# The Hardy inequality and the asymptotic behaviour of the heat equation in twisted domains

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**Joint work with:** **Enrique Zuazua**

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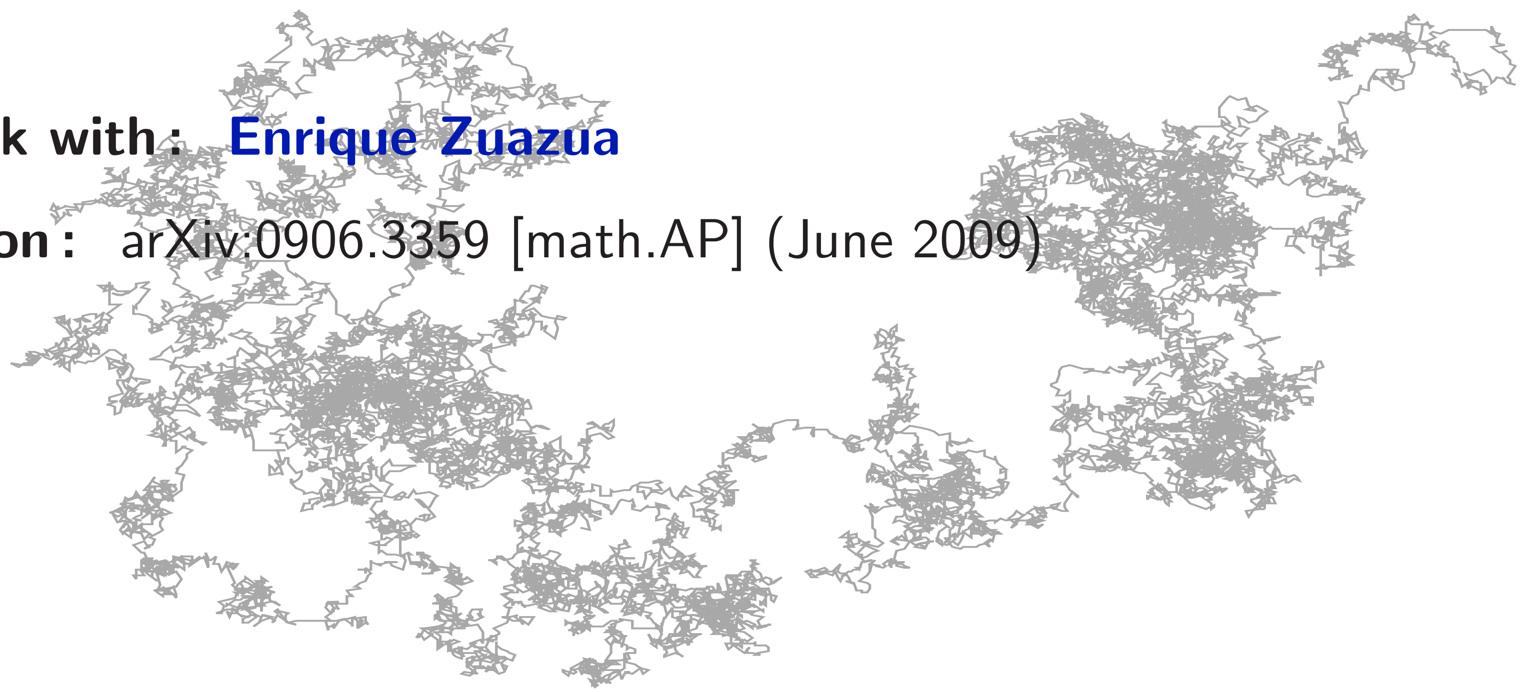
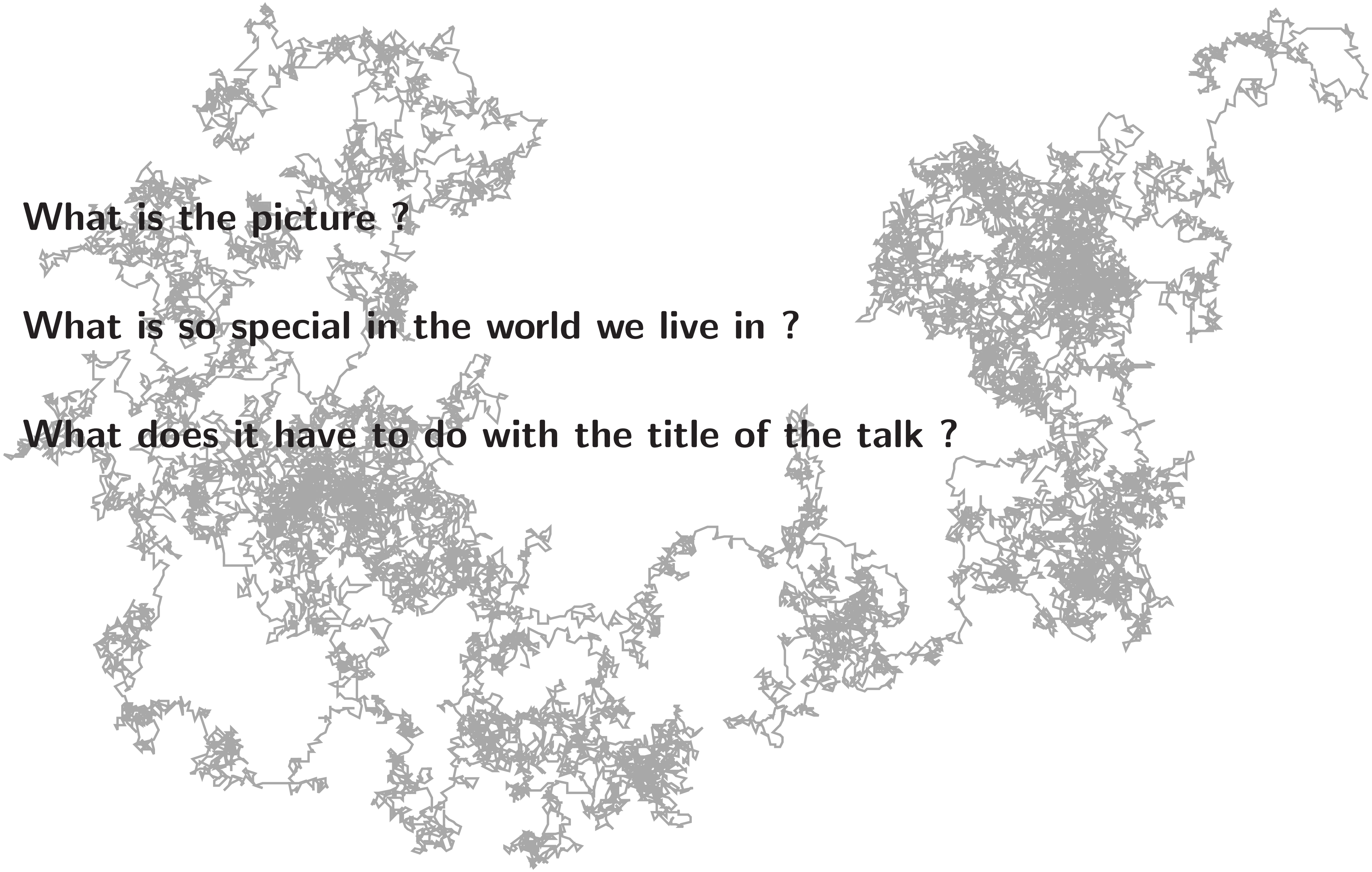


