma 2.2 implies that A is the infinitesimal generator of a C_0 semigroup. Remains to fulfill the requirement that $\mathcal{E}E$ and μH are divergent free. However, taking the divergent of each of the first two equations in 1 we obtain

$$\operatorname{div}(\mathcal{E}E_t(t)) = \operatorname{div}\operatorname{curl} H(t) = 0$$
$$\operatorname{div}(\mu H_t(t)) = -\operatorname{div}\operatorname{curl} E(t) = 0.$$

Consequently

$$\operatorname{div}(\mathcal{E}E(t)) = \operatorname{div}(\mathcal{E}E_0) \quad \forall\, t \geq 0$$

and

$$\operatorname{div}(\mu H(t)) = \operatorname{div}(\mu H_0) \quad \forall\, t \geq 0.$$

Thus, it is natural to consider the subspace

$$W = \big\{(z,w) \in X, \quad \operatorname{div}(\mathcal{E}z) = 0 = \operatorname{div}(\mu\, w)\big\}.$$

Theorem 2.3. *Under the assumptions at the beginning of this section on Ω, $\mathcal{E}$ and μ (here it is only necessary that the entries are in $L^\infty(\Omega)$). If we take initial data $(E_0, H_0) \in \mathcal{D}(A) \cap W$ then, system 1–3 has a unique strong solution*

$$(E,H) \in C([0,\infty); \mathcal{D}(A) \cap W) \cap C^1([0,\infty); W).$$

3. **Stabilization.** In this section we will prove our main result. Our strategy will be to use convenient multipliers with the help of an elliptic problem. The result, then, will follow by assuming a geometric condition on the domain ("substarlike") and an extra technical assumption on the matrices $\mathcal{E}(x)$ and $\mu(x)$.

Let us consider a solution $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann problem

$$\begin{cases} \Delta\Phi(x) = 1 \ \text{ in } \Omega \\ \frac{\partial\Phi}{\partial\eta} = \frac{\operatorname{Vol}(\Omega)}{\operatorname{Area}(\partial\Omega)} \ \text{ on } \partial\Omega. \end{cases}$$

Let $\delta > 0$ and $x_0 \in \Omega$ (to be chosen later) and consider the auxiliary function

$$h(x) = \delta\Phi(x) + \frac{|x-x_0|^2}{2}. \tag{7}$$

Clearly $h(x)$ has the following properties: $\nabla h(x) = \delta\nabla\Phi(x) + (x - x_0)$ on Ω, $\frac{\partial h}{\partial\eta}(x) = \delta\frac{\operatorname{Vol}(\Omega)}{\operatorname{Area}(\partial\Omega)} + (x-x_0)\cdot\eta(x)$ on $\partial\Omega$, $\Delta h(x) = \delta + 3$ on Ω and

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(x) = \delta\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x) + \delta_{ij} \quad \text{ where } \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases}$$

Theorem 3.1. *Let (E,H) be the solution of problem 1–3 with $(E_0,H_0) \in \mathcal{D}(A) \cap W$ as in Theorem 2.3 where all entries of $\mathcal{E}$ and μ belong to $W^{1,\infty}(\Omega)$. Suppose that we can find $k > 0$ in such a way that*

$$k + (x - x_0)\cdot\eta \geq 0 \quad \text{on } \partial\Omega \tag{8}$$

holds for some $x_0 \in \Omega$ and

$$\begin{aligned} &\frac{1}{2}\,(\mathcal{E}v)\cdot v + \nabla h \cdot \left(\left[\frac{\partial\mathcal{E}}{\partial x_1}v\right]\cdot v,\ \left[\frac{\partial\mathcal{E}}{\partial x_2}v\right]\cdot v,\ \left[\frac{\partial\mathcal{E}}{\partial x_3}v\right]\cdot v\right) \geq 0 \\ &\frac{1}{2}\,(\mu v)\cdot v + \nabla h \cdot \left(\left[\frac{\partial\mu}{\partial x_1}v\right]\cdot v,\ \left[\frac{\partial\mu}{\partial x_2}v\right]\cdot v,\ \left[\frac{\partial\mu}{\partial x_3}v\right]\cdot v\right) \geq 0 \end{aligned} \tag{9}$$

are valid for any $v \in \mathbb{R}^3$, $x \in \Omega$ a.e.. Then, there exist positive constants C and α such that, the total energy satisfies

$$\begin{aligned} I(t) &= \frac{1}{2}\left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} + ||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\} \\ &\le CI(0)\exp(-\alpha t) \quad \textit{for any } t \ge 0. \end{aligned}$$

