Quasivelocities and Optimal Control for Underactuated Mechanical Systems

Leonardo Colombo

Instituto de Ciencias Matemáticas(CSIC-UAM-UC3M-UCM)

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(Work in progress)

This is a joint work with David Martin de Diego

Motivation

- J.Cortés, M. de León, S. Martínez, D. Martín de Diego, Geometric description of vakonomic and nonholonomic dynamics. Comparison of solutions, SIAM J. Control Optim. 41, 1389-1412 (2003).
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- M. Barbero-Liñán, A.Echeverría Enríquez, D. Martín de Diego, M.C Muños-Lecanda and N. Román-Roy. Skinner-Rusk unified formalism for optimal control systems and applications. J. Phys. A: Math Theor. 40 (2007), 12071-12093.

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- Gotay, M. and Nester, J. Presymplectic Lagrangian Systems I. The constraint algorithm and the equivalence theorem, Ann. I.H.P Phys. Theor. 30, 129-142 (1979).

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- Gotay, M. and Nester, J. Presymplectic Lagrangian Systems I. The constraint algorithm and the equivalence theorem, Ann. I.H.P Phys. Theor. 30, 129-142 (1979).
- L.Colombo, D.Martín de Diego, M. Zuccalli. Optimal Control for Underactuated Mechanical Systems: A Geometry Approach, Preprint(2009)

Outline

1. Introducction
2. Quasivelocities
3. Optimal Control for Underactuated Mechanical Systems

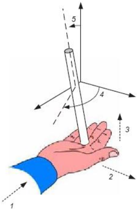
Introduction

Underactuated Mechanical Systems

A Control System is *underactuated* if the number of the control inputs is less than the dimension of the configuration space.

For example

To balance a cylindrical rod on your hand.



Introduction

1. Q configuration space,
2. TQ Velocity space,
3. Lagrangian $L : TQ \to \mathbb{R},$ diferentiable function
4. Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0,$
5. Hamilton equations $\frac{\partial H}{\partial q^i} = -\dot{p}_i, \ \frac{\partial H}{\partial p_i} = \dot{q}^i,$
6. 2-Cartan form $\Omega_L := -d\Theta_L = dq^i \wedge dp_i,$
7. Energy $E_L := \dot{q}^i p_i - L,$
8. Dynamic equations $i_X \Omega_L = dE_L.$

Introduction

Optimal Control for Underactuated Mechanical Systems in standard coordinates

1. Configuration space $Q = Q_1 \times Q_2$
2. Velocity space $TQ = TQ_1 \times TQ_2$
3. Coordinates $(q^A) = (q^a, q^\alpha)$, $1 \leq A \leq n$ in Q; (q^a), $1 \leq a \leq r$, y (q^α), $r+1 \leq \alpha \leq n$ coordinates in Q_1 and Q_2 respectively.
4. Lagrangian $L : TQ \to \mathbb{R}$ regular.

Euler-Lagrange equations with control

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} &= u^a, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} &= 0 \end{aligned} \tag{1}$$

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Disadvantages

1. No included external forces
2. No included control forces

Quasivelocities

- Q, configuration space n-dimensional
- (q^A) coordinates in Q
- $\{X_B\}$ local basis of vector fields defined in the same coordinate neighbourhood.

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Let $(y^1, ..., y^n)$ (the **quasivelocities**) be the components of a velocity vector v on TQ relative to the basis X_B, then

$$v = y^B X_B(q) = y^B X_B^A(q)\frac{\partial}{\partial q^A},$$

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$$v = y^B X_B(q) = y^B X_B^A(q)\frac{\partial}{\partial q^A},$$

therefore, $\dot{q}^A = y^B X_B^A(q)$.

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The Euler-Lagrange equations in quasivelocities or Hamel equations

$$\begin{aligned} \dot{q}^A &= y^B X_B^A(q) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^A}\right) &= \frac{\partial L}{\partial q^B} X_B^A - \mathcal{C}_{AB}^D y^B \frac{\partial L}{\partial y^D} \end{aligned}$$

Optimal Control for Underactuated Mechanical Systems

In standard local coordinates the control equations that we will are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = F_A + u_a \overline{X}^a_A$$

where $F = F^A(q,\dot{q})dq^A$ represents given external forces and $\overline{X}^a = \overline{X}^a_A(q)dq^a$, $1 \leq a \leq m \leq n$, the control forces.

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Complete with 1-form $\overline{X}^\alpha$ to local basis $\{\overline{X}^a, \overline{X}^\alpha\}$ of $\Lambda^1 Q$ and take its dual basis that we denote by $\{X_a, X_\alpha\}$.

