Vakonomic Constraints in Higher-Order Classical Field Theory

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1. Introduction

2. The Skinner-Rusk formalism in CFT

3. The Skinner-Rusk formalism in HOFT

4. Vakonomic constraints
Joint work with:

- Manuel de León, *ICMAT*
- David Martín de Diego, *ICMAT*
- Joris Vankerschaver, *CalTech*

C. M. Campos, M. de León, D. Martín de Diego, J. Vankerschaver. *Unambiguous formalism for higher-order Lagrangian field theories*  

C. M. Campos, M. de León, D. Martín de Diego.  
*Vakonomic constraints in higher-order field theories*  
Work in progress.
Introduction

Classically we have...

- The Euler-Lagrange equations: \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \)
- The Hamilton equations: \( \frac{\partial H}{\partial q_i} = -\dot{p}^i \), \( \frac{\partial H}{\partial p^i} = \dot{q}_i. \)
- The Cartan form: \( \Omega_L := -d\Theta_L = dq^i \wedge d\hat{p}_i. \)
- The Legendre transform: \( FL(q^i, \dot{q}^i) = (q^i, \hat{p}_i = \frac{\partial L}{\partial q^i}). \)
- There is an equivalence between the Lagrangian and Hamiltonian formalisms.
Question: ¿what can we do in degenerate cases?

Mark J. Gotay, James M. Nester, and George Hinds. 
*Presymplectic manifolds and the Dirac-Bergmann theory of constraints.*

Alternative: to combine the phase space and space of velocities.

Ray Skinner and Raymond Rusk.
*Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$.*

Adaptation: Classical Field Theory.

M. de León, J. C. Marrero, and D. Martín de Diego.
*A new geometric setting for classical field theories.*

General framework that recovers the well known tools of mechanics \((m = 1)\).

Nice description of the Euler-Lagrange equations.

Cartan forms may be obtained, but not canonically.

There is no well defined Legendre transform.

D. J. Saunders and M. Crampin.  
\textit{On the Legendre map in higher-order field theories}  
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Different approaches

3. Classical field theory (and higher-order): multisymplectic framework.

Huge literature on regard to these subjects.


Let $\pi : E \rightarrow M$ be a fiber bundle (dim $M = m$ and dim $E = m + n$).

$J^1\pi$ denotes its first prolongation, the first jet bundle.

$J^1\pi^\dagger$ denotes its affine dual.

Coordinates:
- $(x^i)$ for $M$, $d^m x = dx^1 \wedge \cdots \wedge dx^m$, $d^{m-1} x_i = i_{\partial_i} d^m x$,
- $(x^i, u^\alpha)$ for $E$,
- $(x^i, u^\alpha, u^\alpha_i)$ for $J^1\pi$,
- $(x^i, u^\alpha, p, p_i^\alpha)$ for $J^1\pi^\dagger$.

The Lagrangian function $L : J^1\pi \rightarrow \mathbb{R}$.

The mixed space of velocities and momenta: $W = J^1\pi \times_E J^1\pi^\dagger$.

The pairing $\Phi : w \in W \mapsto \langle pr_2(w), pr_1(w) \rangle = p + p_i^\alpha u^\alpha_i \in \mathbb{R}$.

The dynamical function $H = \Phi - L \circ pr_1$.

The premultisymplectic $(m + 1)$-form $\Omega = pr_2^* \Omega_{J^1\pi^\dagger}$.

The dynamical equation $i_X \Omega = dH$.

**Note:** we have not mentioned neither the Legendre transform, nor the Cartan form.
The multi-index notation

**Definition**

- A multi-index is an \( m \)-tuple \( I \in \mathbb{N}^m \) whose \( i \)-th component is \( I(i) \).
- The sum and “substraction” is defined componentwise
  \[(I \pm J)(i) = I(i) \pm J(i)\].
- The length of \( I \) is the sum \(|I| = \sum_i I(i)\), and its factorial \( I! = \prod_i I(i)! \).
- \( 1_i = (\delta^i_j) = (0, \ldots, 1, \ldots, 0) \).

The partial derivatives of a function \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) are

\[
f_I = \frac{\partial^{||I||} f}{\partial x^I} := \frac{\partial^{I(1)+I(2)+\cdots+I(m)} f}{\partial x_1^{I(1)} \partial x_2^{I(2)} \cdots \partial x_m^{I(m)}}.
\]

For instance, given \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \),

\[
f_{(2,1,0)} = \frac{\partial^3 f}{\partial x_1^2 \partial x_2} = f_{(1,1,0) + 1_1} = f_{(2,0,0) + 1_2}.
\]
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Let $\pi : E \longrightarrow M$ be a fiber bundle (dim $M = m$ and dim $E = m + n$).