# Outline

*3 fundamental questions*



1. **What is the picture ?**

2. **What is so special in the world we live in ?**

3. **What does it have to do with the title of the talk ?**

# The uniqueness of $\mathbb{R}^3$

- Brownian motion is recurrent in $\mathbb{R}^1$ and $\mathbb{R}^2$,
  *i.e.* the Brownian particle visits every region infinitely many times.

- Brownian motion is transient in $\mathbb{R}^d$ with $d \geq 3$,
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Remark. The Brownian motion (Wiener process) can be rigorously constructed via the heat kernel associated with the diffusion equation $u_t - \frac{1}{2}\Delta u = 0$.

# Analytic background of transience

[Grigor'yan 1999]

**Theorem.** Let $M$ be a complete non-compact Riemannian manifold.
The following properties are **equivalent**:

1. Brownian motion on $M$ is **transient**.
2. $M$ is non-parabolic, *i.e.* $\exists$ non-constant positive superharmonic function.
3. $-\Delta$ is subcritical in $M$, *i.e.* $\exists$ Green function, finite off the pole.
4. $\forall$ precompact open subset of $M$ has positive capacity.
5. $-\Delta$ has no virtual bound state, *i.e.* $-\Delta - \varepsilon V \geq 0$ for all small $\varepsilon$. $[V \in C_0^\infty(M)]$
6. $\exists$ **Hardy inequality** for $-\Delta$, *i.e.* $-\Delta \geq w(\cdot)$ with some positive $w$.

Remark. $M = \mathbb{R}^d$ $(d \geq 3)$: $\quad w(x) = \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2}$ (classical HI)

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Remark. 5 reflects the **spectral-stability** effect of transience.

[non-trivial $V \geq 0$, $\varepsilon > 0$]

$\sigma(-\Delta^{\mathbb{R}^d} - \varepsilon V)$


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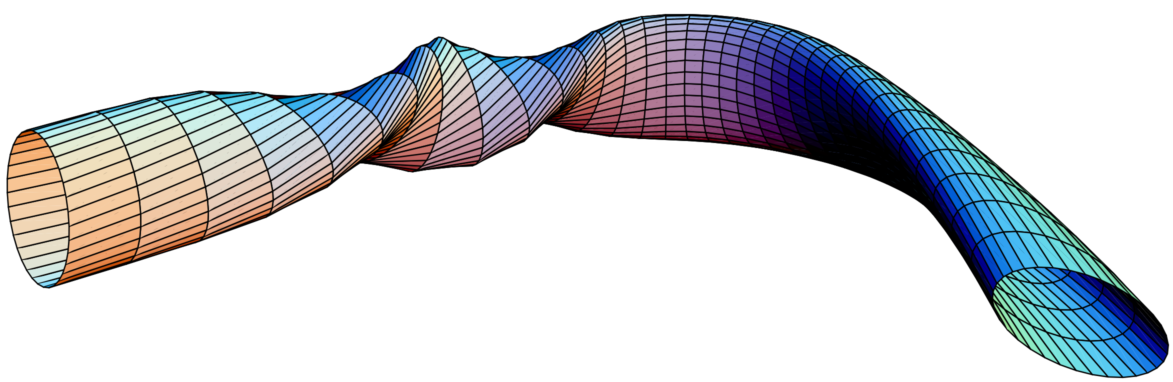
Natural candidates: TUBES



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study of *fine* properties of the transience

$$\boxed{S_D^\Omega(t) := e^{(\Delta_D^\Omega + \lambda_1)t}}$$

**Conjecture.** Better time-decay of $S_D^\Omega(t)$ iff $-\Delta_D^\Omega - \lambda_1 \geq w(\cdot) > 0$ holds.

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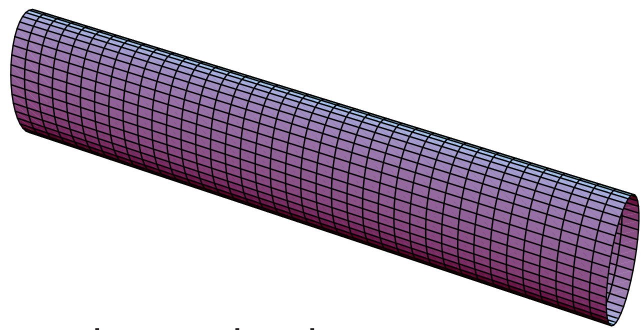
**2.** polynomial decay-rate inherited from 'dimensional' properties of $\Omega$

