Entire maximal graphs

Worskshop of Geometry and Physics, Benasque 2009

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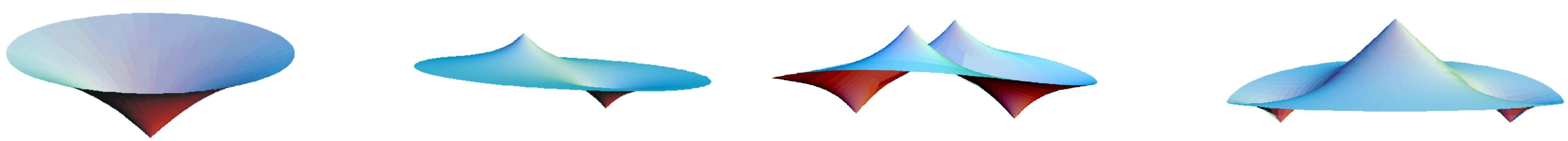
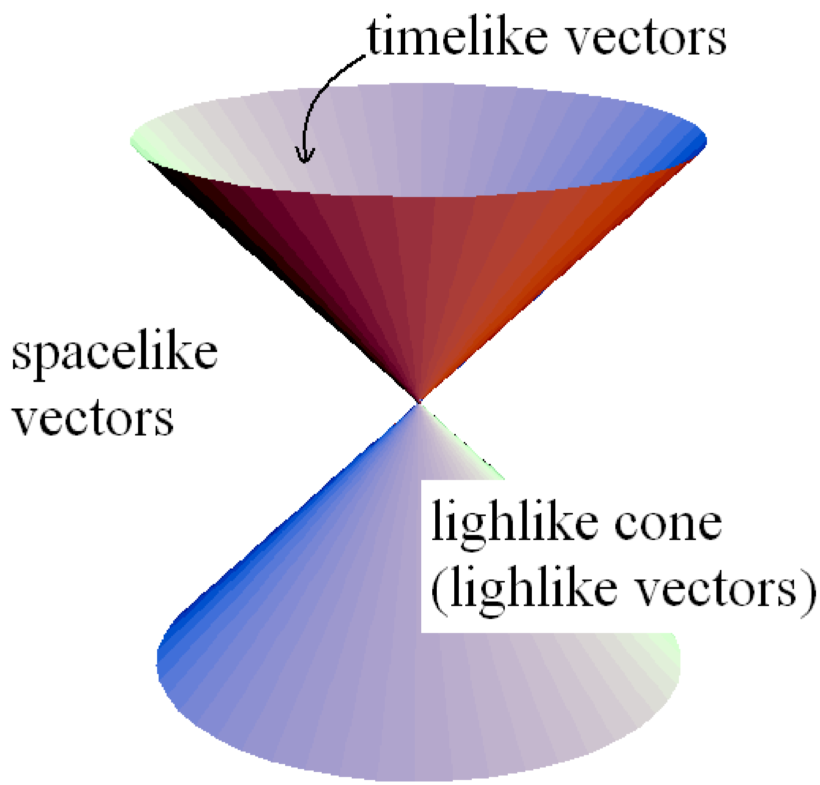


Set up

Maximal surfaces

- $\mathbb{L}^3 = (\mathbb{R}^3, ds^2 = dx^2 + dy^2 - dt^2)$ Lorentz-Minkowski space.



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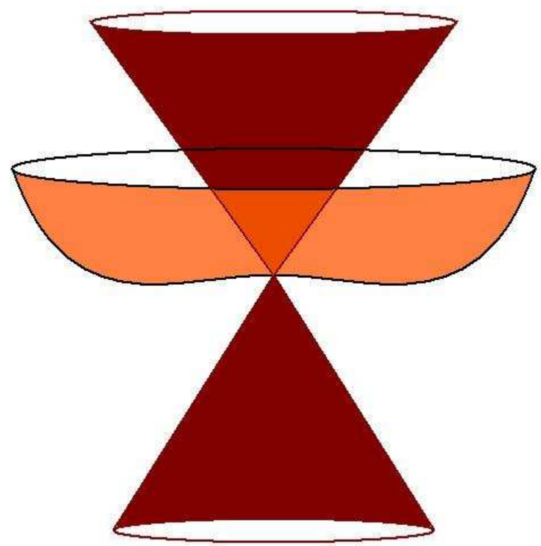
  - spacelike = the induced metric is Riemannian



