Rapid stabilization of the Korteweg-de Vries equation on a periodic domain

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1. Introduction
2. Control of KdV
3. Main results

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Korteweg-de Vries equation

- Korteweg-de Vries equation (1895):

$$\partial_t u \, + \partial_x^3 u \, + u \, \partial_x u = 0$$

- $x \in I \; (I = \mathbb{R}, \mathbb{T}, (0, L)), \;\; t \in \mathbb{R}, \;\; u = u(x, t) \in \mathbb{R}$.

- Classical model for propagation of small amplitude long waves in nonlinear dispersive media (e.g. water waves, plasma physics,...)

Well-posedness

- $x \in I = \mathbb{R}$
  Temam [1969], Saut-Temam [1976], Bona-Scott [1976], Kato [1983], Kruzhkov-Faminskii [1983], Kenig-Ponce-Vega [1991,1996], Bourgain [1993],
  Colliander-Keel-Staffilani-Takaoka-Tao [2003]: GWP in $H^s(\mathbb{R})$, $s > -3/4$.
- $x \in I = \mathbb{T}$
  Bourgain [1993,1996]
  Kappeler-Topalov [2006]: GWP in $H^{-1}(\mathbb{T})$
- $x \in I = (0, L)$ + 3 b.c.
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Basic issues in Control Theory

$$(S) \qquad \frac{du}{dt} = f(u,h), \; u(0) = u_0$$

- **Exact Controllability**
  Given $T > 0$, u_0 and u_1 in some space, can we find a control input $h = h(t)$ driving system (S) from u_0 at $t = 0$ to u_1 at $t = T$?

- **Stabilization**
  Can we find a closed-loop control $h = h(u)$ such that the origin is asymptotically (or exponentially) stable for (S)?

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Tools in Control Theory

- Nonharmonic Fourier analysis (1D PDEs, Schrödinger)
- Multipliers (wave, plates)
- Carleman estimates (heat, Navier-Stokes)
- Microlocal analysis (wave)

Control properties of KdV: bounded domain

$$\partial_t u + \partial_x^3 u + \partial_x u + u\,\partial_x u = 0$$

$$u(0,t) = h_1(t),\ u(L,t) = h_2(t),\ u_x(L,t) = h_3(t)$$

- **Control on the right** ($h_1 = h_2 = 0$, R. [1997])
  Local exact controllability if

$$L \notin \mathcal{N} = \{2\pi\sqrt{\frac{k^2+l^2+kl}{3}},\ k,l \in \mathbb{N}^*\}$$

- **Control on the left** ($h_2 = h_3 = 0$, R. [2004])
  Local null controllability.
- **J. Bona' observation**
  For a solution $u(x,t) = e^{i(kx-\omega t)}$ of $u_t + u_{xxx} + u_x = 0$, $\omega = k - k^3$ (dispersion relation). Hence $\omega < 0$ for $k >> 1$: High frequencies propagate to the LEFT.

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Litterature (bounded domain)

- **Controllability:** Zhang [1999], Coron-Crépeau [2004], Cerpa [2007], Glass-Guerrero [2008,preprint], Cerpa-Crépeau [2009], Chapouly [preprint]
- **Stabilization:** Perla Menzala-Vasconcellos-Zuazua [2002], Pazoto [2005], R.-Zhang [2006], Linares-Pazoto [2007], Cerpa-Crépeau [2009]
- **Main tools:** Compactness argument based upon **Kato smoothing effect**: assume $h_i = 0$ ($i = 1, 2, 3$).

$$u_{|t=0} \in L^2(0, L) \quad \Rightarrow \quad u \in L^2(0, T, H^1(0, L))$$

Control properties of KdV: periodic domain

- $$\partial_t u + \partial_x^3 u + u\,\partial_x u = f(x,t), \quad x \in \mathbb{T} = \mathbb{R}/(2\pi)\mathbb{Z}$$

