Data assimilation for a large scale ocean circulation model

Jean-Pierre Puel

Almost Retired, Emeritus Professor

Laboratoire de Mathématiques de Versailles,
Université de Versailles Saint-Quentin,
45 avenue des Etats Unis, 78035 Versailles Cedex, France
jppuel@math.uvsq.fr

Work in collaboration with Galina Garcia and Axel Osses

Benasque, August 24., 2009

Outline

1. Linear quasi-geostrophic ocean model
   - Velocity-Pressure formulation
   - Stream Function formulation
   - Existence results. Regularity
2. Data Assimilation. Theory
   - Preliminaries
   - Data assimilation method
   - Approximation by Optimal Control Problems
3. Numerical computations
   - Discretization
   - First series of numerical tests
   - Second series of numerical tests

Linear quasi-geostrophic ocean model

We use the so-called β-plane approximation, with $\beta = 2\Omega_0 R^{-1} \cos\tilde{\theta}_0$, where Ω_0 and R are the angular velocity and radius of the Earth, respectively, and $\tilde{\theta}_0$ a reference latitude.

$$
\begin{cases}
u_t - A_H \Delta u + \gamma u + (f_0 + \beta x_2)\,\mathrm{k} \wedge u + \dfrac{1}{\rho_0} \nabla p = \mathcal{T} \quad \text{in } \Omega \times (0, T), \\
\operatorname{div} u = 0 \quad \text{in } \Omega \times (0, T), \\
u = 0 \quad \text{on } \Gamma \times (0, T), \\
u(0) = u_0 \quad \text{in } \Omega,
\end{cases}
$$

$u(x,t)$ and $p(x,t)$, respectively, denote the velocity and the pressure of the fluid at $(x,t) = (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}_+$.
A_H represents the horizontal *eddy viscosity* coefficient,
γ is the bottom *friction* coefficient,
ρ_0 is the fluid density,
$\mathcal{T}$ is the wind stress,
$(f_0 + \beta x_2)\mathrm{k} \wedge u$ is the Coriolis term, with $\mathrm{k} \wedge u = (-u_2, u_1)$.

Formulation in terms of the stream function $\psi(x, t)$

Since $\operatorname{div} u = 0$, $u = 0$ on $\Gamma \times (0, T)$, and Ω is a connected subset of $\mathbb{R}$, we can introduce the stream function $\psi(x, t)$

$$\left\{ \begin{aligned} R_o \frac{\partial}{\partial t}(\Delta\psi) - \epsilon_m \Delta^2\psi + \epsilon_s \Delta\psi + \frac{\partial \psi}{\partial x_1} = -\operatorname{curl} \mathcal{T} \quad \text{in } \Omega \times (0, T), \\ \psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T), \\ \Delta\psi(0) = -\operatorname{curl} u_0 = \Delta\psi_0 \quad \text{in } \Omega, \end{aligned} \right. \tag{1}$$

where the coefficients R_o, ϵ_s and ϵ_m are the non-dimensional Rossby, Stommel and Munk numbers, respectively:

$$R_o = \frac{U}{\beta L^2}, \quad \epsilon_m = \frac{A_H}{\beta L^3}, \quad \epsilon_s = \frac{\gamma}{\beta L}. \tag{2}$$

U denotes a typical horizontal velocity, L is a representative horizontal length scale of ocean circulation. $\Omega = [0,1] \times [0,1]$ and $T_0 = 0.05$. For the Rossby, Munk and Stommel numbers, we consider :

$$R_o = 1.5 \times 10^{-3}, \qquad \epsilon_m = 1 \times 10^{-4}, \qquad \epsilon_s = 5 \times 10^{-3},$$

which correspond to

$$\gamma = 1 \times 10^{-7}\,\mathrm{s}^{-1}, \quad A_H = 2 \times 10^3 \mathrm{m}^2\mathrm{s}^{-1},$$
$$L = 10^6\,\mathrm{m}, \quad T = 1 \text{ year},$$
$$U = 0.03\mathrm{m\,s}^{-1}, \quad \beta = 2 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}, \quad D_0 = 800\,\mathrm{m}.$$