Remark 1. Condition 8 is called "substarlike". It is a geometric condition on the region Ω. It was first introduced by B. Kapitonov in [10]. If $k = 0$ then 8 is reduced to the well known starlike condition on Ω. Likely, the final result of Theorem 3.1 may not be optimal with respect to assumptions 8 and 9.

Remark 2. Suppose that instead of 7 we just choose $h(x) = \frac{1}{2}|x - x_0|^2$ then condition 9 can be written as

$$(\mathcal{E}v)\cdot v + (2(x - x_0)\cdot\nabla)((\mathcal{E}v)\cdot v) \ge 0$$

$$\text{and} \qquad (\mu v)\cdot v + (2(x - x_0)\cdot\nabla)((\mu v)\cdot v) \ge 0$$

which are the assumptions introduced by M. Eller in [5] to prove a boundary observability inequality. See also [6] (Theorem 1.1) for the isotropic case.

Proof of Theorem 3.1. We present the proof in three steps.
Step 1: Let $h(x)$ as in 7. We use the multipliers $\nabla h \times \mu H$ and $\nabla h \times \mathcal{E}E$ to obtain an identity for $\int_0^T \left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} + ||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\} dt$. Those kind of multipliers were previously used by M. Eller and J. Masters in [7] to obtain a result on exact boundary controllability for Maxwell equations and by M. Eller in [5] to deduce a continuous observability for the anisotropic Maxwell system. Although the stabilization problem was not considered in [5] it could have been studied in the presence of boundary dissipation.

We take the inner product of $\mathcal{E}E_t - \operatorname{curl} H = 0$ with $\nabla h \times \mu H$ and $\mu H_t + \operatorname{curl} E = 0$ with $\nabla h \times \mathcal{E}E$. By adding and integrating in space/time $\Omega \times (0,T)$ we obtain

$$\begin{aligned} \int_0^T \int_\Omega \big\{ &\operatorname{curl} H \cdot (\nabla h \times \mu H) - \mathcal{E}E_t \cdot (\nabla h \times \mu H) \\ &+ \mu H_t \cdot (\nabla h \times \mathcal{E}E) + \operatorname{curl} E \cdot (\nabla h \times \mathcal{E}E)\big\}\, dx\, dt = 0. \end{aligned} \tag{10}$$

We can rewrite 10 as

$$\begin{aligned} \int_0^T \int_\Omega \Big\{ &\operatorname{curl} H \cdot (\nabla h \times \mu H) + \frac{d}{dt}\big[(\nabla h \times \mathcal{E}E)\cdot \mu H\big] \\ &+ \operatorname{curl} E \cdot (\nabla h \times \mathcal{E}E)\Big\}\, dx\, dt = 0 \end{aligned}$$

which implies

$$\begin{aligned} &\int_\Omega (\nabla h \times \mathcal{E}E(T))\cdot \mu H(T)\, dx - \int_\Omega (\nabla h \times \mathcal{E}E_0)\cdot \mu H_0\, dx \\ &\quad + \int_0^T \int_\Omega \big\{\operatorname{curl} H \cdot (\nabla h \times \mu H) + \operatorname{curl} E \cdot (\nabla h \times \mathcal{E}E)\big\}\, dx\, dt = 0. \end{aligned} \tag{11}$$

Let us rewrite the integrand of the last term on the left hand side of 11 in a convenient way: direct calculation give us the identity

$$\begin{aligned} 2(\mathcal{E}E \times \nabla h)\cdot \operatorname{curl} E = 2[&\operatorname{grad}(E\cdot\nabla h)]\cdot \mathcal{E}E - 2U \cdot \mathcal{E}E \\ &- [\operatorname{grad}(\mathcal{E}E \cdot E)]\cdot \nabla h + K \cdot \nabla h \end{aligned} \tag{12}$$

where
$$K=\left(\frac{\partial\mathcal{E}}{\partial x_1}E\cdot E,\ \frac{\partial\mathcal{E}}{\partial x_2}E\cdot E,\ \frac{\partial\mathcal{E}}{\partial x_3}E\cdot E\right)$$
and
$$U=[h_{x_ix_j}]_{3\times 3}\begin{bmatrix}E_1\\E_2\\E_3\end{bmatrix},$$
here $[h_{x_ix_j}]$ denotes the Hessian matrix of $h(x)$.

