Local solvability beyond condition (Ψ)

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Joint works

- F.C.-S.Spagnolo ('89)

- F.C.-L.Pernazza-F.Treves ('03)

- F.C.-P.Cordaro-L.Pernazza (in progress)

Let

$$P = \sum_{|\alpha| \le m} a_\alpha(y) \partial_y^\alpha, \qquad y \in \Omega \subset \mathbb{R}^n$$

P (pseudo)differential operator **locally solvable**:

$$\forall y_0 \in \Omega, \quad \exists V \text{ neighbourhood of } y_0 \text{ such that}$$

$$\forall f \in \mathcal{D}(V) \quad \exists u \in \mathcal{D}'(V) \text{ verifying}$$

$$Pu = f \text{ in } V$$

Remark. Otherwise $f \in H^s$, $u \in H^{s'}$, $s, s' \in \mathbb{R}$

Theorem. *(Nirenberg-Treves, Beals-Fefferman, Moyer, Hörmander, Lerner, Dencker)*

- P *differential* **loc.solv.** $\Leftrightarrow$ P *verifies condition* (P)

- P *pseudodifferential* **loc.solv.** $\Leftrightarrow$ P *verifies condition* (Ψ)

$$p_m = \sum_{|\alpha|=m} a_\alpha(y)\xi^\alpha \quad \text{principal symbol of } P$$

Bicharacteristics of $\Re p_m$

$$\begin{cases} \dfrac{dy}{ds} &= \nabla_\xi \Re p_m(y,\xi) \\[2ex] \dfrac{d\xi}{ds} &= -\nabla_y \Re p_m(y,\xi) \end{cases}$$

(P) on every null-bicharacteristic of $\Re p_m$, $\Im p_m$ does not change sign

(Ψ) on every null-bicharacteristic of $\Re p_m$, $\Im p_m$ does not change sign from $-$ to $+$

Remark. P differential $\Rightarrow$ (P) equivalent to (Ψ)

In Theorem "**loc.solv.** $\Leftrightarrow (P)$" we need 2 conditions:

$A)$ some regularity of the coefficients

$B)$ P operator of principal type (i.e. if $p_m(y_0, \xi_0) = 0$ for some $\xi_0 \neq 0 \Rightarrow \nabla_\xi p_m(y_0, \xi_0) \neq 0$)

We will consider some cases of operators violating condition $A)$ or $B)$

Not A)

- (F.C.-S.Spagnolo, $'89$)

Consider the strictly hyperbolic operator

$$P = \partial_t^2 - \partial_x(A(t,x)\partial_x)$$

with

$$0 < \lambda^{-1} \leq A(t,x) \leq \lambda$$

If $|\partial_t A(t,x)| \leq M$, it is well known that the Cauchy Problem

$$\begin{cases} \quad Pu = f \\ \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x) \end{cases} \tag{CP}$$

is uniquely solvable, that is

$$\forall \ u_0 \in H^1, \ u_1 \in H^0, \ f \in C([0,T], H^0)$$

$$\exists! \ u \text{ solution of (CP)} \Rightarrow \ P \text{ loc. solv.}$$

A less regular (F.C.-E.De Giorgi-S.Spagnolo, '79)

$$A(t,x) \equiv a(t)$$

$$|a(t+\tau) - a(t)| \leq c|\tau||\log|\tau||, \quad |\tau| \leq 1/2$$

$\Rightarrow$ (CP) well posed in H^∞, that is:

$$\forall\ u_0 \in H^{s+1}, u_1 \in H^s \quad \text{(for simplicity } f \equiv 0\text{)}$$

$\exists!\ u$ solution of (CP) such that for some $\beta > 0$ and any s

$$\begin{aligned} \|u(t,\cdot)\|_{H^{s+1-\beta t}} &+ \|u_t(t,\cdot)\|_{H^{s-\beta t}} \\ &\leq \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} \end{aligned} \qquad \text{(EE)}$$

that is we have a **loss of derivatives**

- (F.C.-N.Lerner, '95)

