Introduction to the study of generic dynamics and its relation with more classical PDE results

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Benasque, september 2009

Outline of the talk

- **Short introduction to generic dynamics**
  Motivation and definitions.
  Overview of ODE's results.

- **Generic dynamics of PDEs**
  Which kind of properties are used ?
  Some open problems

Generic dynamics of ODEs

$$\begin{cases} \dot{X}(t) = G(X(t)) \\ X(0) \in \mathbb{R}^d \end{cases}$$

Suitable conditions on $G \Rightarrow$ the flow of the ODEs generates a dynamical system $S(t)$ on $\mathbb{R}^d$:
• existence of global solutions
• existence of a compact global attractor (i.e. an invariant set which attracts the bounded sets of $\mathbb{R}^d$).

Generic dynamics of ODEs

Let $k \geq 1$. We endow $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$ with Whitney topology (or classical $\mathcal{C}^k$ topology).
For a generic $G \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$ (i.e. belonging to a countable intersections of dense open subsets) :

- **How simple/complicated are the dynamics ?**
  Gradient dynamics or nice Morse decomposition versus chaotic dynamics

- **Are the dynamics stable with respect to perturbations of the system ?**
  Local stability, global stability.

Hyperbolicity of equilibrium points

Let E be an equilibrium point that is $G(E) = 0$.
E is **hyperbolic** if $DG(E)$ has no spectrum on the vertical line $\{z \in \mathbb{C},\ Re(z) = 0\}$.

$\Rightarrow$ existence of **stable and unstable manifolds**

$$W^s(E) = \{X_0 \in \mathbb{R}^d,\ \lim_{t \to +\infty} S(t)X_0 = E\}$$

$$\begin{aligned} W^u(E) = &\{X_0 \in \mathbb{R}^d,\ \exists \text{ a backward solution } X(t) \\ &\text{with } X(0) = X_0 \text{ and } \lim_{t \to -\infty} X(t) = E\} \end{aligned}$$

and **local stability** of the dynamics.

Hyperbolicity of periodic orbits

Let $P(t)$ be a periodic orbit of minimal period T. We introduce the linearized map

$$U_0 \longmapsto \Pi(T)U_0 = U(T)$$

where $U(t)$ solves

$$\dot{U}(t) = DG(P(t))U(t), \quad U(0) = U_0 \ .$$

$P(t)$ is **hyperbolic** if $\Pi(T)$ has no spectrum on the unit circle $\{z \in \mathbb{C}, \ |z| = 1\}$ except the eigenvalue 1 which is simple.

NB : $\dot{P}(0)$ is always an eigenvector for the eigenvalue 1.

Kupka-Smale property

Definition

$S(t)$ satisfies **Kupka-Smale property** *if :*

- *all the equilibrium points or periodic orbits are hyperbolic,*
- *their stable and unstable manifolds intersect transversally.*

Kupka-Smale property implies the **local stability of the dynamics** with respect to perturbations of the system.

Morse-Smale property

Definition

$S(t)$ satisfies **Morse-Smale property** *if :*

- *it satisfies Kupka-Smale property,*
- *there is only a finite number of equilibrium points and periodic orbits,*
- *there is no other non-wandering points.*

A point $X \in \mathbb{R}^d$ is wandering if for any neighborhood $\mathcal{N} \ni X$, $S(t)\mathcal{N} \cap \mathcal{N} = \emptyset$ for t large enough.

Morse-Smale property implies the **global stability of the dynamics** with respect to perturbations of the system $S(t)$: if $\tilde{G} \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$ is close to G then there exists a homeomorphism h which maps the trajectories of $S(t)$ onto the trajectories of $\tilde{S}(t)$ (Palis 1968).

