Optimal Control in Coefficients for Degenerate Elliptic Equations

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Main Notation

Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary.

Degenerate Weight Function

A weight function $\rho : \Omega \to \mathbb{R}_+$ is called to be degenerate on Ω if

$$\rho + \rho^{-1} \in L^1_{loc}(\mathbb{R}^N). \tag{1}$$

Non-Degenerate Weight Function

We say that a nonnegative function $\rho(x) \geq 0$ is a non-degenerate weight if

$$\rho + \rho^{-1} \in L^\infty(\Omega). \tag{2}$$

Definition of W and H-spaces

With the degenerate weight function $\rho(x) \geq 0$ we will associate two weighted Sobolev spaces

1. $W = W(\Omega, \rho\, dx)$ is the set of functions $y \in W_0^{1,1}(\Omega)$ for which the norm

$$\|y\|_\rho = \left(\int_\Omega \left(y^2 + |\nabla y|^2 \right) \rho\, dx \right)^{1/2} \tag{3}$$

   is finite;
2. $H = H(\Omega, \rho\, dx)$ is the closure of $C_0^\infty(\Omega)$ in $\|\cdot\|_\rho$-norm.

For a "typical" degenerate weight ρ we have: W and H are Hilbert spaces, $H \subseteq W$, and the identity $W = H$ is not valid in general.

Admissible Controls

Definition of Admissible Controls

We say that a matrix $A = [a_{ij}]$ is an admissible control ($A \in U_{ad}$) if $A = [\vec{a}_1, \dots, \vec{a}_N] \in M_\alpha^\beta(\Omega)$ (i.e., $A \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ and $A(x) \geq \alpha I$, $(A(x))^{-1} \geq \beta^{-1} I$, a.e. in Ω , $0 < \alpha \leq \beta$), and

$$\left|\operatorname{div}_\rho \vec{a}_i\right| \leq \gamma_i, \;\; \rho\, dx - \text{a.e. in} \;\; \Omega, \;\; \forall\, i = 1, \dots, N. \tag{4}$$

where the elements $\operatorname{div}_\rho \vec{a}_i \in L^2(\Omega, \rho\, dx)$ are defined as

$$\int_\Omega \operatorname{div}_\rho \vec{a}_i\, \varphi \rho\, dx = -\int_\Omega (\vec{a}_i, \nabla\varphi)_{\mathbb{R}^N} \rho\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega), \tag{5}$$

and $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ is a given strictly positive vector.

Statement of the Optimal Control Problem

For given functions $y_d \in L^2(\Omega, \rho\, dx)$, $f \in C_0^\infty(\mathbb{R}^N)$, and a penalization parameter $\zeta > 0$, we consider the following problem:

$$I(A, y) = \zeta \int_\Omega |y(x) - y_d(x)|^2 \rho\, dx + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} \rho\, dx \to \inf, \tag{6}$$

subject to the constraints

$$A \in U_{ad}, \tag{7}$$

$$-\operatorname{div}\left(\rho A(x) \nabla y\right) + \rho y = f \quad \text{in} \quad \Omega, \qquad y \in W(\Omega, \rho\, dx). \tag{8}$$

Remark 1.

The boundary value problem (8) can exhibit the Lavrentiev phenomenon and nonuniqueness of the weak solutions. As a result, the corresponding optimal control problem can be stated in different forms.

Classification of the Solutions to the Dirichlet BVP

Definition 1

We say that a function $y = y(A, f) \in W$ is a weak solution to BVP (7)–(8) if the integral identity

$$\int_\Omega \left(\left(A(x)\, \nabla y, \nabla \varphi \right)_{\mathbb{R}^N} + y\varphi \right) \rho \, dx = \int_\Omega f\varphi \, dx \quad \forall \, \varphi \in C_0^\infty(\Omega). \tag{9}$$

Definition 2.