$J^k\pi$ is the $k$th-jet bundle and $J^k\pi^\dagger$ is its affine dual.

Coordinates:
- $(x^i, u^\alpha_j)$ for $J^1\pi^\dagger$, where $|J| \leq k$,
- $(x^i, u^\alpha_i, p, p^I_{\alpha;i})$ for $J^1\pi$, where $|I| \leq k - 1$.

The Lagrangian function $L : J^k\pi \longrightarrow \mathbb{R}$.

The mixed space: $W := J^k\pi \times J^{k-1}\pi \ J^k\pi^\dagger$.

The pairing $\Phi(x^i, u^\alpha_i, u^\alpha_K, p_{\alpha;i}, p) = p_{\alpha;i}u^\alpha_i + p$.

The dynamical function $H := \Phi - L \circ pr_1$.

The canonical multisymplectic $(m + 1)$-form

$\Omega = -dp \wedge dm^x - dp_{\alpha;i}^I \wedge du^\alpha_i \wedge dm^{-1}x_i$.

The premultisymplectic $(m + 1)$-form $\Omega_H := \Omega + dH \wedge \eta$.

We look for Ehresmann conexions in the fiber bundle $\pi_W,M : W \longrightarrow M$ whose horizontal projector $h$ satisfies

$$i_h\Omega_H = (m - 1)\Omega_H.$$
Solving the dynamical equation

In first place, we restrict to the space where such solutions exist:

\[ W_1 := \{ w \in W / \exists h : T_w W \rightarrow T_w W \text{ lineal } \text{t.q. } h^2 = h, \quad \ker h = (V_{\pi_{W,M}})_w, \quad i_h \Omega_H(w) = (m - 1)\Omega_H(w) \} \]

The projectors must be of the form:

\[ h = \left( \frac{\partial}{\partial x^i} + A_{ji}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + B_{j\alpha}^{l,i} \frac{\partial}{\partial p_{l,j}^{\alpha}} + C_j \frac{\partial}{\partial p} \right) \otimes dx^i. \]

After some computation...
Solving the dynamical equation

... we finally obtain:

\[ B_{\alpha j}^j = \frac{\partial L}{\partial u_{\alpha j}}; \]  
\[ \sum_{l+1_i=J} p_{\alpha}^{l,i} = \frac{\partial L}{\partial u_{\alpha j}} - B_{\alpha j}^j, \text{ where } |J| = 1, \ldots, k - 1; \]  
\[ \sum_{l+1_i=K} p_{\alpha}^{l,i} = \frac{\partial L}{\partial u_{\alpha K}}, \text{ where } |K| = k; \]  
\[ A_{\alpha i}^i = u_{\alpha i}^{l+1}, \text{ where } |l| = 0, \ldots, k - 1. \]

Adding the condition \( H(w) = 0 \) (\( p = L - p^{l,i} u_{l+1_i} \)), we define

\[ W_2 := \{ w \in W_1 : H(w) = 0 \} = \{ w \in W : (3) \text{ and } H(w) = 0 \}. \]
Framework

\[ W = J^k\pi \times J^{k-1}\pi J^k\pi^\dagger \]

\[ W_2 = \left\{ w \in W : \sum_{l+1, i=K} p^{l,i}_\alpha = \frac{\partial L}{\partial u^K_\alpha}, \quad p = L - p^{l,i} u_{l+1,i} \right\} \]
Tangency conditions

\[ W_2 := \{ w \in W_1 : H(w) = 0 \} = \{ w \in W : (3) \text{ and } H(w) = 0 \} \]

Now, we have to guarantee that the solutions are tangent to \( W_2 \), that is
that \( h(T_w W) \subset i_\ast(TW_2) \forall w \in W_2 \), which is equivalent to the following equations

\[
\sum_{l+1_i=K} B_{\alpha j}^{li} = \frac{\partial^2 L}{\partial x^i \partial u^\alpha_K} + u_{l+1_j}^\beta \frac{\partial^2 L}{\partial u^\beta_l \partial u^\alpha_K} + A_{K'j}^{\alpha'} \frac{\partial^2 L}{\partial u^{\alpha'}_{K'} \partial u^\alpha_K}, \tag{4}
\]