Analagous result for the case

$$A(t,x) \in \mathsf{LogLip}([0,T] \times \mathbb{R}^d_x)$$

(EE) true for $|s| \leq 1$, $s - \beta t > -1$

- (F.C.-G.Métivier, '08)

Local result: $A(t,x)$ defined in $\Omega \subset \mathbb{R}_t \times \mathbb{R}^d_x$ open, $A \in$ LogLip $\Rightarrow$ (CP) locally well posed with loss of derivatives. In particular, for $A(t,x) \in$ LogLip, P is locally solvable.

It is possible to show that LogLip is the minimal possible regularity in order to have (CP) well posed:

$$\exists A(t) \in \bigcap_{\alpha<1} C^{0,\alpha}, \quad \lambda^{-1} \leq A \leq \lambda, \quad \exists u_0, u_1 \in H^\infty$$

such that (CP) has no distributional solution in $[0,T]\times\mathcal{A}$ $\forall T > 0$, $\forall\mathcal{A}$ open with $0 \in \mathcal{A}$.

More precisely $\forall\omega(\tau)$ with $\omega(\tau) \overset{\tau\to 0^+}{\longrightarrow} +\infty$, there exists $A(t) \in \omega -$ LogLip:

$$|A(t+\tau) - A(t)| \leq C|\tau||\log|\tau||\omega(|\tau|), \quad |\tau| \leq 1/2$$

such that (CP) has no solution.

Question: local solvability for $\partial_t^2 - \partial_x(A(t,x)\partial_x)$?

Evidently if $A = A(t)$ or if $A = A(x)$ we have local solvability.

But for $A(t,x)$, with $\lambda^{-1} \leq A(t,x) \leq \lambda$?

Theorem. *There exists* $a(t)$ *with* $0 < \lambda^{-1} \le a \le \lambda$

$$a \in \bigcap_{\alpha<1} C^{0,\alpha}(\mathbb{R}), \quad a \in C^{\infty}(\mathbb{R} \setminus \{0\})$$

such that the equation

$$\left(\partial_t^2 - \partial_x \left(\frac{a(t)}{a(x)} \partial_x\right)\right) u = x, \tag{1}$$

has no solution in any neighbourhood Ω *of* 0*:*

$\nexists$ *solution* $u \in C^1(\Omega)$

$\nexists$ $u \in \mathcal{D}'(\Omega)$ *solution in* $\Omega \cap \{t \neq 0\} \cap \{x \neq 0\}$

More precisely, for any $\omega(\tau)$ *such that* $\omega(\tau) \overset{\tau \to 0^+}{\longrightarrow} +\infty$*, there exists* $a \in \omega-\text{LogLip}$ *such that* (1) *has no solution.*

Idea of the construction (the case $a \in \bigcap_{\alpha<1} C^{0,\alpha}$)

Let w_ϵ the solution of

$$\begin{cases} w_\epsilon''(\tau) + \alpha_\epsilon w_\epsilon(\tau) = 0 \\ w_\epsilon(0) = 1 \quad w_\epsilon'(0) = 0 \end{cases}$$

where

$$\alpha_\epsilon(\tau) \simeq 1 - \epsilon \sin 2\tau + \epsilon^2 \sin^2 \tau$$

Then $|\alpha_\epsilon - 1| \le M\epsilon$, $|\alpha_\epsilon'(\tau)| \le M\epsilon$,

$$w_\epsilon(\tau) = p_\epsilon(\tau) e^{-\epsilon|\tau|}$$

for some p_ϵ 2π-periodic on $\{\tau > 0\}$ and on $\{\tau < 0\}$

Moreover

$$|w_\epsilon| + |w'_\epsilon| + |w''_\epsilon| \leq c$$

$$\int_0^{2\pi} w_\epsilon \, d\tau \geq \gamma\epsilon \qquad (\gamma > 0)$$

As a consequence we have, for $\tau = \pm 2\pi\nu$, $\nu \in \mathbb{N}$,

$$w_\epsilon(\tau) = e^{-\epsilon|\tau|}, \quad w'_\epsilon(\tau) = 0, \quad w''_\epsilon(\tau) = e^{-\epsilon|\tau|}$$