The terms $2[\operatorname{grad}(E\cdot\nabla h)]\cdot\mathcal{E}E$ and $[\operatorname{grad}(\mathcal{E}E\cdot E)]\cdot\nabla h$ on the right hand side of 12 can be replaced by using the identities
$$\begin{aligned}\operatorname{div}[(\mathcal{E}E\cdot E)\nabla h]&=[\operatorname{grad}(\mathcal{E}E\cdot E)]\cdot\nabla h+(\mathcal{E}E\cdot E)\Delta h\\&=[\operatorname{grad}(\mathcal{E}E\cdot E)]\cdot\nabla h+(3+\delta)(\mathcal{E}E\cdot E)\end{aligned}$$
and
$$\begin{aligned}2\operatorname{div}[(E\cdot\nabla h)\mathcal{E}E]&=2[\operatorname{grad}(E\cdot\nabla h)]\cdot\mathcal{E}E+2(\nabla h\cdot E)\operatorname{div}(\mathcal{E}E)\\&=2[\operatorname{grad}(E\cdot\nabla h)]\cdot\mathcal{E}E\end{aligned}$$
because $\operatorname{div}(\mathcal{E}E)=0$. Thus, from 12 and the above identities we obtain
$$\begin{aligned}2(\mathcal{E}E\times\nabla h)\cdot\operatorname{curl}E&=2\operatorname{div}[(E\cdot\nabla h)\mathcal{E}E]-2U\cdot\mathcal{E}E\\&\quad-\operatorname{div}[(\mathcal{E}E\cdot E)\nabla h]+(3+\delta)(\mathcal{E}E\cdot E)+K\cdot\nabla h.\end{aligned}\tag{13}$$
Similar calculations give us the identity
$$\begin{aligned}2(\mu H\times\nabla h)\cdot\operatorname{curl}H&=2\operatorname{div}[(H\cdot\nabla h)\mu H]-2\tilde{U}\cdot\mu H\\&\quad-\operatorname{div}[(\mu H\cdot H)\nabla h]+(3+\delta)(\mu H\cdot H)+\tilde{K}\cdot\nabla h\end{aligned}\tag{14}$$
where
$$\tilde{K}=\left(\frac{\partial\mu}{\partial x_1}H\cdot H,\ \frac{\partial\mu}{\partial x_2}H\cdot H,\ \frac{\partial\mu}{\partial x_3}H\cdot H\right)$$
and
$$\tilde{U}=[h_{x_ix_j}]_{3\times 3}\begin{bmatrix}H_1\\H_2\\H_3\end{bmatrix}.$$
By substituting identities 13 and 14 into 11 and using the divergence theorem give us
$$\begin{aligned}&(3+\delta)\int_0^T\left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})}+||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\}dt\\&\quad+\int_0^T\int_\Omega\left\{K\cdot\nabla h+\tilde{K}\cdot\nabla h\right\}dx\,dt+\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\left[\mu H\cdot H+\mathcal{E}E\cdot E\right]d\Gamma\,dt\\&=2\int_\Omega(\nabla h\times\mathcal{E}E(T))\cdot\mu H(T)\,dx-2\int_\Omega(\nabla h\times\mathcal{E}E_0)\cdot\mu H_0\,dx\\&\quad-2\int_0^T\int_{\partial\Omega}(\nabla h\times\mu H)\cdot(H\times\eta)\,d\Gamma\,dt\\&\quad-2\int_0^T\int_{\partial\Omega}(\nabla h\times\mathcal{E}E)\cdot(E\times\eta)\,d\Gamma\,dt+2\int_0^T\int_\Omega\left\{\tilde{U}\cdot\mu H+U\cdot\mathcal{E}E\right\}dx\,dt\end{aligned}\tag{15}$$
here we used the vector identities
$$(H\cdot\nabla h)(\mu H\cdot\eta)=\frac{\partial h}{\partial\eta}(\mu H\cdot H)+(\nabla h\times\mu H)\cdot(H\times\eta)$$