We say that $y = y(A, f) \in V$ is a V-solution ($H \subseteq V \subseteq W$) to BVP (7)–(8) if

$$\int_\Omega \left(\left(A(x)\, \nabla y, \nabla \varphi \right)_{\mathbb{R}^N} + y\varphi \right) \rho \, dx = \int_\Omega f\varphi \, dx \quad \forall \, \varphi \in V. \tag{10}$$

or the energy equality

$$\int_\Omega \left(\left(A(x)\, \nabla y, \nabla y \right)_{\mathbb{R}^N} + y^2 \right) \rho \, dx = \int_\Omega fy \, dx \tag{11}$$

holds true.

Remark 2

For a "typical" degenerate weight function ρ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in W, and hence there is no uniqueness of the weak solutions.

Hence for the given control object we have the continuum of the different statements of the original OCP, namely

$$\left\{ \left\langle \inf_{(A,y)\in\, \Xi^W} I(A,y) \right\rangle, \left\langle \inf_{(A,y)\in\, \Xi_V} I(A,y) \right\rangle,\ H \subseteq V \subseteq W \right\}, \tag{12}$$

where

$$\Xi_V = \{(A,y) \in U_{ad} \times V \mid y \in V, (A,y) \text{ are related by (11)}\}, \tag{13}$$

$$\Xi^W = \{(A,y) \in U_{ad} \times W \mid y \in W, (A,y) \text{ satisfy (9)} \ \ \forall \varphi \in C_0^\infty(\Omega)\}. \tag{14}$$

Definition of Optimal Solutions

Definition 3

We say that a pair $(A^0, y^0) \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W(\Omega, \rho\, dx)$ is a weak optimal (H- and W-optimal, resp.) solution to the problem (6)–(8), if (A^0, y^0) is a minimizer for $\left\langle \inf_{(A,y)\in\,\Xi^w} I(A,y)\right\rangle$ (for $\left\langle \inf_{(A,y)\in\,\Xi_H} I(A,y)\right\rangle$, and for $\left\langle \inf_{(A,y)\in\,\Xi_W} I(A,y)\right\rangle$, resp.), i.e.,

$$(A^0, y^0) \in \Xi^w \quad \text{and} \quad I(A^0, y^0) = \inf_{(A,y)\in\,\Xi^w} I(A,y),$$

$$(A^0, y^0) \in \Xi_H \quad \text{and} \quad I(A^0, y^0) = \inf_{(A,y)\in\,\Xi_H} I(A,y),$$

$$(A^0, y^0) \in \Xi_W \quad \text{and} \quad I(A^0, y^0) = \inf_{(A,y)\in\,\Xi_W} I(A,y).$$

For any V $(H \subseteq V \subseteq W)$ we have $\emptyset \neq \Xi_V \subseteq \Xi^w$. Hence

$$\inf_{(A,y)\in\,\Xi^w} I(A,y) \leq \inf_{(A,y)\in\,\Xi_V} I(A,y) \;\forall\, V \text{ s.t. } H \subseteq V \subseteq W. \tag{15}$$

Proposition 1

Assume that $H \neq W$, and variational problems

$$\left\langle \inf_{(A,y)\in\,\Xi_V} I(A,y) \right\rangle \quad \text{and} \quad \left\langle \inf_{(A,y)\in\,\Xi^w} I(A,y) \right\rangle \tag{16}$$

are solvable for any $f \in C_0^\infty(\mathbb{R}^N)$ and $y_d \in L^2(\Omega, \rho\, dx)$. Then there are a constant $\zeta^* > 0$ and functions $f^* \in C_0^\infty(\mathbb{R}^N)$, $y_d^* \in L^2(\Omega, \rho\, dx)$ such that the corresponding optimal solutions to (16) are different and $\inf_{(A,y)\in\,\Xi^w} I(A,y) < \inf_{(A,y)\in\,\Xi_V} I(A,y)$.