Existence results

We can easily prove, by adapting the arguments of Bernardi-Godlewski-Raugel to the presence of a skew-symmetric Coriolis term in the equations, the following existence result:

Theorem

For a given $\psi_0 \in H_0^1(\Omega)$ *and* $\mathcal{T} \in L^2(H^{-1}(\Omega))^2$, *problem (1) has a unique solution* ψ, *with* $\psi \in L^2(H_0^2(\Omega)) \cap C^0(H_0^1(\Omega))$ *and* $\Delta\psi \in H^1(H^{-2}(\Omega))$. *Moreover,*

$$\left|\psi\right|_{L^2(H_0^2(\Omega))\cap C^0(H_0^1(\Omega))} + \left|\Delta\psi_t\right|_{L^2(H^{-2}(\Omega))} \leq \left|\mathcal{T}\right|_{L^2(H-1(\Omega))^2} + \left|\psi_0\right|_{1,\Omega}.$$

In fact, we can improve the regularity of the solution obtained in Theorem 1 .

Theorem

For a given $\psi_0 \in H_0^2(\Omega)$ *and* $\mathcal{T} \in L^2(L^2(\Omega))^2$, *problem (1) has a unique solution* $\psi \in L^2(H^3(\Omega) \cap H_0^2(\Omega)) \cap C^0(H_0^2(\Omega))$ *and* $\psi_t \in L^2(H_0^1(\Omega))$.

Corollary

For any $\psi_0 \in H_0^1(\Omega)$, $\mathcal{T} \in L^2(L^2(\Omega))^2$, *and* $\delta > 0$, *the solution of (1) satisfies* $\psi \in C^0([\delta, T], H_0^2(\Omega))$.

Preliminaries

The problem encountered in oceanology is to simulate the evolution of the ocean circulation. Using the previous (simplified) model, everything is known, EXCEPT the initial value at time $t = 0$.

On the other hand, we know a history of measurements of the solution (observations ψ_{obs}) in some subdomain $\mathcal{O}$ during the time period $(0, T_0)$.

A classical method, called Variational Data Assimilation, based on Optimal Control, is to take the unknown initial value as a control and try to minimize, with respect to this control, the error between the actual measurements and the solution associated with the given control.

This problem is known to be ill-posed and requires to be regularized by a Tychonov method.

If $\psi(\psi_0)$ is the solution corresponding to the initial value ψ_0, we consider the functional

$$\tilde{H}_r(\psi_0) = \int_0^{T_0}\int_{\mathcal{O}} |\psi_{obs} - \psi(\psi_0)|^2 \, dx \, dt + r \Big|\psi_0\Big|^2_{1,\Omega}.$$

and the minimization problem : Find ${\psi_0}^r$ such that

$$\tilde{H}_r({\psi_0}^r) = \min_{\psi_0} \tilde{H}_r(\psi_0).$$

Here $r > 0$ is the Tychonov parameter.

New method

The idea is to recover the final state value (state value at time $t = T_0$) without knowledge of ψ_0.
For the reconstruction of $\psi(T_0)$ we will introduce a control problem for the following backward adjoint system: For $z(T_0)$ in $H_0^1(\Omega)$ and h in $L^2(L^2(\mathcal{O}))$, let us consider the following equation:

$$\begin{cases} -R_o \dfrac{\partial}{\partial t}(\Delta z) - \epsilon_m \Delta^2 z + \epsilon_s \Delta z - \dfrac{\partial z}{\partial x_1} = -h 1_{\mathcal{O}} \quad \text{in } \Omega \times (0, T_0), \\ z = \dfrac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ z(T_0) = \varphi_0 \quad \text{in } \Omega. \end{cases} \tag{3}$$

For the existence of a solution of (3), we will use the transposition method.