and

$$(E\cdot\nabla h)(\mathcal{E}E\cdot\eta)=\frac{\partial h}{\partial\eta}\,(\mathcal{E}E\cdot E)+(\nabla h\times\mathcal{E}E)\cdot(E\times\eta).$$

Step 2: We estimate all terms on the right hand side of identity 15 by $I(0)$ (the initial energy) and $|E\times\eta|$, $|H\times\eta|$ on $L^2(0,T;L^2(\partial\Omega))$. We estimate

$$F(T)\equiv 2\int_\Omega(\nabla h\times\mathcal{E}E(T))\cdot\mu H(T)\,dx-2\int_\Omega(\nabla h\times\mathcal{E}E_0)\cdot\mu H_0\,dx.$$

Clearly

$$\begin{aligned}|F(T)|&\le 2\int_\Omega|\nabla h|\,|\mathcal{E}E(T)|\,|\mu H(T)|\,dx+2\int_\Omega|\nabla h|\,|\mathcal{E}E_0|\,|\mu H_0|\,dx\\&\le C_1\int_\Omega\big\{|\mathcal{E}E(T)|\,|\mu H(T)|+|\mathcal{E}E_0|\,|\mu H_0|\big\}\,dx\end{aligned}$$

where $C_1=2\sup_{x\in\Omega}|\nabla h(x)|$. Using the equivalence of the norms $||\cdot||_{L^2(\Omega;\mathcal{E})}$ (and $||\cdot||_{L^2(\Omega;\mu)}$) with $||\cdot||_{[L^2(\Omega)]^3}$ we obtain

$$\begin{aligned}|F(T)|&\le C_2\big\{||E(\cdot,T)||^2_{L^2(\Omega;\mathcal{E})}+||H(\cdot,T)||^2_{L^2(\Omega;\mu)}\\&\qquad+||E_0||^2_{L^2(\Omega;\mathcal{E})}+||H_0||^2_{L^2(\Omega;\mu)}\big\}\\&\le 2C_2\big\{||E_0||^2_{L^2(\Omega;\mathcal{E})}+||H_0||^2_{L^2(\Omega;\mu)}\big\}\end{aligned}\tag{16}$$

because $I(t)$ is decreasing. Let Φ defined as in the beginning of this Section and consider the number ρ given by

$$\rho=\sup_{\substack{x\in\overline{\Omega}\\ \xi\in\overline{B}_1(0)}}\;2\sum_{i,j=1}^{3}\left|\frac{\partial^2\Phi(x)}{\partial x_i\partial x_j}\right||\xi_i|\,|\xi_j|$$

where $\xi=(\xi_1,\,\xi_2,\,\xi_3)\in\{y\in\mathbb{R}^3,\quad |y|\le 1\}=\overline{B}_1(0)$.

Next, we estimate the term $2\int_0^T\int_\Omega U\cdot\mathcal{E}E\,dx\,dt$. Since $\frac{\partial^2 h}{\partial x_i\partial x_j}=\delta\frac{\partial^2\Phi}{\partial x_i\partial x_j}+\delta_{ij}$, we have the identity