M, c, γ constants independent on ϵ.

w_ϵ **exponentially decreasing**

Now we define the function $a(t)$

Let us consider the sequences

$$\rho_k = 4\pi 2^{-k}, \quad h_k = 2^{2^{Nk}}, \quad \epsilon_k = h_k^{-1}(\log h_k)^3$$

N so large that, for any k, we have

$$\epsilon_k \leq \frac{1}{2M}$$

$$4M \sum_{j=1}^{k-1} \epsilon_j h_j \rho_j \leq \epsilon_k h_k \rho_k$$

$$2M \sum_{j=k+1}^{\infty} \epsilon_j \rho_j \leq \epsilon_k \rho_k$$

Now let us define

$$t_k = \frac{\rho_k}{2} + \sum_{j=k+1}^{\infty} \rho_j$$

$$I_k = \left[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2} \right]$$

so I_k and I_{k+1} are contiguous, and

$$I_k \longrightarrow \{0\} \text{ for } k \to \infty$$

Finally let $a(t)$ be defined

$$a(t) = \begin{cases} \alpha_{\epsilon_k}(h_k(t - t_k)) & t \in I_k \\ \\ 1 & t \in \mathbb{R} \setminus \bigcup_{k=1}^{\infty} I_k \end{cases}$$

It is easy to see that $a \in C^{0,\alpha}$, $\forall \alpha < 1$

Let us pose

$$A(t,x) = \frac{a(t)}{a(x)}, \quad \psi_k(t) = w_{\epsilon_k}(h_k(t - t_k))$$

so that

$$\psi_k'' + h_k^2 a(t)\psi_k = 0 \qquad \text{in } I_k$$

and so

$$\psi_k(t_k \pm \frac{\rho_k}{2}) = e^{-\epsilon_k \rho_k h_{k/2}}, \qquad \psi_k'(t_k \pm \frac{\rho_k}{2}) = 0$$

$$\int_{I_k} \psi_k(t)\, dt \geq 2\gamma\epsilon_k h_k^{-1}$$

and finally

$$v_k(t,x) = \psi_k(t)\psi_k'(x)$$

Then

$$(v_k)_{tt} - (A(t,x)(v_k)_x)_x = 0 \quad \text{on } Q_k = I_k \times I_k \qquad (\star)$$

Let be now $u(t,x) \in C^1(W)$ solution of

$$u_{tt} - (A(t,x)u_x)_x = x \qquad (\star\star)$$

in a neighbourhood W of $(0,0)$.

By pairing $(\star)$ and $(\star\star)$ we obtain, for k large,

$$\int_{\partial Q_k} [(u_t v_k - u(v_k)_t)\nu_t - A(t,x)(u_x v_k - u(v_k)_x)\nu_x]\, d\sigma$$
$$= \iint_{Q_k} x v_k \, dt dx$$
$$(\square)$$

where (ν_t, ν_x) is the exterior normal to $\partial(I_k \times I_k) = \partial Q_k$ and $d\sigma$ the one-dimensional measure.

But (□) becomes false for k large enough.