Definition

For each $\varphi_0 \in L^2(\Omega)$ and $h \in L^2(L^2(\mathcal{O}))$, we say that (z, z_0) is a weak solution of (3) if $z \in L^2(H_0^1(\Omega))$, $z_0 \in H_0^1(\Omega)$ and

$$\int_0^{T_0} < f, z > dt - R_o \int_\Omega \nabla\theta_0 \cdot \nabla z_0 \, dx =$$
$$- \int_0^{T_0} \int_{\mathcal{O}} h\theta \, dx \, dt + R_o \int_\Omega \varphi_0 \Delta\theta(T_0) \, dx,$$

for every $f \in L^2(H^{-1}(\Omega))$ and $\theta_0 \in H_0^1(\Omega)$, where θ is the solution of

$$\begin{cases} R_o \dfrac{\partial}{\partial t}(\Delta\theta) - \epsilon_m \Delta^2\theta + \epsilon_s \Delta\theta + \dfrac{\partial\theta}{\partial x_1} = f \quad \text{in } \Omega \times (0, T_0), \\ \theta = \dfrac{\partial\theta}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ \theta(0) = \theta_0 \quad \text{in } \Omega. \end{cases} \tag{4}$$

Here, θ satisfies the same regularity as in (1).

Theorem

For every $\varphi_0 \in L^2(\Omega)$ *and* $h \in L^2(L^2(\mathcal{O}))$*, there exists a unique solution* (z, z_0) *in* $L^2(H_0^1(\Omega)) \times H_0^1(\Omega)$*. Moreover,* $\Delta z \in C^0(H^{-2}(\Omega))$ *and* $\Delta z(0) = \Delta z_0$*.*

We now give a result of null controllability for this adjoint system, namely we claim that we can find h (control) such that the corresponding solution z satisfies $z(0) = 0$.

Theorem

For any non empty $\mathcal{O} \subset \Omega$, $T_0 > 0$ *and* $\varphi_0 \in L^2(\Omega)$, *there exists* $h = h(\varphi_0)$ *in* $L^2(L^2(\mathcal{O}))$ *such that the solution* z *of problem* (3)-(4) *satisfies*

$$z(0) = 0 \qquad \text{in } \ \Omega. \tag{5}$$

This result requires a global Carleman estimate for the adjoint of this adjoint system...., essentially the original system with zero right hand side.

Using this control h we can prove the following reconstruction for the component of $\Delta\psi(T_0)$ on the function φ_0.

Theorem

For any unknown $\psi_0 \in H_0^1(\Omega)$*, for any given* $\mathcal{T} \in L^2(L^2(\Omega))^2$*,*

$$\forall \varphi_0 \in L^2(\Omega), \quad (\Delta\psi(T_0), \varphi_0) = \frac{1}{R_o}\Big\{ \int_0^{T_0}\int_{\mathcal{O}} \psi_{obs} h(\varphi_0)\, dx\, dt + \int_0^{T_0}\int_{\Omega} \mathcal{T} \cdot \mathrm{curl}\, z(\varphi_0)\, dx\, dt \Big\}. \tag{6}$$

Moreover, there exists a positive constant C *depending on* Ω, $\mathcal{O}$ *and* T_0 *such that (stability estimate)*

$$\Big|\Delta\psi(T_0)\Big|_{0,\Omega}^2 \leq C\left\{ \int_0^{T_0}\int_{\mathcal{O}} |\psi_{obs}|^2\, dx\, dt + \int_0^{T_0}\int_{\Omega} |\mathcal{T}|^2\, dx\, dt \right\}. \tag{7}$$

Formally, the first equality can be shown essentially by multiplying the equation for ψ by $z(\varphi_0)$ and integrating by parts. As we have $z(\varphi_0)(0) = 0$, this kills the unknown term containing ψ_0.

Taking successively for φ_0 elements of a Hilbert basis of $L^2(\Omega)$, we can therefore reconstruct exactly $\Delta\psi(T_0)$.