$$\begin{aligned}2U\cdot\mathcal{E}E&=2(E_1h_{x_1x_1}+E_2h_{x_1x_2}+E_3h_{x_1x_3})(\mathcal{E}_{11}E_1+\mathcal{E}_{12}E_2+\mathcal{E}_{13}E_3)\\&\quad+2(E_1h_{x_2x_1}+E_2h_{x_2x_2}+E_3h_{x_2x_3})(\mathcal{E}_{21}E_1+\mathcal{E}_{22}E_2+\mathcal{E}_{23}E_3)\\&\quad+2(E_1h_{x_3x_1}+E_2h_{x_3x_2}+E_3h_{x_3x_3})(\mathcal{E}_{31}E_1+\mathcal{E}_{32}E_2+\mathcal{E}_{33}E_3)\\&=2\delta(E_1\Phi_{x_1x_1}+E_2\Phi_{x_1x_2}+E_3\Phi_{x_1x_3})(\mathcal{E}_{11}E_1+\mathcal{E}_{12}E_2+\mathcal{E}_{13}E_3)\\&\quad+2\delta(E_1\Phi_{x_2x_1}+E_2\Phi_{x_2x_2}+E_3\Phi_{x_2x_3})(\mathcal{E}_{21}E_1+\mathcal{E}_{22}E_2+\mathcal{E}_{23}E_3)\\&\quad+2\delta(E_1\Phi_{x_3x_1}+E_2\Phi_{x_3x_2}+E_3\Phi_{x_3x_3})(\mathcal{E}_{31}E_1+\mathcal{E}_{32}E_2+\mathcal{E}_{33}E_3)\\&\quad+2E\cdot\mathcal{E}E.\end{aligned}\tag{17}$$

By integrating identity 17 over $\Omega\times(0,T)$ gives us several terms. One is

$$2\int_0^T\int_\Omega\mathcal{E}E\cdot E\,dx\,dt=2\int_0^T||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})}\,dt.$$

All others are of the form

$$2\delta\int_0^T\int_\Omega E_i\Phi_{x_jx_i}\mathcal{E}_{jk}E_k\,dx\,dt$$

where $i, j, k = 1, 2, 3$. We can estimate each one of these terms as follows: Let $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 = \{x \in \Omega,\ |E(x,t)| \neq 0\},\ \Omega_2 = \Omega \backslash \Omega_1$ for $t \geq 0$

$$\left| 2\delta \int_0^T \int_\Omega E_i \Phi_{x_j x_i} \mathcal{E}_{jk} E_k \, dx\, dt \right| \leq C_3 \delta \int_0^T \int_{\Omega_1} |E|^2 |\Phi_{x_j x_i}| \frac{|E_i|}{|E|} \frac{|E_k|}{|E|} \, dx\, dt$$
$$\leq A_1^{-1} C_3 \delta \rho \int_0^T ||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} \, dt$$

where we used the equivalence of the norms $||\cdot||_{L^2(\Omega;\mathcal{E})}$ and $||\cdot||_{[L^2(\Omega)]^3}$ and C_3 is a positive constant depending on $\mathcal{E}$. Thus, from 17 we deduce the estimate

$$\begin{aligned} 2\int_0^T \int_\Omega U \cdot \mathcal{E}E \, dx\, dt \leq A_1^{-1} C_3 \delta \rho \int_0^T ||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} \, dt \\ + 2\int_0^T ||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} \, dt. \end{aligned} \tag{18}$$

In a similar procedure we can estimate

$$\begin{aligned} 2\int_0^T \int_\Omega \tilde{U} \cdot \mu H \, dx\, dt \leq A_2^{-1} C_4 \delta \rho \int_0^T ||H(\cdot,t)||^2_{L^2(\Omega;\mu)} \, dt \\ + 2\int_0^T ||H(\cdot,t)||^2_{L^2(\Omega;\mu)} \, dt \end{aligned} \tag{19}$$

where C_4 is a positive constant depending on μ.

Next, we estimate the term

$$-2\int_0^T \int_{\partial\Omega} (\nabla h \times \mathcal{E}E) \cdot (E \times \eta) \, d\Gamma\, dt.$$

Let $\gamma_1 > 0$. Clearly

$$\begin{aligned} &\left| -2\int_0^T \int_{\partial\Omega} (\nabla h \times \mathcal{E}E) \cdot (E \times \eta) \, d\Gamma\, dt \right| \\ &\leq \gamma_1 \int_0^T \int_{\partial\Omega} |\nabla h \times \mathcal{E}E|^2 \, d\Gamma\, dt + \gamma_1^{-1} \int_0^T \int_{\partial\Omega} |E \times \eta|^2 \, d\Gamma\, dt. \end{aligned} \tag{20}$$

