Conformal metrics of constant curvature on planar domains

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The Liouville equation

Consider the following classical nonlinear problem:

\[
\begin{cases}
\Delta u + f(u) = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : t > 0\}, \\
\frac{\partial u}{\partial t} = g(u) & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
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Here, we will make the following choices:

\[
\begin{align*}
\Delta u + 2Ke^u &= 0 \quad \text{in } \mathbb{R}^2_+, \\
\frac{\partial u}{\partial t} &= -2\kappa e^{u/2} \quad \text{on } \partial \mathbb{R}^2_+ \equiv \mathbb{R}, \quad K, \kappa \in \mathbb{R}.
\end{align*}
\]

The equation \( \Delta u + 2Ke^u = 0 \) is called the \textit{Liouville equation}. 
Geometrical interpretation

A conformal metric \( ds^2 = e^u(dx^2 + dy^2) \) on a planar domain \( \Omega \subset \mathbb{R}^2 \) satisfies

\[ \Delta u + 2K e^u = 0, \]

where \( K \) is the Gaussian curvature of \( ds^2 \).

The Liouville equation describes conformal metrics of constant curvature \( K \) on planar domains.
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$$\Delta u + 2Ke^u = 0,$$

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NOTE: the Liouville equation is conformally invariant: if $u$ is a solution and $\Phi$ is a regular conformal map on $\mathbb{R}^2$, then $u \circ \Phi$ is also a solution.
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The boundary condition $u_t = -2\kappa e^{u/2}$ on $\mathbb{R} \equiv \partial \mathbb{R}^2_+$ means that $ds^2$ has constant geodesic curvature $\kappa \in \mathbb{R}$ on the boundary.
The half-plane problem

constant curvature \( K \)

constant geodesic curvature \( \kappa \)
Previous results

(1) Y.Y. Li, M. Zhu (1995): Any solution to \((P)\) for \(K = 1\) with

\[
\int_{\mathbb{R}^2_+} e^u < +\infty, \quad \int_{\mathbb{R}} e^{u/2} < +\infty
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is a canonical solution.
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(3) Y.Y. Li, M. Zhu (1995), and Chipot-Shafir-Fila (1996): Any solution to
\[
\begin{cases}
\Delta u + a u^{\frac{n+2}{n-2}} = 0, & u > 0, \quad \text{in } \mathbb{R}^n_+,
\frac{\partial u}{\partial x_n} = c u^{\frac{n}{n-2}} & \text{on } \partial \mathbb{R}^n_+
\end{cases}
\]
is a canonical solution.
Our objectives...

(I) To solve problem (P) without additional hypotheses.
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(I) To solve problem \((P)\) without additional hypotheses.

(II) To solve the analogous problem in \(\mathbb{D}^*\):

\[
\begin{cases}
\Delta u + 2K e^u = 0 & \text{in } \mathbb{D}^* = \{z \in \mathbb{R}^2 \equiv \mathbb{C} : 0 < |z| < 1\}, \\
\frac{\partial u}{\partial \nu} = -2\kappa e^{u/2} + 2 & \text{on } S^1 = \{z : |z| = 1\}.
\end{cases}
\]
Other related theories

- Complex analysis.
- Minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{L}^3$.
- Constant mean curvature surfaces in $\mathbb{H}^3$ and $S^3_{1}$.
- Flat surfaces in $\mathbb{H}^3$ and $S^3_{1}$.
- Linear Weingarten surfaces.
The Neumann problem in $\mathbb{R}^2_+$
The Liouville equation and complex analysis

We fix $K \in \{-1, 1\}$ and identify $\mathbb{C} \equiv \mathbb{R}^2$ and $\mathbb{C}_+ \equiv \mathbb{R}^2_+$. 
The Liouville equation and complex analysis

We fix $K \in \{-1, 1\}$ and identify $\mathbb{C} \equiv \mathbb{R}^2$ and $\mathbb{C}_+ \equiv \mathbb{R}^2_+$.

**Liouville’s theorem:**
Solutions to $\Delta u + 2Ke^u = 0$ on $\Omega \subset \mathbb{C}$ simply connected are:

$$u = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}.$$ 

Here $g$ is meromorphic (holomorphic with $|g| < 1$ if $K = -1$) with $g' \neq 0$ (and conversely).