We can also show that this reconstruction is equivalent to finding the (final) state value for ψ which minimizes the functional

$$H(\psi) = \int_0^{T_0} \int_{\mathcal{O}} |\psi_{obs} - \psi|^2 \, dx \, dt \tag{8}$$

among the trajectories ψ of the system (without initial values) such that

$$\int_0^{T_0} \int_{\mathcal{O}} |\psi|^2 \, dx \, dt < +\infty.$$

The price of this strategy is to solve a null controllability problem for the adjoint system for every element φ_0 of a Hilbert basis. We can consider an approximation of this null controllability problem by a standard optimal control problem : Let $z = z(h)$ be the solution of the adjoint system corresponding to the control h. We now fix a parameter $\alpha > 0$ and we define the cost functional

$$J_\alpha(h) = \int_0^{T_0}\int_{\mathcal{O}} |h|^2 \; dxdt + \frac{1}{2\alpha}\Big|z(0)\Big|^2_{1,\Omega}, \tag{9}$$

where we have penalized the final condition (5). We look for $h_\alpha \in L^2(L^2(\mathcal{O}))$ such that

$$J_\alpha(h_\alpha) = \min_{h \in L^2(L^2(\mathcal{O}))} J_\alpha(h). \tag{10}$$

This problem has a unique solution h_α and we call $z_\alpha = z(h_\alpha)$.

We can prove the following convergence

$$\left\{\int_0^{T_0}\int_{\mathcal{O}} \psi_{obs} h_\alpha(\varphi_0)\,dx\,dt - \int_0^{T_0}\int_\Omega \mathcal{T}\cdot \mathrm{curl}\, z_\alpha(\varphi_0)\,dx\,dt\right\} \\ \to (\Delta\psi(T_0),\varphi_0).$$

Therefore, we will have to compute $h_\alpha(\varphi_0)$ and $z_\alpha(\varphi_0)$ for different values of φ_0, elements of a Hilbert basis. Of course we will take a finite number of these elements for computations. It is therefore important to use a reduced basis. We will present here computations for a choice of basis given by eigenfunctions of the Laplace operator. Other basis like POD basis could be interesting but would require large computations for choosing the basis functions.

We will use an implicit Euler scheme for time discretization, and for the space discretization, a finite element method with a regular family of triangulations $\{\mathcal{T}_h\}$ of $\overline{\Omega}$. We take approximations of $L^2(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ by piecewise P_1 polynomials.
In fact we will have an underlying finite element method with which we make all computations, but the basis functions on which we want to recover the final state $\psi(T_0)$ is not the finite element basis, but a reduced basis, here eigenfunctions of the Laplace operator.

The following series of test problems have been done with $\Delta t = T_0/50$ and the choice for the penalty parameter is $\alpha = 0.025$
As we have no real measurements for testing our method, we will compare the results of our experiments with the results of the original model (1), i.e., we compute the ocean circulation using (1) over the time interval $(0, T_0)$, for initial given value $\Delta\psi(0) = -\sin(\pi x_1)\sin(\pi x_2)$ and surface wind stress (12). Then, we save ψ_h in the observatory $\mathcal{O} \times (0, T_0)$ and ψ_h^N, which will be our exact target values.

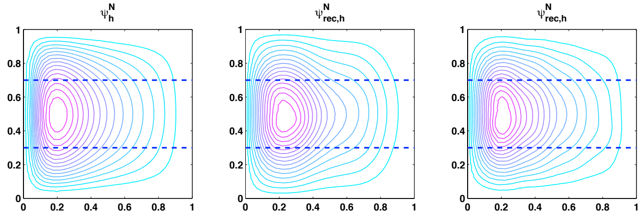
First test

For a first test we use the following wind stress

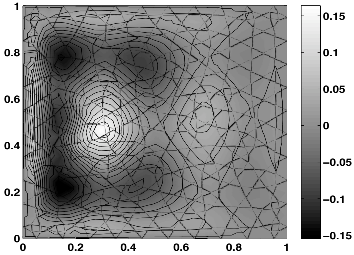
$$\mathcal{T} = (\tau_1, \tau_2) = \exp(\pi^2 t)\left(-\frac{1}{\pi}\cos(\pi\frac{x_2}{L}), 0\right).$$

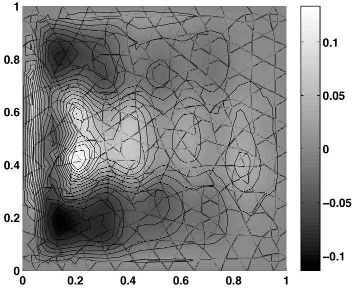
The observatory for this test is

$$\mathcal{O} = [0,1] \times [0.3, 0.7].$$



Contour lines of stream function at T_0 using real physical parameters. Exact (left), recovered using 11 eigenvalues (center) and recovered using 64 eigenvalues (right). The area between the dotted lines corresponds to the observatory $\mathcal{O}$. The regularizing parameter is $\alpha = 0.025$.