We can choose a positive constant C_0 in such a way that $\frac{\partial h}{\partial \eta}(x) \geq C_0 |\nabla h(x)|$ for any $x \in \partial\Omega$. In fact, by 8 we know that $\frac{\partial h}{\partial \eta}(x) \geq 0$ for any $x \in \partial\Omega$, then if $\nabla h(x) = 0$ the inequality always holds. If $\nabla h(x) \neq 0$ then we define $C_0 = \min_{x \in \partial\Omega} \frac{\frac{\partial h}{\partial \eta}(x)}{|\nabla h(x)|} \geq 0$ because $\frac{\partial h}{\partial \eta}(x)$ is a continuous function and $\partial\Omega$ is compact.

Using the estimate 20 we obtain

$$\begin{aligned} &\left| -2\int_0^T \int_{\partial\Omega} (\nabla h \times \mathcal{E}E) \cdot (E \times \eta) \, d\Gamma\, dt \right| \\ &\leq \gamma_1 \int_0^T \int_{\partial\Omega} |\nabla h|^2 \, |\mathcal{E}E|^2 \, d\Gamma\, dt + \gamma_1^{-1} \int_0^T \int_{\partial\Omega} |E \times \eta|^2 \, d\Gamma\, dt \\ &\leq \gamma_1 C_0^{-2} C_5 \int_0^T \int_{\partial\Omega} \frac{\partial h}{\partial \eta} \, |E|^2 \, d\Gamma\, dt + \gamma_1^{-1} \int_0^T \int_{\partial\Omega} |E \times \eta|^2 \, d\Gamma\, dt \end{aligned} \tag{21}$$

due to the choice of C_0 and our assumption $\mathcal{E}_{ij}(x)$ belong to $L^\infty(\Omega)$.

From 21 we deduce the estimate

$$\begin{aligned}&\left|-2\int_0^T\int_{\partial\Omega}(\nabla h\times\mathcal{E}E)\cdot(E\times\eta)\,d\Gamma\,dt\right|\\&\leq\gamma_1C_0^{-2}C_5A_1^{-1}\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\,\mathcal{E}E\cdot E\,d\Gamma\,dt+\gamma_1^{-1}\int_0^T\int_{\partial\Omega}|E\times\eta|^2\,d\Gamma\,dt\end{aligned}\tag{22}$$

where $A_1>0$ was defined in Section 2. We choose $\gamma_1=\frac{C_0^2A_1}{2C_5}>0$ in 22 to obtain

$$\begin{aligned}&\left|-2\int_0^T\int_{\partial\Omega}(\nabla h\times\mathcal{E}E)\cdot(E\times\eta)\,d\Gamma\,dt\right|\\&\leq\frac{1}{2}\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\,\mathcal{E}E\cdot E\,d\Gamma\,dt+\gamma_1^{-1}\int_0^T\int_{\partial\Omega}|E\times\eta|^2\,d\Gamma\,dt.\end{aligned}\tag{23}$$

Similar discussion proves that

$$\begin{aligned}&\left|-2\int_0^T\int_{\partial\Omega}(\nabla h\times\mu H)\cdot(H\times\eta)\,d\Gamma\,dt\right|\\&\leq\frac{1}{2}\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\,\mu H\cdot H\,d\Gamma\,dt+\gamma_2^{-1}\int_0^T\int_{\partial\Omega}|H\times\eta|^2\,d\Gamma\,dt\end{aligned}\tag{24}$$

where $\gamma_2=\frac{C_0^2A_2}{2C_6}>0$ (C_6 depends on $||\mu_{ij}||_{L^\infty(\Omega)}$).

Step 3: The conclusion.