  Forcing term f supported in some given open set $\omega \subset \mathbb{T}$.
- To keep the mass $\int_{\mathbb{T}} u(x,t)dx$ conserved, we impose the condition $[f] := (2\pi)^{-1}\int_{\mathbb{T}} f\,dx = 0$. We shall assume that

  $$f(x,t) = [Gh](x,t) = g(x)\left(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)\,dy\right)$$

  g being a fixed nonnegative, smooth function supported in ω with $\int_{\mathbb{T}} g(x)dx = 1$.
- Main contributions
  Zhang [1990], Komornik-Russell-Zhang [1991], Russell-Zhang [1993,1996]

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  For all $T > 0$ and $u_0, u_1 \in H^s(\mathbb{T})$ ($s \geq 0$) with $[u_0] = [u_1]$, there exists a control input $h \in L^2(0, T, H^s(\mathbb{T}))$ s.t. $u(., 0) = u_0$, $u(., T) = u_1$.
- **Exponential Stabilization**
  Let $h = -G^* u = -Gu$. Then there is some $\mu > 0$ such that for all $s \geq 0$
  $$||u(., t) - [u_0]||_s \leq Ce^{-\mu t}||u_0 - [u_0]||_s$$
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Ideas of the proof

- Kato smoothing effect **no longer valid** on the torus. Fortunately, Bourgain [1993] discovered a more subtle smoothing effect thanks to which he proved the GWP in $L^2(\mathbb{T})$.
- Tools for the proofs: contraction mapping principle in Bourgain spaces
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Bourgain spaces

- For $b, s \in \mathbb{R}$ and $u(x,t) = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \hat{u}(k,\tau) e^{i(kx+\tau t)}\, d\tau$, let

$$
\begin{aligned}
||u||^2_{X_{b,s}} &= ||W(-t)u||^2_{H^b(\mathbb{R};H^s(\mathbb{T}))} \qquad [W(t) = e^{-t\partial_x^3}] \\
&= \sum_k \int_{\mathbb{R}} \langle k\rangle^{2s} \langle \tau - k^3\rangle^{2b} |\hat{u}(k,\tau)|^2 d\tau \\
||u||^2_{Y_{b,s}} &= \sum_k \left(\int_{\mathbb{R}} \langle k\rangle^{s} \langle \tau - k^3\rangle^{b} |\hat{u}(k,\tau)| d\tau \right)^2
\end{aligned}
$$

- $X_{b,s}$ (resp. $Y_{b,s}$) completion of $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ for the norm $||\cdot||_{X_{b,s}}$ (resp. $||\cdot||_{Y_{b,s}}$). Finally $Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$. Let $X^T_{b,s}$, $Z^T_{b,s}$ be the restriction spaces to $(0,T)$.

$$||u||_{X^T_{b,s}} = \inf\{||v||_{X_{b,s}} | v = u \text{ on } \mathbb{T}\times(0,T)\}$$

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Bourgain spaces (2)

- $Z^T_{\frac{1}{2},s} \subset C([0,T], H^s(\mathbb{T}))$.
- $||W(t)\phi||_{Z^T_{b,s}} \leq C||\phi||_s \; \forall b, s$
- $|| \int_0^t W(t-s)f(s)ds||_{Z^T_{\frac{1}{2},s}} \leq C||f||_{Z^T_{-\frac{1}{2},s}}$
- **Bilinear estimate**

$$||(uv)_x||_{Z^T_{-\frac{1}{2},s}} \leq CT^\theta ||u||_{X^T_{\frac{1}{2},s}} ||v||_{X^T_{\frac{1}{2},s}}.$$

The results

Joint work with
Camille Laurent, *Université Paris-Sud (France), and*
Bing-Yu Zhang, *University of Cincinnati*
Consider

$$\partial_t u + u\partial_x u + \partial_x^3 u = Gh = g(x)(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)\,dy)$$
$$u(.,0) = u_0$$

To simplify the exposition, assume $[u(.,t)] = [u_0] = 0$.