**Note:** the developing map $g$ gives a global isometric immersion of $(\Omega, e^u|dz|^2)$ into $\mathbb{Q}^2(K)$. 
The extension lemma

Let $u$ be a solution to

\[\begin{align*}
\Delta u + 2Ke^u &= 0 \quad \text{in } \mathbb{R}^2_+, \\
\frac{\partial u}{\partial t} &= -2\kappa e^{u/2} \quad \text{on } \partial \mathbb{R}^2_+ \equiv \mathbb{R}, \quad K, \kappa \in \mathbb{R}.
\end{align*}\]

(P)

Then it holds

\[u_{zz} - \frac{1}{2}u_z^2 = \{g, z\} := \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \quad (=: Q).\]

By the boundary condition, $\text{Im}Q = 0$ on $\mathbb{R}$.

By Schwarz’s reflection principle, $Q$ (and $g$) can be meromorphically extended to $\mathbb{C}$. 
The Cauchy problem for $\Delta u + 2Ke^u = 0$

The unique solution to the Cauchy problem

\[
\begin{align*}
\Delta u + 2Ke^u &= 0, \\
u(s, 0) &= a(s), \\
u_t(s, 0) &= d(s)
\end{align*}
\]

can be constructed as follows.
The Cauchy problem for $\Delta u + 2Ke^u = 0$

The unique solution to the Cauchy problem

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\begin{cases}
\Delta u + 2Ke^u &= 0, \\
u(s, 0) &= a(s), \\
\dot{u}(s, 0) &= d(s)
\end{cases}
\]

can be constructed as follows. Let $\alpha(s)$ be the unique curve in $Q^2(K)$ with

\[
v(s) = \int e^{a(r)/2} dr, \quad \kappa_g(s) = \frac{-d(s)}{2e^{a(s)/2}}.
\]

Let $g(s) := \pi(\alpha(s))$ denote its stereographic projection on $\bar{C}$, and extend it holomorphically to $g(z)$. Then,

\[
u = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}.
\]
The Cauchy problem for $\Delta u + 2Ke^u = 0$

The unique solution to the Cauchy problem

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can be constructed as follows. Let $\alpha(s)$ be the unique curve in $Q^2(K)$ with

$$v(s) = \int e^{a(r)/2} dr, \quad \kappa_g(s) = \frac{-d(s)}{2e^{a(s)/2}}.$$

If $u_t(s, 0) = -2\kappa e^{u(s, 0)/2}$, then $\kappa_g(s) = \kappa \equiv \text{constant} !!!
The solution for $K = 1$

Let $u : \mathbb{C}_+ \to \mathbb{R}$ be a solution to $(\mathbf{P})$ for $K = 1$. Then, its developing map $g$ is, over $\mathbb{R}$, of the form

$$g(s) = \alpha \exp \left( i \int_{\kappa}^{s} \mu(r) dr \right),$$

where

$$\alpha := \frac{\text{sgn}(\kappa)}{\sqrt{\kappa^2 + 1 - |\kappa|}}, \quad \mu(s) := \text{sgn}(\kappa) \sqrt{\kappa^2 + 1} e^{a(s)/2}.$$

By the properties of $g$ we see that $h(s) := 1/\mu(s)$ satisfies:

- $h(s) \neq 0$ and it can be extended to an entire function $h(z)$.
- $h(z)$ only has simple zeros with $h'(z_0) = \pm i$ at them.

(And conversely...)
The solution for $K = -1$ and $\kappa \geq 1$

If $\kappa > 1 \implies$ similar to $K = 1$.

If $\kappa = 1$, then $g(\mathbb{C}_+) \subset \mathbb{D}$ and

$$g(s) = \frac{h(s)}{h(s) + 2i}, \quad h(s) := \int_{s_0}^{s} e^{a(r)/2} dr, \quad a(s) := u(s, 0)$$
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$$g(s) = \frac{h(s)}{h(s) + 2i}, \quad h(s) := \int_{s_0}^{s} e^{a(r)/2} dr, \quad a(s) := u(s, 0)$$

$h(s)$ extends to $\mathbb{C}$ with $h(\mathbb{C}_+) \subset \mathbb{C}_+$ and $h(\mathbb{C}_-) \subset \mathbb{C}_-$.

By the little Picard theorem, $h(z) = h_0 z + h_1$ and so

$$u(s, t) = \log \left( \frac{h_0^2}{(1 + h_0 t)^2} \right).$$
\(K = -1\) and \(|\kappa| < 1\) is impossible

- \(g(\mathbb{C}_+) \subset \mathbb{D}\).
- \(g(\mathbb{R}) \subset C_\kappa\).
- \(g(\bar{z}) = J(g(z))\).