1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
0.15
0.1
0.05
0
−0.05
−0.1
−0.15



Relative error in percentage between the recovered stream function using 11 eigenvalues (above) and 64 eigenvalues (below) and the exact solution. The observatory is $\mathcal{O} = [0, 1] \times [0.3, 0.7]$. The regularizing parameter is $\alpha = 0.025$.

For the following experiment, we consider different observatory sizes and some perturbation in the observations $\psi^n_{obs,h}$ given by:

$$\hat{\psi}^n_{obs,h} = \psi^n_{obs,h} + A \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin(wt), \qquad (11)$$

where A is the noise amplitude $(0.5 * \max(\psi^n_{obs,h}))$ and k, m, w were taken big enough. In Table 6, we present the relative errors in $L^2(\Omega)$ and $H^1(\Omega)$ for the final recovered stream function in both case, with noise and in absence of it in the observatory set, using 64 eigenvalues. We obtain small values for each case.

$\mathcal{O} = [0, L]\times$	$\dfrac{\left\lvert \psi_h^N - \psi_{rec,h} \right\rvert_{0,\Omega}}{\left\lvert \psi_h^N \right\rvert_{0,\Omega}}$	$\dfrac{\lvert \psi_h^N - \psi_{rec,h} \rvert_{1,\Omega}}{\lvert \psi_h^N \rvert_{1,\Omega}}$	$\dfrac{\left\lvert \psi_h^N - \psi_{rec,h}^{noise} \right\rvert_{0,\Omega}}{\left\lvert \psi_h^N \right\rvert_{0,\Omega}}$	$\dfrac{\lvert \psi_h^N - \psi_{rec,h}^{noise} \rvert_{1,\Omega}}{\lvert \psi_h^N \rvert_{1,\Omega}}$
$[0.4L, 0.6L]$	0.1043	0.2561	0.1054	0.2582
$[0.3L, 0.7L]$	0.0634	0.2260	0.0668	0.2284
$[0.2L, 0.8L]$	0.0540	0.2137	0.0562	0.2159
$[0.1L, 0.9L]$	0.0460	0.2069	0.0471	0.2082
$[0, L]$	0.0454	0.2063	0.0465	0.2077

Relative errors in $L^2(\Omega)$ and $H^1(\Omega)$ for the final recovered stream function with noise in the observatory set ($\psi_{rec,h}^{noise}$) and in the absence of noise ($\psi_{rec,h}$) versus the observatory size $\mathcal{O}$.

Second test

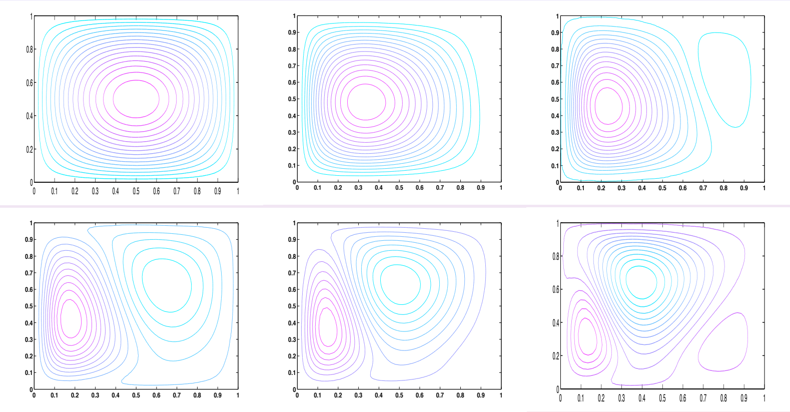
Second set of numerical experiments
Here we divide the wind stress by 10^2 in order to have less forcing in the system.