Definition 4

We say that a sequence $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of a weight weight function ρ if:

$$\rho^\varepsilon + (\rho^\varepsilon)^{-1} \in L^\infty(\Omega), \quad \forall \varepsilon > 0, \tag{17}$$

$$\rho^\varepsilon \to \rho, \quad (\rho^\varepsilon)^{-1} \to \rho^{-1} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{18}$$

Definition 5

A bounded sequence $\left\{v_\varepsilon \in L^2(\Omega, \rho^\varepsilon dx)\right\}$ converges weakly to $v \in L^2(\Omega, \rho\, dx)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon \varphi\, \rho^\varepsilon dx = \int_\Omega v\varphi\, \rho\, dx \text{ for any } \varphi \in C_0^\infty(\Omega),$$

and it is written as $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \rho^\varepsilon dx)$.

Compensated Compactness Lemma in Variable Spaces

Lemma 1

Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) \geq 0$. Let $\left\{\vec{f}_\varepsilon \in L^2(\Omega, \rho^\varepsilon dx)^N\right\}_{\varepsilon>0}$ and $\{g_\varepsilon \in W^{1,2}(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ be such that

1. $\left\{\vec{f}_\varepsilon\right\}_{\varepsilon>0}$ is bounded in the variable space

$$X(\Omega, \rho^\varepsilon dx) = \left\{\vec{f} \in L^2(\Omega, \rho^\varepsilon dx)^N \mid \mathrm{div}_{\rho^\varepsilon} \vec{f} \in L^2(\Omega, \rho^\varepsilon dx)\right\},$$

   and $\vec{f}_\varepsilon \rightharpoonup \vec{f}$ in $L^2(\Omega, \rho^\varepsilon dx)^N$;

2. $g_\varepsilon \rightharpoonup g$ in $L^2(\Omega, \rho^\varepsilon dx)$, and $\nabla g_\varepsilon \rightharpoonup \nabla g$ in $L^2(\Omega, \rho^\varepsilon dx)^N$.

Then $\lim_{\varepsilon\to 0} \int_\Omega \varphi \left(\vec{f}_\varepsilon, \nabla g_\varepsilon\right)_{\mathbb{R}^N} \rho^\varepsilon dx = \int_\Omega \varphi \left(\vec{f}, \nabla g\right)_{\mathbb{R}^N} \rho\, dx$, $\forall \varphi \in C_0^\infty(\Omega)$.

Remark 3

In the previous Lemma the supposition "let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) \geq 0$" can be replaced by the following one: let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence with properties:

1. $\rho^\varepsilon(x) \geq 0$, $\forall\ \varepsilon > 0$;
2. $\rho^\varepsilon \to \rho$, $(\rho^\varepsilon)^{-1} \to \rho^{-1}$ in $L^1(\Omega)$ as $\varepsilon \to 0$;
3. for every $\varepsilon > 0$ the subspace $C_0^\infty(\Omega)$ is dense in $W(\Omega, \rho^\varepsilon dx)$ with respect to the norm $\|\cdot\|_{\rho^\varepsilon}$.

Existence Theorem for H-optimal solutions

Theorem 1

Let $\rho(x) \geq 0$ be a degenerate weight function. Then the optimal control problem

$$I(A,y) = \zeta \int_\Omega |y(x) - y_d(x)|^2 \rho\, dx + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} \rho\, dx \to \inf, \tag{19}$$

$$A \in U_{ad}, \quad y \in H(\Omega, \rho\, dx), \tag{20}$$

$$\int_\Omega \left(\left(A(x)\, \nabla y, \nabla \varphi \right)_{\mathbb{R}^N} + y\varphi \right) \rho\, dx = \int_\Omega f\varphi\, dx \;\forall\, \varphi \in H(\Omega, \rho\, dx) \tag{21}$$

admits at least one H-solution

$$(A^{opt}, y^{opt}) \in \Xi_H \subset L^\infty(\Omega; \mathbb{R}^{N\times N}) \times H(\Omega, \rho\, dx)$$

for every $f \in L^2(\Omega, \rho^{-1} dx)$.