\[
C_j = \frac{\partial L}{\partial x^j} + A_{j}^{\alpha} \frac{\partial L}{\partial u^{\alpha_j}} + A_{l+1_i j}^{\alpha} p_{\alpha}^{l,i} + B_{\alpha j}^{li} u_{l+1_i}^{\alpha}.
\]

where \( |K| = k \).
The higher-order Euler-Lagrange equations

Let consider an Ehresmann connexion in $\pi_{WM} : W \longrightarrow M$ along $W_2$ whose horizontal projector $\mathbf{h}$ is a solution of the dynamical equation

$$i_\mathbf{h} \Omega_H = (m - 1)\Omega_H.$$

Theorem

Let $\sigma$ be a section of $\pi_{W_2M} : W_2 \longrightarrow M$ and denote $\bar{\sigma} = i \circ \sigma$ and $\phi = \pi_{W_2E} \circ \sigma$. If $\sigma$ is an integral section of $\mathbf{h}$, then $\sigma$ is holonomic,

$$pr_1 \circ \bar{\sigma} = j^k \phi,$$

and satisfies the higher-order Euler-Lagrange equations:

$$j^{2k} \phi^* \left( \sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u^\alpha_J} \right) = 0.$$
Some results

Theorem

Let $\Gamma$ be a connexion in $\pi_{WM} : W \to M$ along $W_2$ whose horizontal projector $h$ satisfies

$$i_h \Omega_H = (m - 1) \Omega_H.$$

The integral sections of $\Gamma$ satisfy the DeDonder equations.

Theorem

$(W_2, \Omega_2)$ is multisymplectic iff $L$ is regular ($\det \left( \frac{\partial^2 L}{\partial u_\alpha^K \partial u_\alpha'^{K'}} \right) \neq 0$).
Some results

Consider the equations in which appear $B^l_j$'s with $|l| = k - 1$ (equations (2) y (4)),

$$B^l_j = \frac{\partial L}{\partial u^\alpha_j} - \sum_{l+1_i=J} p^{l,i}_{\alpha},$$

where $|J| = k - 1$;

$$\sum_{l+1_i=K} B_{\alpha j} = \frac{\partial^2 L}{\partial x^j \partial u^\alpha_K} + u^\beta_{l+1_j} \frac{\partial^2 L}{\partial u^\beta_l \partial u^\alpha_K} + A_{K'j}^{\alpha'} \frac{\partial^2 L}{\partial u^{\alpha'}_{K'} \partial u^\alpha_K},$$

where $|K| = k$.

This is a linear system of equations in the $B$'s which is

- overdetermined when $k = 1$ or $m = 1$,
- completely determined when $k = m = 2$,
- not determined otherwise.

**Theorem**

*If $k, m \geq 2$, the above system has maximal rank.*
Some results

Theorem

*If* $k, m \geq 2$, the above system has maximal rank.

Case $k = 2$ and $m = 3$: 11 equations with 12 unknowns.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Case $k = 5$ and $m = 6$: 1638 equations with 4536 unknowns.
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Introducing constraints

- The Lagrangian function $L : J^k \pi \longrightarrow \mathbb{R}$.
- The constraint submanifold $C = \{ \psi^\mu = 0 \} \hookrightarrow J^k \pi$, $1 \leq \mu \leq l$.
- The mixed space $W = J^k \pi \times J^{k-1} \pi \cdot J^{k \pi \dagger}$.
- The restricted mixed space $W^C_0 = \{ w \in \text{pr}_1^{-1}(C) : H(w) = 0 \}$.

**Theorem**

Given $w \in W^C_0$ and $X \in \Lambda^m_d(T_w W^C_0)$, let $\bar{X} = i_* X \in \Lambda^m_d(T_w W)$. We have

$$i_X \Omega^C_0 = 0 \iff i_{\bar{X}} \Omega \in T^0_w W^C_0,$$

where $T^0_w W^C_0$ is the annihilator of $i_* (T_w W^C_0)$ in $T_w W$.