Optimal Control for Underactuated Mechanical Systems

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Lagrangian Control Equation in Quasivelocities

$$\begin{aligned}
\dot{q}^A &= y^B X_B^A(q) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) - \frac{\partial L}{\partial q^B} X_a^B + \mathcal{C}_{aB}^D y^B \frac{\partial L}{\partial y^D} &= F_A X_a^A + u_a, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) - \frac{\partial L}{\partial q^B} X_\alpha^B + \mathcal{C}_{\alpha B}^D y^B \frac{\partial L}{\partial y^D} &= F_A X_\alpha^A.
\end{aligned}$$

Optimal Control Problem for Underactuated Mechanical Systems

Problem

The optimization problem deals with the problem of finding a control law for the system such that a certain optimality criterion is achieved. Usually, the optimization criterion is given by a cost functional of the type

$$\mathcal{A} = \int_{t_0}^{t_f} C(q^A(t), y^A(t), u_a(t))dt.$$

Optimal Control Problem for Underactuated Mechanical systems

This optimal control problem is equivalent to the following constrained variational problem

$$\text{Minimize } \overline{\mathcal{A}} = \int_{t_0}^{t_f} \widetilde{L}\left(q^A(t), y^A(t), \dot{y}^A(t)\right) dt$$

subject to constraints

$$\Phi^{\alpha}(q^A, y^A, \dot{y}^A(t)) = \frac{d}{dt}\left(\frac{\partial L}{\partial y^{\alpha}}\right) - \frac{\partial L}{\partial q^B} X_{\alpha}^B + \mathcal{C}_{\alpha B}^D y^B \frac{\partial L}{\partial y^D} - F_A X_{\alpha}^A = 0,$$

where $\widetilde{L}$ is defined as

$$\widetilde{L}(q^A, y^A, \dot{y}^A) = C\left(\frac{d}{dt}\left(\frac{\partial L}{\partial y^{\alpha}}\right) - \frac{\partial L}{\partial q^B} X_{\alpha}^B + \mathcal{C}_{\alpha B}^D y^B \frac{\partial L}{\partial y^D} - F_A X_{\alpha}^A\right).$$

Optimal Control for Underactuated Mechanical Systems

Geometrically, we have that $(q^A, y^A, \dot{y}^A)$ are coordinates on $T^{(2)}Q$, and the constraints Φ^α determines a submanifold $\mathcal{M}$ of $T^{(2)}Q$ and $\widetilde{L}$ is a lagrangian function also defined in $T^{(2)}Q$, that is $\widetilde{L} : T^{(2)}Q \to \mathbb{R}$.

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The canonical inmersion $j_2 : T^{(2)}Q \to T(TQ)$ in the induced coordinates $(q^A, y^A, \dot{y}^A)$ is

$$
\begin{aligned}
T^{(2)}Q &\to TTQ \\
(q^A, y^A, \dot{y}^a) &\mapsto (q^A, y^A, X^A_B y^B, \dot{y}^a)
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$$\begin{array}{rl} T^{(2)}Q \to TTQ & \\ (q^A, y^A, \dot{y}^a) & \mapsto (q^A, y^A, X_B^A y^B, \dot{y}^a) \end{array}$$

Assume that the matrix $\left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)_{1\leq\alpha,\beta\leq n-m}$ is regular, then we

can rewrite the constraints in the form $\dot{y}^\beta = G^\alpha(q^A, y^A, \dot{y}^a)$ and, coordinates $(q^A, y^A, \dot{y}^a)$ on $\mathcal{M}$.

Optimal Control for Underactuated Mechanical Systems

Let us define $\widetilde{L}_{\mathcal{M}}$ by $\widetilde{L}_{\mathcal{M}} = \widetilde{L}\mid_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ and consider $W_0 = \mathcal{M} \times_{TQ} T^*TQ$ with induced coordinates $(q^A, y^A, \dot{y}^a, p_A, \widetilde{p}_A)$.

Let us define the 2-form $\Omega = pr_2^*(\omega_{TQ})$ on W_0, where ω_{TQ} is the canonical symplectic form on T^*TQ, and $\widetilde{H}(v_x, \alpha_q) = \langle \alpha_x, (j_2)\mid_{\mathcal{M}}(v_x)\rangle - \widetilde{L}_{\mathcal{M}}(v_x)$ where $x \in TQ, v_x \in \mathcal{M}_{\S}(\tau_{TQ}\mid_{\mathcal{M}})^{-1}(x)$ and $\alpha_x \in T^*_xTQ$.