Classical results

- $d = 1$
  The dynamics are gradient
  Morse-Smale property holds generically
- $d = 2$
  Poincaré-Bendixson property holds
  Morse-Smale property holds generically (Peixoto 1962)
- $d \geq 3$
  Kupka-Smale property holds generically
  (Kupka 1963, Smale 1967)
  There exists chaotic dynamics (Smale 1965)
  Non-density of stable dynamics
  (Guckenheimer and Williams 1979)
- $d \geq 1$, $G = -\nabla V$
  The dynamics are gradient
  Morse-Smale property holds generically (Smale 1961)

Dynamics of parabolic PDEs

Let Ω be a regular bounded domain of $\mathbb{R}^N$, let $p > N$ and $\alpha \in (N/p + 1, 2)$.
With suitable assumptions on f, the scalar parabolic equation

$$u_t = \Delta u + f(x, u, \nabla u)$$

generates a global dynamical system in $W^{\alpha,p}(\Omega)$ (+boundary conditions) and admits a compact global attractor $\mathcal{A}$.

NB : often $\mathcal{A}$ is finite-dimensional but its dimension may be as large as wanted.

Generic dynamics of parabolic PDEs

Generically with respect to f :

- $u_t = u_{xx} + f(x, u, u_x)$ on $(0, 1)$ is Morse-Smale (Henry 1985)
- $u_t = \Delta u + f(x, u)$ is Morse-Smale (Brunovský-Poláčik 1997)
- $u_t = u_{xx} + f(x, u, u_x)$ on S^1 is Morse-Smale (Czaja-Rocha 2008 + RJ-Raugel 2009) (non gradient PDE)
- $u_t = \Delta u + f(x, u, \nabla u)$ is Kupka-Smale (Brunovský-RJ-Raugel in preparation) (non gradient PDE)

NB : for $\Omega = S^1$ Poincaré-Bendixson property hold

Need of unique continuation properties

Let $p(t)$ be a periodic solution of

$$u_t = \Delta u + f(x, u, \nabla u)$$

of minimal period $T > 0$.

Theorem – Brunovský-RJ-Raugel (2009)

Assume $f \in \mathcal{C}^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Then there exists a dense open set of points $(x_0, t_0) \in \Omega \times \mathbb{R}$ such that if

$$(x_0, p(x_0, t_1), \nabla p(x_0, t_1)) = (x_0, p(x_0, t_0), \nabla p(x_0, t_0))$$

then $t_1 = t_0 + nT$ with $n \in \mathbb{Z}$.

Need of unique continuation properties

Tools : if $v(x,t) = p(x,t) - p(x,t+(t_1-t_0))$ is not trivial, then it solves a linear parabolic equation and thus cannot have a zero of infinite order. Then estimate sharply the size of the singular nodal set

$$\{(x,t) \in \Omega \times \mathbb{R},\ v(x,t) = 0 \text{ and } \nabla v(x,t) = 0\}$$

(method of Hardt and Simon).

Open problem : similar result for systems of parabolic equations $U_t = \Delta U + F(x,U)$.

Generically with respect to f :

- $u_{tt} + \gamma u_t = \Delta u + f(x, u)$ is Morse-Smale (Brunovský-Raugel 2003)
- $u_{tt} + \gamma(x) u_t = \Delta u + f(x, u)$ (or boundary damping) on the segment $(0, 1)$ is Morse-Smale (RJ 2005)

Need of punctual asymptotic

Let $u(t)$ be a global solution in $H_0^1((0,1)) \times L^2((0,1))$ of

$$u_{tt} + \gamma(x)u_t = u_{xx} + f(x,u), \quad (x,t) \in (0,1) \times \mathbb{R}\ .$$

Assume that $u(t)$ converges to an equilibrium e when t goes to $+\infty$. Let A_e be the operator corresponding to the linearization of the equation at $(e,0)$.

Theorem – RJ (2005)

There exists a generic set of points $x_0 \in (0,1)$ such that

$$\lim_{t\to\infty} \frac{\ln \|u(t) - e\|_{H^1}}{t} = \lim_{t\to\infty} \frac{\ln |u(x_0,t) - e(x_0)|}{t} = \lambda$$

where λ is either $-\frac{1}{2}\int_0^1 \gamma(x)dx$ or the negative real part of an eigenvalue of A_e.

Need of punctual asymptotic

Tools : use the existence of a Riesz basis of eigenfunction of A_e and the asymptotics of the high frequencies (Cox-Zuazua 1994).

Open problem : dimension higher than one ?