We have indeed

$$|v_k| + |(v_k)_t| + |(v_k)_x| \leq c h_k^2 e^{-\epsilon_k h_k \rho_{k/2}} \quad \text{on } \partial Q_k$$

Introducing this estimate in (□), we obtain

$$\left| \iint_{Q_k} x v_k \, dt dx \right| \leq c h_k^2 e^{-\epsilon_k h_k \rho_{k/2}}$$

On the other hand

$$\iint_{Q_k} x v_k \, dt dx = - \iint_{Q_k} \psi_k(t) \psi_k(x) \, dt dx + \int_{\partial Q_k} x \psi_k(t) \psi_k(x) \, d\sigma$$

Now

$$\iint_{Q_k} \psi_k(t)\psi_k(x)\,dtdx = \left(\int_{I_k} \psi_k(s)\,ds\right)^2 \geq 4\gamma^2\epsilon_k^2 h_k^{-2}$$

while

$$\left|\int_{\partial Q_k} x\psi_k(t)\psi_k(x)\,d\sigma\right| \leq ce^{-\epsilon_k h_k \rho_{k/2}}.$$

In conclusion we get

$$\gamma^2\epsilon_k^2 h_k^{-2} \leq ch_k^2 e^{-\epsilon_k h_k \rho_{k/2}}, \quad \text{false for } k \to \infty$$

Not B)

- (F.C.-L.Pernazza-F.Treves, '03)

- (F.C.-P.Cordaro-L.Pernazza, in progress)

We will consider

$$Lu = \partial_t u - \sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}\partial_{x_j}u) - \sum_{j=1}^{d} b_j \partial_{x_j}u - cu = f \quad (\star)$$

a_{ij}, b_j, c smooth functions in $\Omega \subset \mathbb{R}^{d+1}$

a_{ij} real valued

($\star$) may be not locally solvable:

- Kannai '71: $L = \partial_t + t \sum_{j=1}^{d} \partial_{x_j}^2$

- Similarly: $L = \partial_t + t^{2k+1} \sum_{j=1}^{d} \partial_{x_j}^2$

The main point here is that the symbol $A(t, \xi) = t|\xi|^2$ (respectively $= t^{2k+1}|\xi|^2$) changes sign from $-$ to $+$ with t.

This simple observation would lead one to believe that the key resides in the condition (Ψ) (Nirenberg-Treves)

For $(\star)$ the condition (Ψ) could be taken

$$(\tilde{\Psi}) \begin{cases} \nexists \xi \in \mathbb{R}^d \text{ s. t. the real function} \\ \sum a_{ij}(t,x)\xi_i\xi_j \text{ change sign from } - \text{ to } + \\ \text{along the integral curves of the vector field} \\ X = \partial_t - \sum_{j=1}^{d} \Re b_j(t,x)\partial_{x_j} \end{cases}$$

Remark. Property $(\tilde{\Psi})$, as stated here, is not invariant.

The necessity of $(\tilde{\Psi})$ is given credence by the following weakly hyperbolic example, due to N.Lerner-K.Pravda Starov:

$$P_k = \partial_t^2 - \alpha_k(x_2)\partial_{x_1}^2 + \partial_{x_2}, \quad k = 1, 2, \ldots$$

where, for any k, $\alpha_k(x_2)$ is a decreasing C^k function with

$$\alpha_k(x_2) > 0 \text{ for } x_2 < 0, \quad \alpha_k(x_2) \equiv 0 \text{ for } x_2 \geq 0$$

They show directly P_k not locally solvable near 0.

But this is not a good condition: there are examples not verifying $(\tilde{\Psi})$, but locally solvable:

$$P = \partial_t + (\partial_{x_1}\partial_{x_2} + t\partial_{x_2}^2) \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^2$$

Here the symbol of A is $\xi_2(\xi_1 + t\xi_2)$: $(\tilde{\Psi})$ is not verified. But $x_2 - tx_1 \mapsto x_2$ transforms the operator P in

$$\partial_t - x_1\partial_{x_2} + \partial_{x_1}\partial_{x_2}$$

Then

$$e^{-x_1^2/2}P(e^{x_1^2/2}u) = (\partial_t + \partial_{x_1}\partial_{x_2})u$$

costant coefficients!