In coordinates

$$\Omega = dq^A \wedge dp_A + dy^A \wedge d\widetilde{p}_A,$$

$$\widetilde{H} = p_A X^A_B(q) y^B + \widetilde{p}_a \dot{y}^a + \widetilde{p}_\alpha G^\alpha(q^A, y^A, \dot{y}^a) - \widetilde{L}_{\mathcal{M}}(q^A, y^A, \dot{y}^a).$$

Optimal Control for Underactuated Mechanical Systems

The dynamics of this variational constrained problem is determining by the solution of the equation

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Following the Gotay-Nester-Hinds algorithm we obtain the primary constraints $d\widetilde{H}\left(\frac{\partial}{\partial \dot{y}^a}\right) = 0$, that is

$$\varphi_a = \frac{\partial \widetilde{H}}{\partial \dot{y}^a} = \widetilde{p}_a + \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial \dot{y}^a} - \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a} = 0$$

Optimal Control for Underactuated Mechanical Systems

The dynamics is restricted to the manifold W_1 determined by the vanishing of the constraints $\varphi_a = 0$. Observe that $\dim W_1 = 4n$ with induced coordinates $(q^A, y^A, \dot{y}^a, p_A, \widetilde{p}_a)$.

A curve solution of dynamic equations must verify the following system of diferential equations

Optimal Control for Underactuated Mechanical Systems

$$\frac{dq^A}{dt} = X_B^A(q(t))y^B(t) \quad (2)$$

$$\frac{dy^\alpha}{dt} = G^\alpha(q^A(t), y^A(t), \frac{dy^a}{dt}) \qquad \frac{dy^a}{dt} = \dot{y}^a(t) \quad (3)$$

$$\begin{aligned}\frac{dp_A}{dt} &= -p_C(t)\frac{\partial X_B^C}{\partial q^A}(q(t))y^B(t) - \widetilde{p}_\alpha(t)\frac{\partial G^\alpha}{\partial q^A}(q^B(t), y^B(t), \dot{y}^b) \\ &\quad + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial q^A}(q^B(t), y^B(t), \dot{y}^b(t))\end{aligned} \quad (4)$$

$$\begin{aligned}\frac{d\widetilde{p}_A}{dt} &= -p_C(t)X_A^C(q(t)) - \widetilde{p}_\alpha(t)\frac{\partial G^\alpha}{\partial y^A}(q^B(t), y^B(t), \dot{y}^b) \\ &\quad + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^A}(q^B(t), y^B(t), \dot{y}^b)\end{aligned} \quad (5)$$

$$\widetilde{p}_a = -\widetilde{p}_\alpha\frac{\partial G^\alpha}{\partial \dot{y}^a} + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a} \quad (6)$$

From Equations (5) and (6) we deduce

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial \dot{y}^a}\right) = -p_C X_a^C - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial y^a} + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^a}$$

Differentiating with respect to time, replacing in the previous equality and using (4) we obtain the following equations system

Optimal Control for Underactuated Mechanical Systems

$$\frac{d^2}{dt^2}\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial \dot{y}^a}\right) - \frac{d}{dt}\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^a} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial y^a}\right)$$

$$+X_a^A\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial q^A} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial q^A}\right)\widetilde{p}_C y^B\left[X_a^D \frac{\partial X_B^C}{\partial q^D} - X_B^D \frac{\partial X_a^C}{\partial q^D}\right] = 0$$

$$\frac{d\widetilde{p}_\alpha}{dt} = -p_\alpha - \widetilde{p}_\beta \frac{\partial G^\beta}{\partial y^\alpha} + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^\alpha}$$

Optimal Control for Underactuated Mechanical Systems

$$\frac{d^2}{dt^2}\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial \dot{y}^a}\right) - \frac{d}{dt}\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^a} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial y^a}\right)$$

$$+X_a^A\left(\frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial q^A} - \widetilde{p}_\alpha \frac{\partial G^\alpha}{\partial q^A}\right)\widetilde{p}_C y^B\left[X_a^D \frac{\partial X_B^C}{\partial q^D} - X_B^D \frac{\partial X_a^C}{\partial q^D}\right] = 0$$

$$\frac{d\widetilde{p}_\alpha}{dt} = -p_\alpha - \widetilde{p}_\beta \frac{\partial G^\beta}{\partial y^\alpha} + \frac{\partial \widetilde{L}_{\mathcal{M}}}{\partial y^\alpha}$$

Let us consider the 2-form $\Omega_{W_1} = i^*_{W_1}\Omega$ where $i_{W_1} : W_1 \hookrightarrow W_0$ is the canonical inclusion.

Optimal Control for Underactuated Mechanical Systems

Theorem

(W_1, Ω_{W_1}) *is symplectic if and only if for any choise of local coordinates* $(q^A, y^A, \dot{y}^a, p_A, \widetilde{p}_A)$ *on* W_0

$$\det\left(\frac{\partial^2 \widetilde{L}_{\mathcal{M}}}{\partial \dot{y}^a \partial \dot{y}^b} - \widetilde{p}_\alpha \frac{\partial^2 G^\alpha}{\partial \dot{y}^a \partial \dot{y}^b}\right)_{(n-m)\times(n-m)} \neq 0.$$

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Thanks for voting me !!!