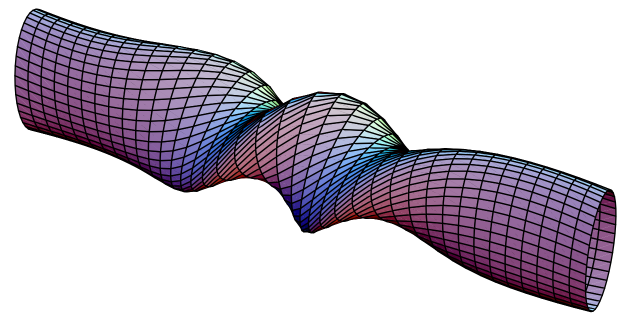
*e.g.* $\|S_D^{\mathbb{R}^d}(t)\|_{L^2(\mathbb{R}^d,K)\to L^2(\mathbb{R}^d)} \sim t^{-d/4}$ as $t \to \infty$, $K(x) := e^{|x|^2/4}$

# The Hardy inequality in twisted tubes

$\Omega_0 := \mathbb{R} \times \omega$ where $\omega$ is a bounded domain in $\mathbb{R}^2$

$$\Omega_\theta := \left\{ \big( x_1, x_2 \cos\theta(x_1) + x_3 \sin\theta(x_1), -x_2 \sin\theta(x_1) + x_3 \cos\theta(x_1) \big) \mid x \in \Omega_0 \right\}$$





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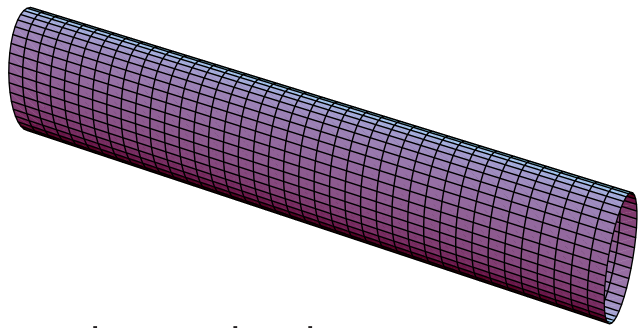

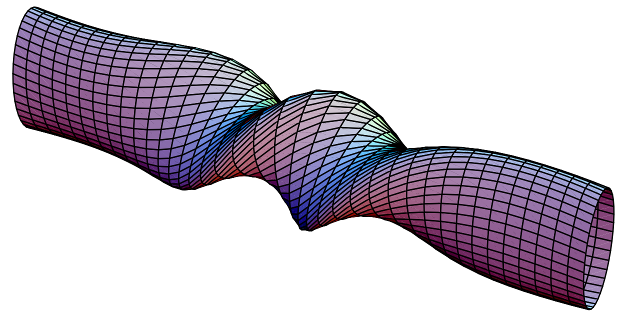

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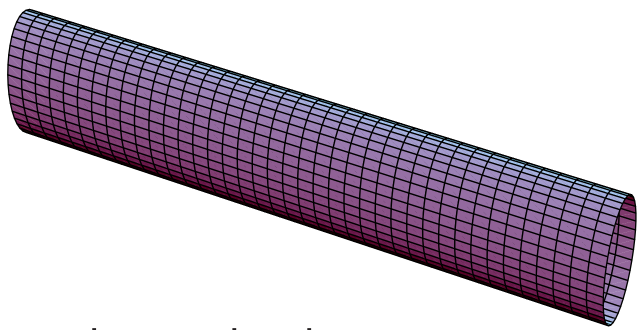