$$\mathcal{T} = (\tau_1, \tau_2) = 10^{-2} \exp(\pi^2 t) \left(-\frac{1}{\pi} \cos(\pi \frac{x_2}{L}), 0 \right). \tag{12}$$

and we assume that the observation data $(\psi^n_{obs,h})$ have certain observation error of random distribution:

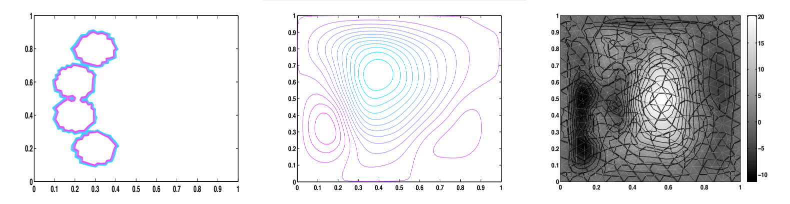
$$\hat{\psi}^n_{obs,h} = \psi^n_{obs,h}(1 + \delta R(x_1, x_2, t)),$$

where $R(x_1, x_2, t)$ denotes a random function varying in the range $[-1, 1]$, and δ is the parameter representing the noise level.
We also consider different observatories which are disconnected.
In a first figure, we show the evolution of the stream function for different interval of time.



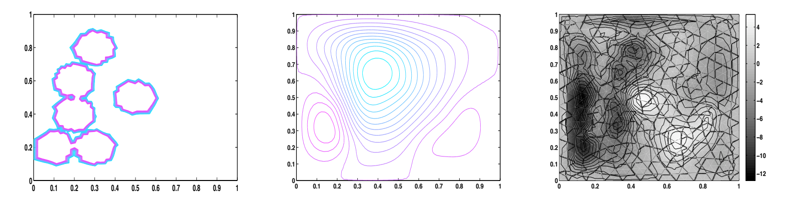
Evolution of the exact stream function in time.

Next figure shows the distribution of the observatories, numerical reconstruction of stream function, and relative percentage of error between recovered stream function and exact solution at T_0.



Left: Location of the observatories. Center: Recovered stream function at T_0 using 4 observatories. Right: Relative percentage of error between recovered stream function and exact solution at T_0. The regularizing parameter is $\alpha = 0.025$.

Using the information of the last figure, we added two observatories in the zones where the errors are important $((0.1, 0.1), (0.5, 0, 5))$. In the next figure we can see how the errors decrease considerably.



Left: Location of the observatories. Center: Recovered stream function at T_0 using 6 observatories. Right: Relative percentage error between recovered stream function and exact solution at T_0. Notice the change of grey scale with respect to Figure 3. The regularizing parameter is $\alpha = 0.025$.

In the Table below, we present the relative errors in $L^2(\Omega)$ and $H^1(\Omega)$ for the final recovered stream function in both cases, with and without noise in the observatory set, using 64 eigenvalues. Notice that, increasing the number of observatories we can increase to the noise level in the observatory maintaining a satisfactory reconstruction of the stream function at T_0.

Number of $\mathcal{O}$	Noise level δ	$\frac{\left\|\psi_h^N - \psi_{rec,h}\right\|_{0,\Omega}}{\left\|\psi_h^N\right\|_{0,\Omega}}$	$\frac{\|\psi_h^N - \psi_{rec,h}\|_{1,\Omega}}{\|\psi_h^N\|_{1,\Omega}}$
4	0	0.1813	0.2776
4	0.01	0.1852	0.2786
4	0.05	0.2034	0.2850
4	0.1	0.2314	0.3018
6	0	0.1007	0.2130
6	0.08	0.1045	0.2232
6	0.1	0.1154	0.2295
6	0.15	0.1357	0.2440

Relative errors in $L^2(\Omega)$ and $H^1(\Omega)$ for the final recovered stream function with noise in the observatory set and in the absence of noise $(\delta = 0)$ versus the observatory size $\mathcal{O}$.