So, \(g(\mathbb{C})\) omits infinitely many points (Contradiction!).
$K = -1$ and $\kappa \leq -1$ is impossible

- $g(\mathbb{C}_+ \setminus \kappa) \subset \mathbb{D}$.
- $g(\mathbb{R}) \subset C_\kappa$.
- $g(\bar{z}) = J(g(z))$.

So, $g(\mathbb{C})$ omits infinitely many points (Contradiction!).
$K = -1$ and $\kappa \leq -1$ is impossible

- $g(\mathbb{C}_+) \subset \mathbb{D}$.
- $g(\mathbb{R}) \subset C_\kappa$.
- $g(\bar{z}) = J(g(z))$.

So, $g(\mathbb{C})$ omits infinitely many points (Contradiction!).
To sum up

We have obtained for the problem

\[
\begin{cases}
\Delta u + 2Ke^u = 0 & \text{in } \mathbb{R}_+^2, \\
\frac{\partial u}{\partial t} = -2\kappa e^{u/2} & \text{on } \partial \mathbb{R}_+^2 \equiv \mathbb{R}.
\end{cases}
\] \quad (P)

- If \( K = -1 \) and \( \kappa < 1 \) \( \Rightarrow \) the problem does not have a solution.
- If \( K = -1 \) and \( \kappa = 1 \) \( \Rightarrow \) the unique solution is

\[
u(s, t) = \log \left( \frac{h_0^2}{(1 + h_0 t)^2} \right).
\]

- In the remaining cases \( \Rightarrow \) there is an enormous family of solutions, all of which can be described by entire functions.
The punctured disc problem
Formulation of the problem

\[
\begin{align*}
\Delta u + 2Ke^u &= 0 \quad \text{in } \mathbb{D}^*, \\
\frac{\partial u}{\partial \nu} &= -2Ke^{u/2} + 2 \quad \text{on } \mathbb{S}^1, \quad K, \kappa \in \mathbb{R}.
\end{align*}
\]
Reduction to the half-plane case
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- Solutions to **(PDP)** come from $2\pi$-periodic solutions to **(P)**.
- In particular, **(PDP)** does not have a solution for $K = -1$ and $\kappa < 1$. 
Reduction to the half-plane case

- Solutions to (PDP) come from $2\pi$-periodic solutions to (P).
- In particular, (PDP) does not have a solution for $K = -1$ and $\kappa < 1$.

What are the finite area solutions to this problem?
The finite area case

**Theorem:** Any solution to (PDP) such that

\[ \int_{\mathbb{D}^*} e^u < \infty \]

is radially symmetric, i.e. \( u = u(r) \) where \( r = |z| \).
The finite area case

**Theorem:** Any solution to (PDP) such that

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\int_{\mathbb{D}^*} e^u < \infty
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is radially symmetric, i.e. \( u = u(r) \) where \( r = |z| \).

Moreover, all solutions can be explicitly given by simple expressions.

For instance, if \( K = 1 \) and \( \kappa \geq 0 \), then

\[
u(r) = 2 \log \frac{2R\beta r^{\beta-1}}{1 + R^2 r^{2\beta}},
\]

where

\[
R := \frac{1}{\sqrt{\kappa^2 + 1} - |\kappa|}.
\]
**Sketch of proof:** the solution $u$ on $\mathbb{D}^*$ is

$$u = \log \frac{4|G'(\zeta)|^2}{(1 + |G(\zeta)|^2)^2}, \quad G(\zeta) \text{ multivalued on } \mathbb{C}.$$

- $Q^* := \{G, \zeta\}$ is holomorphic on $\mathbb{D}^*$, and
  $$\text{Im}(\zeta^2 Q^*) = 0 \quad \text{on } \mathbb{S}^1.$$

- $G(\zeta) = \zeta^\alpha F(\zeta)$, $F$ single valued on $\mathbb{C}^*$. By the finite area assumption, $F$ is meromorphic at 0 (Chou-Wan, 1994).

- $Q^* = r_0/\zeta^2$ and so $G(\zeta) = \mathcal{M}(\zeta^\beta)$.

- By $|G(\zeta)| = R$ on $\mathbb{S}^1$, then $\mathcal{M}(\zeta) = Re^{i\theta} \zeta$ and the result follows.
An open problem
The half-plane problem with corners

Jost, Wang, Zhou \Rightarrow\text{ Classification for } K = 1 \text{ when } \int_{\mathbb{R}^2_+} e^u < \infty, \quad \int_{\mathbb{R}} e^{u/2} < \infty.