Perturbation Approach to the Existence of W-Optimal Solutions

Let $\left\{\rho^\varepsilon = \left((\rho^{-1})_\varepsilon\right)^{-1}\right\}_{\varepsilon>0}$ be an "inverse" smoothing of a degenerate weight function $\rho(x) \geq 0$, i.e. $\rho^\varepsilon = \left((\rho^{-1})_\varepsilon\right)^{-1}$ $\forall\, \varepsilon > 0$, where $(\rho^{-1})_\varepsilon(x) = \int_{\mathbb{R}^N} K(z)\rho^{-1}(x+\varepsilon z)\, dz$.
We introduce the following collection of perturbed optimal control problems in coefficients for non-degenerate elliptic equations:

$$I_\varepsilon(A, y) = \zeta \int_\Omega |y(x) - y_d(x)|^2 \rho^\varepsilon dx + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} \rho^\varepsilon dx \to \inf, \tag{22}$$

$$A \in U^\varepsilon_{ad}, \quad y \in W^{1,2}_0(\Omega, \rho^\varepsilon dx), \tag{23}$$

$$-\operatorname{div}\left(\rho^\varepsilon A(x)\nabla y\right) + \rho^\varepsilon y = f \quad \text{in} \quad \Omega, \tag{24}$$

$$U^\varepsilon_{ad} = \left\{ A = [\vec{a}_1, \dots, \vec{a}_N] \in M^\beta_\alpha(\Omega) \,\middle|\, \left|\operatorname{div}_{\rho^\varepsilon} \vec{a}_i\right| \leq \gamma_i \;\forall\, i = 1, \dots, N \right\}, \tag{25}$$

where the elements $f \in C^\infty_0(\mathbb{R}^N)$, $\zeta > 0$, and $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ are the same as it was for the original problem (6)–(8), (4).

Existence Theorem for W-optimal solutions

Theorem 2

The minimization problem $\left\langle \inf_{(A,y)\in \Xi_W} I(A,y) \right\rangle$ is the weak variational limit of the sequence (22)–(25) with respect to the weak convergence in the variable space $\mathbb{Y}(\Omega,\rho^\varepsilon dx) = L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,2}(\Omega,\rho^\varepsilon dx)$.

Theorem 3

Let $\left\{(A_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal pairs to the perturbed problems (22)–(25). Then

$$A_\varepsilon^0 \overset{*}{\rightharpoonup} A^0 \quad \text{in} \quad L^\infty(\Omega;\mathbb{R}^{N\times N}); \tag{26}$$

$$y_\varepsilon^0 \to y^0 \quad \text{in} \quad L^2(\Omega,\rho^\varepsilon dx), \quad \nabla y_\varepsilon^0 \to \nabla y^0 \quad \text{in} \quad L^2(\Omega,\rho^\varepsilon dx)^N, \tag{27}$$

$$\inf_{(A,y)\in \Xi_W} I(A,y) = I\left(A^0, y^0\right) = \lim_{\varepsilon\to 0} \inf_{(A_\varepsilon, y_\varepsilon)\in \Xi_\varepsilon} I_\varepsilon(A_\varepsilon, y_\varepsilon). \tag{28}$$

where $(A^0, y^0) \in \Xi_W$ is a W-optimal solution to the original problem.

Intermediate Result

Representation Formula

Let $\left\{\rho^{\varepsilon} = \left((\rho^{-1})_{\varepsilon}\right)^{-1}\right\}_{\varepsilon>0}$ be an "inverse" smoothing of a degenerate weight function $\rho(x) \geq 0$, and let $(\vec{a}_k)_{\varepsilon}$ be a "direct" smoothing of k-th column of A, i.e.

$$(\vec{a}_k)_{\varepsilon}(x) = \int_{\mathbb{R}^N} K(z)\vec{a}_k(x + \varepsilon z)\, dz.$$

Then the smoothness of the functions $(\vec{a}_k)_{\varepsilon}$ and ρ^{ε} produces the following representation formula for elements $\mathrm{div}_{\rho^{\varepsilon}}(\vec{a}_k)_{\varepsilon} \in L^2(\Omega, \rho^{\varepsilon} dx)$

$$\mathrm{div}_{\rho^{\varepsilon}}(\vec{a}_k)_{\varepsilon} = (\rho^{\varepsilon})^{-1}\,\mathrm{div}\,\left(\rho^{\varepsilon}(\vec{a}_k)_{\varepsilon}\right) \quad \forall\, k = 1, \ldots, N, \quad \forall\, \varepsilon > 0,$$

where the element $\mathrm{div}\,\left(\rho^{\varepsilon}(\vec{a}_k)_{\varepsilon}\right)$ is defined in the sense of distributions.