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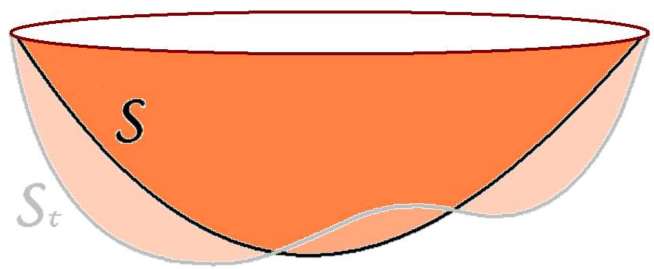
- $S \subset \mathbb{L}^3$ is a maximal surface if it is spacelike and is critical for the area.
  - critical for the area = for any 1-parameter deformation of S, the derivative of the area is zero.



Equivalently, the mean curvature is zero.

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- If S is the graph of a smooth function f, then S is maximal iff

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- Our aim: To study entire maximal graphs.
  (entire graph $=$ defined on the whole plane).

General facts

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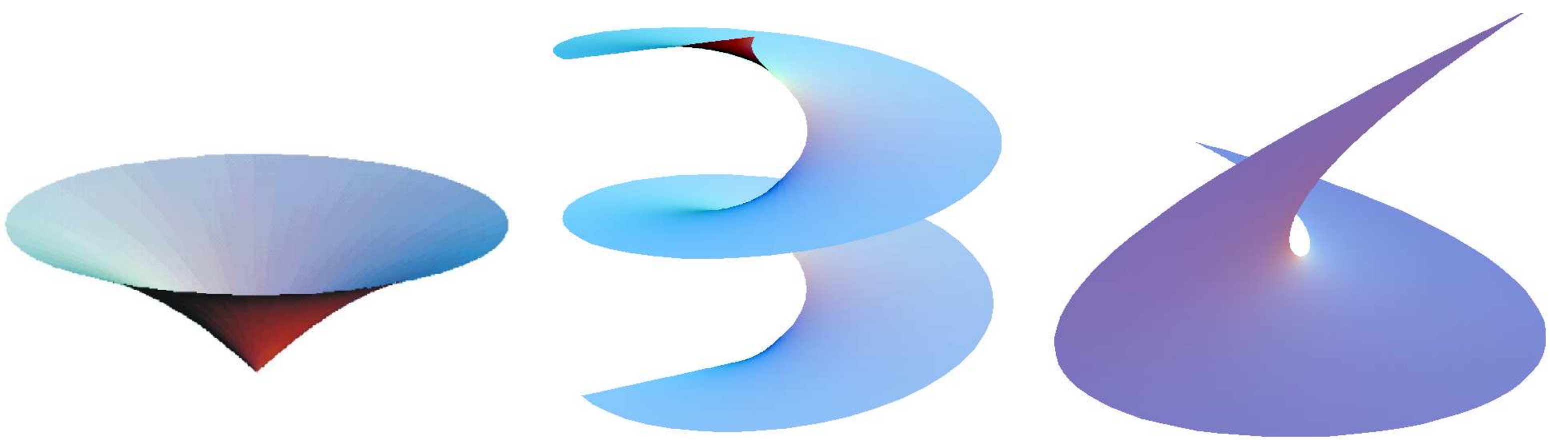