Evolution operators for which condition $(\tilde{\Psi})$ determines solvability

$$L = \partial_t + \epsilon t^\ell \sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}(t,x)\partial_{x_j}) + \sum_{j=1}^{d} b_j(t,x)\partial_{x_j} + c(t,x)$$

where $\epsilon = \pm 1$, $Q(t,x,\xi,\eta) = \sum a_{ij}\xi_i\eta_j$ positive definite for $(t,x) = (0,0)$

Theorem $(*)$.

$$\begin{cases} \ell \text{ even } \Rightarrow L \text{ locally solvable} \\ \ell \text{ odd } \Rightarrow (L \text{ loc. solv. } \Longleftrightarrow \epsilon = -1) \end{cases}$$

Sketch of the proof

If $\ell = 2r$ we consider

$$\Re(-L^* u, e^{-2\lambda t} u) = \lambda ||e^{-\lambda t} u||^2 - \epsilon \int_\Omega t^{2r} Q(\nabla_x u, e^{-2\lambda t} \nabla_x \overline{u})\, dx dt$$
$$+ \Re \int_\Omega e^{-2\lambda t} (Xu) \overline{u}\, dx dt - \frac{1}{2} \int_\Omega c' e^{-2\lambda t} |u|^2\, dx dt$$

where $c' = c - \sum \partial_{x_j} b_j$, $X = \sum b_j \partial_{x_j}$

Then for $|\lambda|$ large:

$$|\Re(L^* u, e^{-2\lambda t} u)| \geq C(||e^{-\lambda t} u||^2 + ||t^r e^{-\lambda t} \nabla_x u||^2)$$

$$\Downarrow$$

local solvability

If $\ell = 2r + 1$, $\epsilon = -1$, then

$$\begin{aligned} 2\Re(-L^* u, tu) &\geq ||u||^2 + 2||t^{r+1}\nabla_x u||^2 \\ &\quad - |(Xu, tu) + (tu, Xu) + 2\Re(c' u, tu)| \\ &\geq ||u||^2 - M\,||\sqrt{|t|}\,u||^2 \end{aligned}$$

where $c' = c - \sum \partial_{x_j} b_j$, $X = \sum b_j \partial_{x_j}$, as before.

Hence, again, we have local solvability.

Finally, if $\ell = 2r + 1$ and $\epsilon = 1$, one argues by contradiction and shows that Hörmander's inequality cannot hold (eikonal equation, transport equations,...)

Invariant formulation of Theorem $(*)$

Consider real smooth operators near $0 \in \mathbb{R}^n$ of the form:

$$Q = -\sum_{j,k=1}^{n} \frac{\partial}{\partial y_j}\left(\varphi(y)^{\ell} a_{jk}(y) \frac{\partial}{\partial y_k}\right) - \sum_{j=1}^{n} b_j(y) \frac{\partial}{\partial y_j} + c(y)$$

where $\ell \in \mathbb{N}$, and

(i) $\varphi(0) = 0$, $d\varphi(0) \neq 0$ on $\varphi^{-1}(0)$

(ii) $\xi \mapsto A(y)(\xi) = \sum a_{jk}(y)\xi_j\xi_k \geq 0$, $\forall y \in \Omega$

(iii) rank $A(0) = n - 1$

(iv) $A(y)(d\varphi) = 0$, $\forall y \in \Omega$

(v) $\theta := \left(\sum_k b_k \frac{\partial \varphi}{\partial y_k} \right)(0) \neq 0$

Thanks to (i)-(iv) the sign of θ is invariantly defined.