Global exact controllability

Thm 1: Assume given $s \geq 0$, $R > 0$. There exists $T > 0$ s.t. for

$$u_0, u_1 \in H^s(\mathbb{T}), \quad [u_0] = [u_1] = 0, \quad ||u_0||_s + ||u_1||_s \leq R$$

one can find $h \in L^2(0, T, H^s(\mathbb{T}))$ driving the system from u_0 at $t = 0$ to u_1 at $t = T$

Global exponential stabilization

Thm 2: $s \geq 0$ given. There exists a constant $\mu > 0$ such that for $u_0 \in H^s(\mathbb{T})$ with $[u_0] = 0$, we have for $t \geq 0$

$$||u(.,t)||_s \leq \alpha(||u_0||_s)e^{-\mu t}||u_0||_s$$

α is a nondecreasing function depending on s.

Sketch of the proofs

- Thm 1 follows from Thm 2 and Russell-Zhang (local) controllability
- To prove Thm 2 for $s = 0$, apply Zuazua's compactness - uniqueness strategy: using the Energy Identity ($||\cdot|| = ||\cdot||_{L^2(\mathbb{T})}$):

$$||u(t)||^2 = ||u_0||^2 - \int_0^t ||Gu(\tau)||^2 d\tau$$

it is sufficient to prove the **Observability Inequality**:

$$||u_0||^2 \leq const \int_0^T ||Gu(\tau)||^2 d\tau$$

This is done by contradiction. If not true there is a sequence $\{u_n\} \subset Z^T_{\frac{1}{2},0}$ with $||u_n(0)|| \leq R_0$ and

$$\int_0^T ||Gu_n(\tau)||^2 d\tau < \frac{1}{n} ||u_n(0)||^2.$$

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Proofs (continued)

Assume $||u_n(0)|| \to \alpha > 0$.

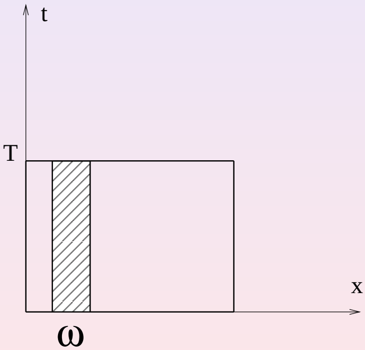
- $w_n = u_n - u \to 0$ weakly in $X^T_{\frac{1}{2},0}$, hence strongly in $X^T_{-\frac{1}{2},-1}$
- $u_n u_{n,x} - u u_x \to 0$ in $X^T_{-\frac{1}{2},-1}$
- $w_n \to 0$ in $L^2(0,T,L^2(\omega))$
- By a **propagation of compactness**, this implies $w_n \to 0$ in $L^2_{loc}(0,T,L^2(\mathbb{T}))$.
- $u = const$ on $\omega \times (0,T)$ hence by a **propagation of regularity**, $u \in C^\infty(\mathbb{T} \times (0,T))$. The UCP yields that $u = const = 0$.
- We get the contradiction from

$$||u_n(0)||_0^2 = ||u_n(t_0)||_0^2 + \int_0^{t_0} ||Gu_n||_0^2 dt$$

and $||u_n(t_0)||_0 \to 0$.

Propagation of compactness/regularity

Introduced in Dehman-Gérard-Lebeau [2006] and Laurent [2009] for NLS.



Propagation of compactness

Assume $T>0$, $\omega\subset\mathbb{T}$, $0\le b'\le b\le 1$ and $u_n\in X^T_{b,0}$, $f_n\in X^T_{-b,-2+2b}$ satisfy

$$\partial_t u_n+\partial_x^3 u_n=f_n,\quad n=1,2,...$$

Assume further that $||u_n||_{X^T_{b,0}}\le const$ and

$$||u_n||_{X^T_{-b,-2+2b}}+||f_n||_{X^T_{-b,-2+2b}}+||u_n||_{X^T_{-b',-1+2b'}}\to 0.$$