Remark 5

The existence of the weak optimal controls in coefficients to the degenerate elliptic equation

$$I(A,y) = \zeta \int_\Omega |y(x) - y_d(x)|^2 \rho\, dx + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} \rho\, dx \to \inf, \tag{29}$$

$$A \in U_{ad}, \quad y \in W(\Omega, \rho\, dx), \tag{30}$$

$$\int_\Omega \Big(\Big(A(x)\, \nabla y, \nabla \varphi \Big)_{\mathbb{R}^N} + y\varphi \Big) \rho\, dx = \int_\Omega f\varphi\, dx \;\forall\, \varphi \in C_0^\infty(\Omega) \tag{31}$$

has not been considered in the literature.
The main reasons:

1. there is no appropriate a priori estimates for the weak solutions of boundary value problem (30)– (31) in $\|\cdot\|_\rho$-norm;
2. the main topological properties of the set of weak admissible solutions Ξ^w such as closedness, compactness, and etc, are unknown in general.

Proposition 2

Let $(A_w^0, y_w^0) \in \Xi^w$ be a weak (but not variational) optimal solution to the problem (29)–(31). Assume that a subset

$$E = \Big\{ x \in \Omega \;\Big|\; [\vec{a}^{\,0}_{w\,1}, \dots, \vec{a}^{\,0}_{w\,N}] =: A_w^0(x) > \alpha I, \quad \left(A_w^0(x)\right)^{-1} > \beta^{-1} I, \quad \left|\operatorname{div}_\rho \vec{a}^{\,0}_{w\,i}(x)\right| < \gamma_i \Big\} \tag{32}$$

has a nonzero Lebesgue measure, and there exists a matrix $A^* \in U_{ad}$ such that

$$\int_\Omega \left(A^*(x)\, \nabla y_w^0, \nabla \varphi \right)_{\mathbb{R}^N} \rho\, dx = 0 \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{33}$$

Then there is a continuum of the weak optimal solutions to the problem (29)–(31), namely, $(A_w^0(x) + \theta \chi_E(x) A^*(x), y_w^0)$ is a weak optimal solution for all $|\theta|$ small enough. Here χ_E denotes the characteristic function of E.

Illustration of the Specific of Weak Optimal Controls in Coefficients

Proposition 3

Assume that $\left\langle \inf_{(A,y)\in\,\Xi^w} I(A,y)\right\rangle$ is a variational weak limit of the sequence of optimal control problems

$$I_\varepsilon(A,y) = \zeta\int_\Omega |y(x)-y_d(x)|^2\rho^\varepsilon\,dx + \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N}\rho^\varepsilon\,dx \to \inf, \qquad (34)$$

$$A\in U^\varepsilon_{ad},\quad y\in W^{1,2}_0(\Omega,\rho^\varepsilon\,dx), \qquad (35)$$

$$-\operatorname{div}\left(\rho^\varepsilon A(x)\nabla y\right)+\rho^\varepsilon y = f \quad \text{in}\quad \Omega, \qquad (36)$$

$$U^\varepsilon_{ad} = \left\{ A=[\vec{a}_1,\dots,\vec{a}_N]\in M^\beta_\alpha(\Omega)\,\middle|\ \left|\operatorname{div}_{\rho^\varepsilon}\vec{a}_i\right|\le\gamma_i\ \forall\, i=1,\dots,N\right\}, \qquad (37)$$

under some non-degenerate perturbation $\{\rho^\varepsilon\}_{\varepsilon>0}$ of a weight function $\rho(x)\ge 0$. Then none of a weak optimal solution $(A^0_w, y^0_w)\in\Xi^w$ (which is not a variational one) can be attained via the optimal solutions to the perturbed problems (34)–(37).

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