If we choose coordinates $(y_1, \ldots, y_n)$ such that $\varphi = y_n$ then (i)-(iv) $\Rightarrow a_{jn} = a_{nj} = 0,\ j = 1, \ldots, n$

$$\Rightarrow \xi \mapsto \sum_{j,k=1}^{n-1} a_{jk}(y) \xi_j \xi_k \text{ positive definite}$$

Theorem $(*)$ becomes:

$$\begin{cases} \quad \ell \text{ even } \Rightarrow Q \text{ locally solvable} \\ \\ \ell \text{ odd } \Rightarrow (Q \text{ loc. solv. } \Longleftrightarrow \theta < 0) \end{cases}$$

For operators like Q we have then

$$Q \text{ locally solvable } \Longleftrightarrow (\Psi')$$

where

$$(\Psi') : \quad \left(\sum_{j=1}^{n} b_j(y) \frac{\partial}{\partial y_j} \right) \operatorname{sgn}(\varphi^{\ell}) \leq 0 \quad \text{as a measure}$$

$$\operatorname{sgn}(\tau) = \begin{cases} 1 & \tau > 0 \\ -1 & \tau < 0 \\ 0 & \tau = 0 \end{cases}$$

More generally, let Y be a C^1 real vector field, $a(y)$ a real analytic function ($a \not\equiv 0$). Then, if we define

$$\mu[Y; a] := Y(\mathrm{sgn}(a))$$

μ can be extended to a real Radon measure.

Moreover

$$a \text{ does not change sign } \Rightarrow \mu[Y; a] = 0$$

$$\mathrm{supp}\ \mu[Y; a] \subset V := \{y : a(y) = 0\}$$

Consider now operators P given by

$$P = X^* a X - Y + g$$

- X and Y real-valued, real analytic vector fields, defined in Ω, neighbourhood of the origin in $\mathbb{R}^n_y$

- $a(0) = 0, \quad Y \neq 0$ in Ω

Now

$$A(y)(\xi) = -\sigma_X(y,\xi)^2 \quad (\sigma_X \text{ purely imaginary})$$

and so (ii) satisfied

Definition. P satisfies (Ψ') if

$$\mu[Y; a] \leq 0$$

Remark. (Ψ') is invariant under real analytic changes of variables and under multiplication of P by a real analytic factor.

Let now be

$$V_0 := \text{ closure of } \{y \in V : a \text{ changes sign near } y\}$$

V_0 is a semianalytic subset of Ω.

$\dim(V_0) = n-1$ when a changes sign (otherwise $V_0 = \emptyset$)

Theorem (•).

$$\left.\begin{array}{r} P \textit{ satisfies } (\Psi') \\[2em] Y \textit{ transversal to } V_0 \end{array}\right\} \Rightarrow P \textit{ locally solvable}$$

Corollary.

$$\operatorname{sgn}(a) \textit{ constant } \Rightarrow P \textit{ locally solvable}$$

A partial converse of **Theorem** ($\bullet$):

Theorem. *Let us consider again*

$$P = X^*aX - Y + g$$

such that

$$\left.\begin{array}{ll} 1) & a^{-1}(0) \textit{ hypersurface} \\[2em] 2) & X \textit{ tangent to } V_0 \\[2em] 3) & (\Psi') \textit{ not satisfied} \end{array}\right\} \Rightarrow P \textit{ not locally solvable}$$

Example. Let us consider the operator

$$P = \epsilon(\partial_x + t\partial_t)^* t^3 \alpha(t,x)(\partial_x + t\partial_t) + (\partial_t + \beta(t,x)\partial_x) + g$$

with $(t,x) \in \mathbb{R} \times \mathbb{R}$, $\alpha(t,x) > 0$, $\epsilon = \pm 1$. Then

$$P \textbf{ locally solvable} \iff \epsilon = 1$$

Solvability when (Ψ') is not necessarily satisfied

Theorem.

$$Xa(0) \neq 0 \Rightarrow P \textit{ locally solvable}$$

The key ingredient in the proof is the *method of concatenations* (Gilioli-Treves)

Question:

$$X \text{ transverse to } V \text{ near the origin}$$
$$\Downarrow ?$$
$$P \text{ locally solvable}$$

Answer negative:

$$P_\alpha = \partial_t^* t^3 \partial_t - \partial_t + \alpha \partial_x$$

with $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\alpha \in \mathbb{R}$. Indeed

$$P_\alpha \text{ **locally solvable** } \Longleftrightarrow \alpha = 0$$