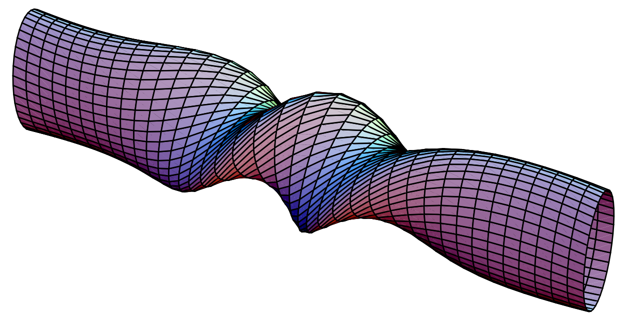

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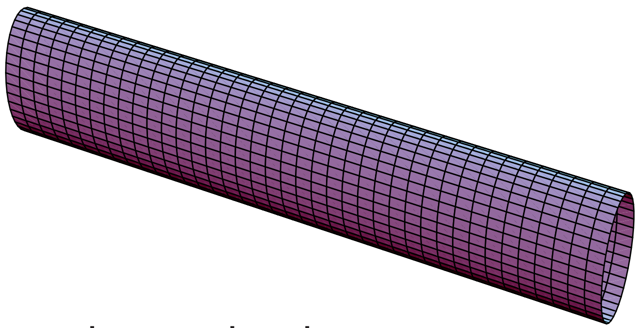

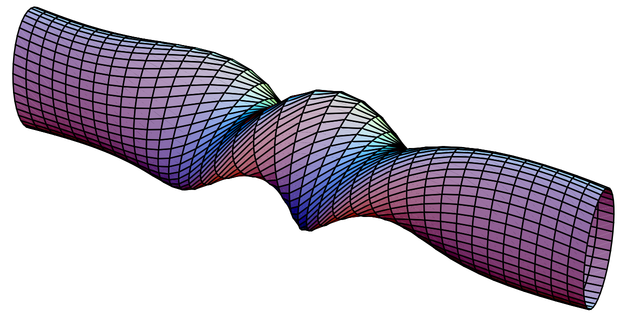

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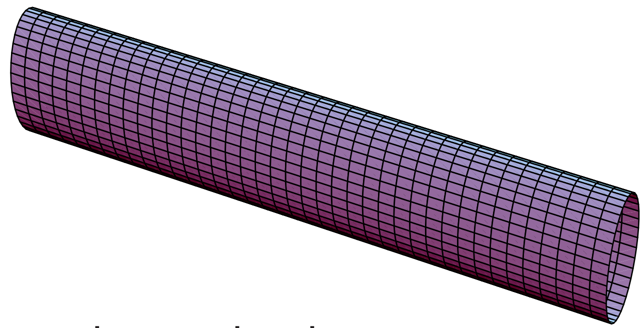

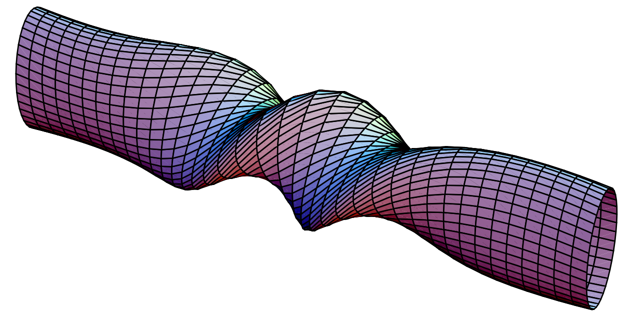

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$\rightsquigarrow$ boom: Briet, Exner, Fraas, Raikov, Sacchetti, Soccorsi, ...