We look for solutions of the equation

$$(-1)^m i_{\bar{X}} \Omega = \lambda_\mu d\psi^\mu + \lambda dH,$$

with $i_{\bar{X}} \eta = 1$. 
Solving the vakonomic dynamical equation

Let \( \bar{X} = \bigwedge_j \left( \frac{\partial}{\partial x^j} + A^\alpha_{ji} \frac{\partial}{\partial u^\alpha_j} + B^l_{\alpha j} \frac{\partial}{\partial p^l_{\alpha j}} + C_j \frac{\partial}{\partial p} \right) \). We obtain that \( \lambda = -1 \), besides the relations of holonomy, dynamics and tangency

\[
\begin{align*}
A^\alpha_{i i} & = u^\alpha_{i+1, i} \\
0 & = \lambda \frac{\partial \psi^\mu}{\partial u^\alpha} + \frac{\partial L}{\partial u^\alpha} - B^j_{\alpha j}, \ |J| = 0 \\
\sum_{l+1, i = J} p^l_{\alpha i} & = \lambda \frac{\partial \psi^\mu}{\partial u^\alpha_j} + \frac{\partial L}{\partial u^\alpha_j} - B^j_{\alpha j}, \ |J| = 1, \ldots, k - 1 \\
\sum_{l+1, i = K} p^l_{\alpha i} & = \lambda \frac{\partial \psi^\mu}{\partial u^\alpha_k} + \frac{\partial L}{\partial u^\alpha_k}, \ |K| = k \\
C_i & = \lambda \left( \frac{\partial \psi^\mu}{\partial x^i} + A^\alpha_{i i} \frac{\partial \psi^\mu}{\partial u^\alpha_i} \right) + \frac{\partial L}{\partial x^i} + A^\alpha_{i i} \frac{\partial L}{\partial u^\alpha_i} + \ldots
\end{align*}
\]
Getting rid of the constraints

Suppose that the constraints are of maximal order, $\Psi^\mu = u^\alpha_{\hat{K}} - \Phi^\alpha_{\hat{K}} = 0$. With a right manipulation on the relations of dynamics, we obtain

$$0 + \sum_{l+1_i=\hat{K}} p^l_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha} = \frac{\partial \tilde{L}}{\partial u^\alpha} - B^j_{\alpha j}, \ |J| = 0$$

$$\sum_{l+1_i=J} p^l_\alpha + \sum_{l+1_i=\hat{K}} p^l_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha_j} = \frac{\partial \tilde{L}}{\partial u^\alpha_j} - B^{jj}_{\alpha j}, \ |J| = 1, \ldots, k - 1$$

$$\sum_{l+1_i=\bar{K}} p^l_\alpha + \sum_{l+1_i=\hat{K}} p^l_\beta \frac{\partial \Phi^\beta_{\hat{K}}}{\partial u^\alpha_{\bar{K}}} = \frac{\partial \tilde{L}}{\partial u^\alpha_{\bar{K}}}, \ |\bar{K}| = k$$

where $\tilde{L}$ is the restricted Lagrangian.
Getting rid of the constraints

More generally, if $\Psi^\mu = u^\mu - \Phi^\mu = 0$, with a right manipulation on the relations of dynamics, we obtain

$$
\sum_{\nu+1_i=\bar{\mu}} p_\nu + \sum_{\nu+1_i=\mu} p_\nu \frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}} = \frac{\partial \tilde{L}}{\partial u^{\bar{\mu}}} + \frac{\partial L}{\partial u^\mu} \frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}} - B^j_{\bar{\mu}j} - B^j_{\mu j} \frac{\partial \Phi^\mu}{\partial u^{\bar{\mu}}},
$$

where $\tilde{L}$ is the restricted Lagrangian.
What we left behind and what is ahead

Conclusions:

- Global framework for field theory.
- There is no well defined Legendre transform or Cartan form.
- The reduction algorithm stops inevitably.
- The cases $k = 1$ (first order), $m = 1$ (mechanics) and $k = m = 2$ are special.

Future work:

- Control.
- Geometric discretization and integration.
- Space plus time decomposition.
- Jets of infinite dimension, $J^\infty \pi$. 
D. J. Saunders.  
*The geometry of jet bundles.*  

C. M. Campos, M. de León, D. Martín de Diego, J. Vankerschaver.  
*Unambiguous formalism for higher-order Lagrangian field theories*  

*On the multisymplectic formalism for first order field theories*  

J. Cortés, M. de León, D. Martín de Diego and S. Martínez.  
*Geometric description of vakonomic and nonholonomic dynamics*  
Lagrangian-Hamiltonian unified formalism for field theory.  

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Presymplectic manifolds and the Dirac-Bergmann theory of constraints.  

X. Gràcia, J. M. Pons, and N. Román-Roy.  
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M. de León, J. C. Marrero, and D. Martín de Diego.  
*A new geometric setting for classical field theories.*  

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*Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$.*  

L. Vitagliano.  
*The Lagrangian-Hamiltonian formalism for higher order field theories*  
In the end

- Yes, we can.
  (Barack Obama)

- ... do so may things. What are we waiting for?

Thank you for your attention and for your votes!