If $u_n\to 0$ in $L^2(0,T,L^2(\omega))$, then

$$u_n\to 0 \text{ in } L^2_{loc}(0,T,L^2(\mathbb{T})).$$

In practice, $b=1/2$, $b'=0$

Propagation of regularity

Assume $T > 0$, $\omega \subset \mathbb{T}$, $0 \leq b < 1$, $r \in \mathbb{R}$ and $f \in X^T_{-b,r}$. Let $u \in X^T_{b,r}$ solve

$$\partial_t u + \partial_x^3 u = f.$$

If $u \in L^2_{loc}(0, T, H^{r+\rho}(\omega))$ for some ρ with

$$0 < \rho \leq \min\{1 - b, \frac{1}{2}\}$$

Then

$$u \in L^2_{loc}(0, T, H^{r+\rho}(\mathbb{T})).$$

Corollary: Let $u \in X^T_{\frac{1}{2},0}$ solves $u_t + u_{xxx} + uu_x = 0$. Then

$$u \in C^\infty(\omega \times (0, T)) \quad \Rightarrow \quad u \in C^\infty(\mathbb{T} \times (0, T))$$

Rapid stabilization

Thm 3. Let $\lambda > 0$ and $s \geq 0$. There exists $\delta > 0$ and $K_\lambda \in \mathcal{L}(H^s(\mathbb{T}), H^s(\mathbb{T}))$ such that for $||u_0||_s \leq \delta$ and $[u_0] = 0$, the solution u of

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GK_\lambda u, \quad u(.,0) = u_0,$$

satisfies

$$||u(.,t)||_s \leq Ce^{-\lambda t}||u_0||_s.$$

$h = -K_\lambda u$ is the feedback law given by Slemrod [1974] for the linearized system.

Time-varying feedback law

- The feedback law $h = -Gu$ (resp. $h = -K_\lambda u$) yields a global (resp. local) exponential stabilization with a given (resp. arbitrary) decay rate.
- Aim: combine both feedback laws to obtain a **global** stabilization with an **arbitrary** decay rate.
- Idea: use the feedback law $h = -Gu$ far from 0 to get the global stabilization, and the feedback law $h = -K_\lambda u$ close do 0 to get a large decay rate.
- In practice: avoid discontinuous feedback laws (difficulty to define a solution!), and use a smooth (periodic) time-varying feedback law, coinciding successively on half periods with $-Gu$ and with $-K_\lambda u$ (at least close to 0).

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Time-varying feedback law (2)

- Pick a 2−periodic smooth function $\theta(t)$ with $\theta(t) = 1$ on $[\delta, 1-\delta]$, $\theta(t) = 0$ on $[1,2]$, and a smooth function $\rho(r)$ with $\rho(r) = 1$ for $0 \le r \le r_0 < 1$, $\rho(r) = 0$ for $r \ge 1$
- Set

$$-K(u,t) := \rho(||u||_s)[\theta(t/T)K_\lambda u + \theta((t-T)/T)\, Gu] + (1-\rho(||u||_s))\, Gu$$

- For $||u||_s > 1$, $K(u,t) = -Gu$
  For $||u||_s < r_0$

$$K(u,t) = \begin{cases} -K_\lambda u & \text{if} \quad \delta T \le t \le (1-\delta)T \mod 2T \\ -Gu & \text{if} \quad (1+\delta)T \le t \le (2-\delta)T \mod 2T. \end{cases}$$

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Time-varying feedback law (3)

Thm 4: Let $\lambda > 0$ be given. There exists $T_0 > 0$ such that for $T > T_0$, for each pair (t_0, u_0) with $[u_0] = 0$, the solution of

$$u_t + uu_x + u_{xxx} = GK(u,t), \quad u(.,t_0) = u_0$$

satisfies

$$||u(.,t)||_s \le \alpha(||u_0||_s)e^{-\frac{1}{2}\lambda(t-t_0)}||u_0||_s, \qquad \text{for } t \ge t_0.$$

Future directions of research

- **Duration** of the control process: can two states u_0 and u_1 be connected by a trajectory of KdV in arbitrarily small time T? If not, how T is related to the magnitude of $||u_0||_s$ and $||u_1||_s$? Same question for NLS.
- Can we design a smooth **time-invariant** feedback law yielding a global stabilization with an arbitrarily large decay rate?
- $x \in (0, +\infty)$. Controllability (open for KdV) R. [2000]; Stabilization: Linares-Pazoto [2009], R.-Pazoto [preprint]: "size" of the support of the controler $a(x)$ in the feedback term $a(x)u$.
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Thank you for your attention!