# The improved decay rate

$$\begin{cases} u_t - \Delta u - \lambda_1 u = 0 & \text{in} \quad \Omega_\theta \times (0,\infty) \\ u = 0 & \text{in} \quad \partial\Omega_\theta \times (0,\infty) \\ u = u_0 & \text{in} \quad \Omega_\theta \times \{0\} \end{cases}$$

$$\boxed{S_D^{\Omega_\theta}(t) := e^{(\Delta_D^{\Omega_\theta} + \lambda_1)\, t}} : u_0 \mapsto u(t)$$

heat semigroup

decay rate

$$\boxed{\Gamma(\Omega_\theta, K) := \sup \left\{ \Gamma \;\middle|\; \exists C_\Gamma > 0,\ \forall t \geq 0,\ \ \|S_D^{\Omega_\theta}(t)\|_{L^2(\Omega_\theta, K) \to L^2(\Omega_\theta)} \leq C_\Gamma\, (1+t)^{-\Gamma} \right\}}$$

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**Theorem (D.K., Zuazua 2009).**

$$\boxed{\Gamma(\Omega_\theta, K) \begin{cases} = 1/4 & \text{if } \Omega_\theta \text{ is untwisted} \\ \geq 3/4 & \text{if } \Omega_\theta \text{ is twisted} \end{cases}}$$

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*Proof.*

$\geq$ self-similarity transformation,* weighted Sobolev spaces, ...

$=$ no twisting $\boxed{\|S(t)\|_{L^2(\Omega_0,K)\to L^2(\Omega_0)} \sim t^{-1/4}}$ as $t \to \infty$

*q.e.d.*

*[Escobedo, Kavian 1987]

# Self-similarity transformation

**1.** Straightening of the tube $\quad \mathcal{L}_\theta : \Omega_0 \to \Omega_\theta$ (curvilinear coordinates)

$$\boxed{u_t - (\partial_1 - \dot\theta\,\partial_\tau)^2 u - \Delta' u - \lambda_1 u = 0} \quad \text{in} \quad \Omega_0$$

$$\mathcal{L}_\theta(x) = \begin{pmatrix} x_1 \\ x_2 \cos\theta(x_1) + x_3 \sin\theta(x_1) \\ -x_2 \sin\theta(x_1) + x_3 \cos\theta(x_1) \end{pmatrix}$$

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**2.** Changing the time $\quad \tilde u(y_1, y_2, y_3, s) = e^{s/4}\, u\big(e^{s/2} y_1, y_2, y_3, \underbrace{e^s - 1}_{t}\big)$

$$\boxed{\tilde u_s - \tfrac{1}{2} y_1 \partial_1 \tilde u - (\partial_1 - \sigma_s\,\partial_\tau)^2 \tilde u - e^s \Delta' \tilde u - \lambda_1 e^s\, \tilde u - \tfrac{1}{4}\tilde u = 0} \qquad \sigma_s(y_1) := e^{s/2}\dot\theta(e^{s/2} y_1)$$

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**3.** Changing the space $L^2(\Omega_0, K), \quad K(y) = e^{y_1^2/4} \implies$ compactness !

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$$\partial_\tau := x_3\partial_2 - x_2\partial_3, \quad \Delta' := \partial_2^2 + \partial_3^2$$

**2.** Changing the time

$$\tilde u(y_1,y_2,y_3,s) = e^{s/4}\, u\big(e^{s/2}y_1, y_2, y_3, \underbrace{e^s - 1}_{t}\big)$$

$$\boxed{\tilde u_s - \tfrac12 y_1\partial_1\tilde u - (\partial_1 - \sigma_s\,\partial_\tau)^2\tilde u - e^s\Delta'\tilde u - \lambda_1 e^s\,\tilde u - \tfrac14\tilde u = 0} \qquad \sigma_s(y_1) := e^{s/2}\dot\theta(e^{s/2}y_1)$$

$$\|u(t)\|_{L^2(\Omega_0)} = \|\tilde u(s)\|_{L^2(\Omega_0)}$$

**3.** Changing the space $\quad L^2(\Omega_0, K), \;\; K(y) = e^{y_1^2/4} \;\Longrightarrow\;$ compactness !

**4.** Asymptotic analysis $\quad$ As $s \to \infty$,

$$|\sigma_s(y_1)| \longrightarrow \|\dot\theta\|_{L^1(\mathbb{R})}\,\delta(y_1)$$

$$\mathbb{R}\times e^{-s/2}\omega \longrightarrow \mathbb{R}$$

$$\boxed{\varphi_s - \tfrac12\, y_1\,\varphi_{y_1} - \varphi_{y_1y_1} - \tfrac14\,\varphi = 0} \quad +$$