Using estimates 16, 18, 19, 23 and 24 together with 15 we obtain

$$\begin{aligned}&(1+\delta)\int_0^T\left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})}+||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\}dt\\&\quad+\int_0^T\int_\Omega\left\{K\cdot\nabla h+\tilde{K}\cdot\nabla h\right\}dx\,dt+\frac{1}{2}\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\big[\mu H\cdot H+\mathcal{E}E\cdot E\big]d\Gamma\,dt\\&\leq 2C_2\left\{||E_0||^2_{L^2(\Omega;\mathcal{E})}+||H_0||^2_{L^2(\Omega;\mu)}\right\}\\&\quad+\gamma_3\int_0^T\int_{\partial\Omega}\left\{|E\times\eta|^2+|H\times\eta|^2\right\}d\Gamma\,dt\\&\quad+(A_1^{-1}C_3+A_2^{-1}C_4)\delta\rho\int_0^T\left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})}+||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\}dt\end{aligned}\tag{25}$$

where $\gamma_3=\max\{\gamma_1^{-1},\gamma_2^{-1}\}$.

Let us choose $0<\delta<\min\left\{\frac{1}{2\rho[A_1^{-1}C_3+A_2^{-1}C_4]},\frac{3}{4}\right\}$. Therefore from 25 we obtain

$$\begin{aligned}&\left(\frac{1}{2}+\delta\right)\int_0^T\left\{||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})}+||H(\cdot,t)||^2_{L^2(\Omega;\mu)}\right\}dt\\&\quad+\int_0^T\int_\Omega\left\{K\cdot\nabla h+\tilde{K}\cdot\nabla h\right\}dx\,dt\\&\quad+\frac{1}{2}\int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\left[\mu H\cdot H+\mathcal{E}E\cdot E\right]d\Gamma\,dt\\&\leq 2C_2\left\{||E_0||^2_{L^2(\Omega;\mathcal{E})}+||H_0||^2_{L^2(\Omega;\mu)}\right\}\\&\quad+\gamma_3\int_0^T\int_{\partial\Omega}\left\{|E\times\eta|^2+|H\times\eta|^2\right\}d\Gamma\,dt.\end{aligned}\tag{26}$$

Using 9 with 26 we deduce

$$\begin{aligned} &\delta \int_0^T \left\{ ||E(\cdot,t)||^2_{L^2(\Omega;\mathcal{E})} + ||H(\cdot,t)||^2_{L^2(\Omega;\mu)} \right\} dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\partial\Omega} \frac{\partial h}{\partial \eta} \left[\mu H \cdot H + \mathcal{E} E \cdot E \right] d\Gamma\, dt \\ &\leq 2C_2 \left\{ ||E_0||^2_{L^2(\Omega;\mathcal{E})} + ||H_0||^2_{L^2(\Omega;\mu)} \right\} \\ &\quad + \gamma_3 \int_0^T \int_{\partial\Omega} \left\{ |E \times \eta|^2 + |H \times \eta|^2 \right\} d\Gamma\, dt. \end{aligned} \tag{27}$$

Thus, we choose $\delta = \delta_0$ in 7 where $\delta_0 = k\, \frac{\text{Area}\,(\partial\Omega)}{\text{Vol}\,(\Omega)}$. Consequently $\frac{\partial h}{\partial \eta} = k + (x - x_0) \cdot \eta \geq 0$ on $\partial\Omega$ due to 8. Now, we use the boundary conditions 2 to obtain the identities

$$\begin{aligned} |H \times \eta|^2 &= (H \times \eta) \cdot (H \times \eta) = [\eta \times (E \times \eta)] \cdot [\eta \times (E \times \eta)] \\ &= (\eta \cdot \eta)|E \times \eta|^2 - |\eta \cdot (E \times \eta)|^2 \\ &= |E \times \eta|^2 - |E \cdot (\eta \times \eta)|^2 = |E \times \eta|^2. \end{aligned}$$

Hence, the last term on the right hand side of 27 reduces to $2\gamma_3 \int_0^T \int_{\partial\Omega} |E \times \eta|^2\, d\Gamma\, dt$. However, due to 4 we know that

$$\int_0^T \int_{\partial\Omega} |E \times \eta|^2\, d\Gamma\, dt \leq \frac{1}{2} \left\{ ||E_0||^2_{L^2(\Omega;\mathcal{E})} + ||H_0||^2_{L^2(\Omega;\mu)} \right\}.$$