A negative result

Theorem – Poláčik (1999)

There exist a domain $\Omega \subset \mathbb{R}^2$ and an open set of nonlinearities $f \in \mathcal{C}^1(\mathbb{R})$ such that the parabolic equation

$$u_t = \Delta u + f(u), \quad u_{|\Omega} = 0$$

admits two equilibrium points e_1 and e_2 for which $W^u(e_1)$ does not intersect $W^s(e_2)$ transversally.

Tools : perturb the disk so that the spectra of $\Delta + f'(e_i)$ are as wanted. Use the fact that the spaces of even and odd functions are invariant by the flow.

Open problem : perturbations with respect to the domain Ω give enough freedom to obtain generic stability results ?

Additional open problems

- Go beyond Kupka-Smale property for the dynamics of the scalar parabolic equations (Pugh closing lemma).
- Strongly damped wave equations
  $u_{tt} - \Delta u_t = \Delta u + f(x, u)$.
- Equations of fluids mechanic.
- ...

Appendix

Appendix

Relations between the parabolic equation and low-dimensional flows

$u_t = u_{xx} + f(x, u, u_x)$ on $(0, 1)$

and $\dot{X}(t) = G(X(t))$ on $\mathbb{R}$.

- Gradient dynamics
- Convergence to an equilibrium point
- Automatic transversality of stable and unstable manifolds
- Genericity of Morse-Smale property
- Knowledge of the equilibrium points implies knowledge of the whole dynamics
- Realization of the flow of the ODE in the dynamics of the PDE

Relations between the parabolic equation and low-dimensional flows

$u_t = u_{xx} + f(x, u, u_x)$ on S^1
and $\dot{X}(t) = G(X(t))$ on $\mathbb{R}^2$.

- Poincaré-Bendixson property
- Automatic transversality of stable and unstable manifolds of two orbits if one of them is a hyperbolic periodic orbit
- Non-existence of homoclinic orbits for periodic orbits
- Genericity of Morse-Smale property
- Realization of the flow of the ODE in the dynamics of the PDE

Relations between the parabolic equation and low-dimensional flows

$u_t = u_{xx} + f(u, u_x)$ on S^1

and $\dot{X}(t) = G(X(t))$ on $\mathbb{R}^2$ and radial symmetry.

- Automatic transversality
- No homoclinic orbit
- Knowledge of the eq. points and of the periodic orbits implies knowledge of the whole dynamics
- Genericity of Morse-Smale property
- Realization of the ODE in the PDE

Relations between the parabolic equation and low-dimensional flows

$u_t = \Delta u + f(x, u, \nabla u)$ on Ω with $dim(\Omega) \geq 2$

and $\dot{X}(t) = G(X(t))$ on $\mathbb{R}^d$, $d \geq 3$.

- Existence of persistent chaotic dynamics
- Genericity of Kupka-Smale property
- Realization of the ODE in the PDE

Relations between the parabolic equation and low-dimensional flows

$u_t = \Delta u + f(x, u)$ on any Ω

and $\dot{X}(t) = -\nabla V(X(t))$.

- Gradient dynamics
- Genericity of Morse-Smale property
- Realization of the ODE in the PDE

Property of the number of zeros

The number of zeros of a solution of a one-dimensional linear parabolic equation satisfies a very special property.
For example, let Ω be the circle S^1.

Theorem – Sturm, Nickel, Matano, Angenent, Fiedler... (1836 and ∼1980)

Let $a(x,t)$ and $b(x,t)$ be in $\mathcal{C}^2(S^1 \times \mathbb{R}_+, \mathbb{R})$. Let w be a non-trivial solution in $\mathbb{L}^2(S^1)$ of

$$\partial_t w = \partial_{xx}^2 w + a(x,t)w + b(x,t)\partial_x w$$

Then, then number of zeros of $w(t)$ is finite and non-increasing in time and strictly decreases at $t = t_0$ if and only if $w(t_0)$ has a multiple zero.

Application :
if u and v are two solutions of $u_t = u_{xx} + f(x, u, u_x)$,
then $w = u - v$ satisfies the above result.