Dirichlet b.c. at $y_1 = 0$ **iff** $\Omega_\theta$ twisted

# Conclusions

Moral :

$\longrightarrow$ straight and twisted tubes have the same spectrum


$\longrightarrow$ Hardy inequality iff the tube is twisted $\qquad -\Delta_D^{\Omega_\theta} - \lambda_1 \geq w(\cdot)$

$\Updownarrow$

$\longrightarrow$ improved decay rate iff the tube is twisted $\qquad \left\| e^{(\Delta_D^{\Omega_\theta}+\lambda_1)\,t} \right\| \leq (1+t)^{-(1/4+\gamma)}$

$\Rightarrow$ fine effect of transience, faster cool down

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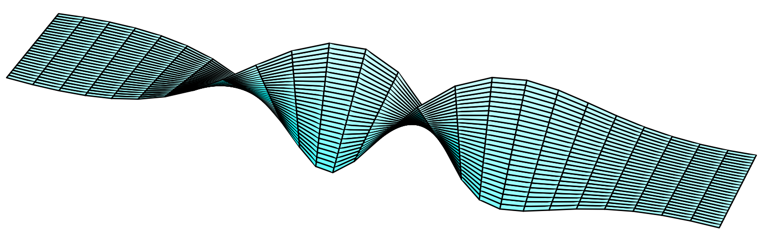
(other twisted systems)



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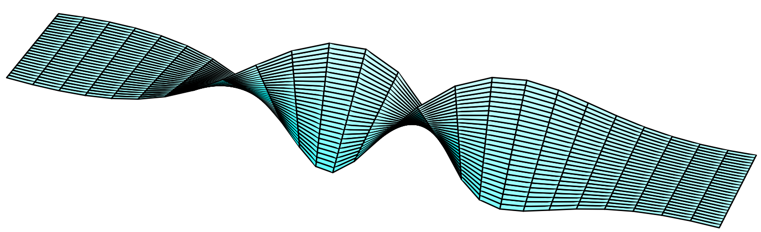


## Open problems :

¿ better topology than $L^2(\Omega_\theta, K) \to L^2(\Omega_\theta)$ ? $\qquad$ *NB* $K(x) = e^{x_1^2/4}$

¿ $\Gamma(\Omega_\theta, K) = 3/4$ if $\Omega_\theta$ is twisted ? (optimality)

¿ direct proof of the equivalence $\Updownarrow$ ? (energy methods fail)

¿ general quasi-cylindrical domains ? ($\exists$ Hardy inequality $\Leftrightarrow$ improved decay rate)

# Our conjecture

[D.K., Zuazua 2009]

Let $\Omega$ be an open connected subset of $\mathbb{R}^d$. Let $H$ and $H_+$ be two self-adjoint operators in $L^2(\Omega)$ such that $\inf\sigma(H) = \inf\sigma(H_+) = 0$. Assume that there is a positive smooth function $\varrho : \Omega \to \mathbb{R}$ such that $H_+ \geq \varrho$, while $H - V$ is a negative operator for any non-negative non-trivial $V \in C_0^\infty(\Omega)$. Then there exists a positive function $K : \Omega \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\|e^{-H_+ t}\|_{L^2(\Omega,K)\to L^2(\Omega)}}{\|e^{-Ht}\|_{L^2(\Omega,K)\to L^2(\Omega)}} = 0\,.$$