Which together with 27 implies that

$$\int_0^\infty \left\{ \|E(\cdot,t)\|^2_{L^2(\Omega;\mathcal{E})} + \|H(\cdot,t)\|^2_{L^2(\Omega;\mu)} \right\} dt \leq C \left\{ \|E_0\|^2_{L^2(\Omega;\mathcal{E})} + \|H_0\|^2_{L^2(\Omega;\mu)} \right\} \tag{28}$$

for some positive constant C. A classical result due to R. Datko [2] says that 28 implies the conclusion of Theorem 3.1. □

Remark 3. We observe that once we got the crucial estimate 25 and by choosing $\delta = \delta_0$ in the appropriate interval we could assume

$$K \cdot \nabla h + \tilde{K} \cdot \nabla h \geq 0 \quad \text{in } \Omega \times (0,T)$$

to conclude the above proof. However, assumption 9 allows us the possibility that the quantity $K \cdot \nabla h + \tilde{K} \cdot \nabla h$ may assume small negative values.

Acknowledgements: Part of this Research was done while the second author was visiting the Department of Mathematics at Brown University, USA in May 2008. He would like to thank Prof. W. Strauss for the kind invitation and warm hospitality. This Research was partially supported by Grants of the Brazilian Government, CNPq, Proc. 305086/2006 − 5, Universal, Proc. 39293 − 6 and PROSUL 490329/2008 − 0. We also would like to thank both referees of the Journal for their comments and suggestions which allow us to improve our original version.

REFERENCES

[1] H. T. Banks, R. C. Smith and Y. Wang, "Smart Materials Structures, Modeling, Estimation and Control," RAM, Wiley, Chichester, Masson, Paris, 1996.

[2] R. Datko, *Extending a theorem of A.M. Liapunov to Hilbert space*, J. Math. Anal. Appl., **32** (1970), 610–616.

[3] R. Dautray and J. L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Vol **3**, Springer-Verlag, Berlin, 1990.

[4] M. M. Eller, *Unique continuation for solutions to Maxwell's system with non-analytic anisotropic coefficients*, J. Math. Anal. Appl., **284** (2003), 698–710.

[5] M. M. Eller, *Continuous observability for the anisotropic Maxwell system*, Appl. Math. Optim., **55** (2007), 185–201.

[6] M. M. Eller, J. E. Lagnese and S. Nicaise, *Decay rates for solutions of a Maxwell system with nonlinear boundary damping*, Comput. Appl. Math., **21** (2002), 135–165.

[7] M. M. Eller and J. E. Masters, *Exact boundary controllability of electromagnetic fields in a general region*, Appl. Math. Optim., **45** (2002), 99–123.

[8] M. M. Eller and M. Yamamoto, *A Carleman inequality for the stationary anisotropic Maxwell system*, J. Math. Pures Appl., **86** (2006), 449–462.

[9] M. Fabrizio and A. Morro, *A boundary condition with memory in electromagnetism*, Arch. Rational Mech. Anal., **136** (1996), 359–381.

[10] B. V. Kapitonov, *Stabilization and exact boundary controllability for Maxwell's equations*, SIAM J. Control Optim., **32** (1994), 408–420.

[11] B. V. Kapitonov and G. Perla Menzala, *Uniform stabilization for Maxwell's equations with boundary conditions with memory*, Asymptot. Anal., **26** (2001), 91–104.

[12] M. Kline and D. Kay, "Electromagnetic Theory and Geometric Optics," Interscience John and Sons Wiley, New York, 1965.

[13] V. Komornik, *Boundary stabilization, observation and control of Maxwell's equations*, Panamer. Math. J., **4** (1994), 47–61.

[14] T. Okaji, *Strong unique continuation property for time harmonic Maxwell equations*, J. Math. Soc. Japan, **54** (2002), 89–122.

[15] G. Strang, "Introduction to Linear Algebra," Wellesley, Cambridge, 1993.

[16] V. Vogelsang, *On the strong unique continuation principle for inequalities of Maxwell type*, Math. Ann., **289** (1991), 285–295.

[17] V. G. Yakhno, T. M. Yakhno and M. Kasap, *A novel approach for modeling and simulation of electromagnetic waves in anisotropic dielectric*, Internat. J. Solids Structures, **43** (2006), 6261–6276.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: `cleber.ufsc@bol.com.br`
E-mail address: `perla@lncc.br`