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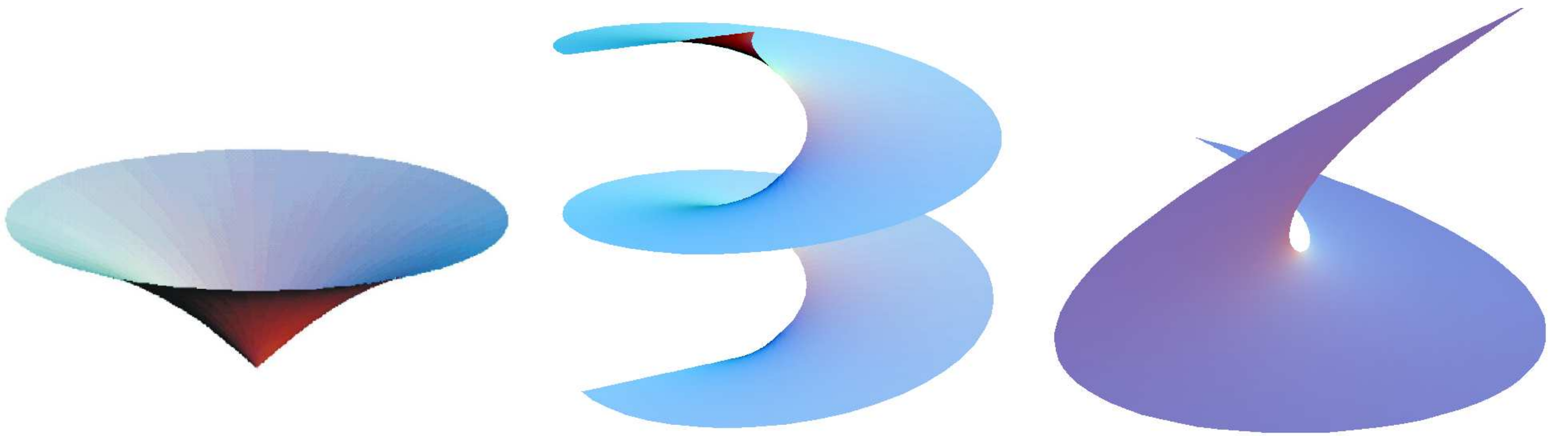
  In particular, there are no non-trivial entire maximal graphs.

$\rightsquigarrow$ Complete maximal surfaces have singularities (points where the metric degenerates).

- Our (new) aim: Study entire maximal graphs with the smallest set of singularities (a finite number of points).

Singularities of maximal surfaces

- There are different types of singularities.



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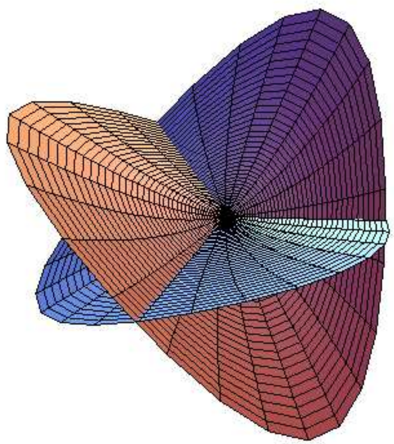
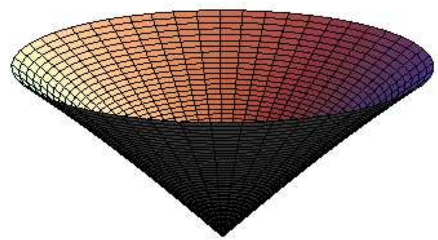
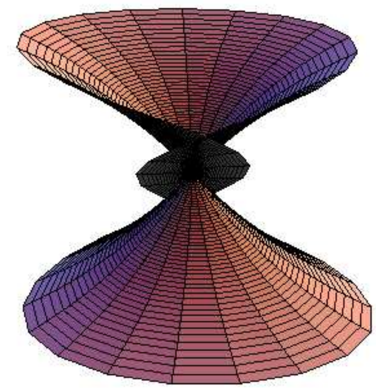
- Isolated singularities

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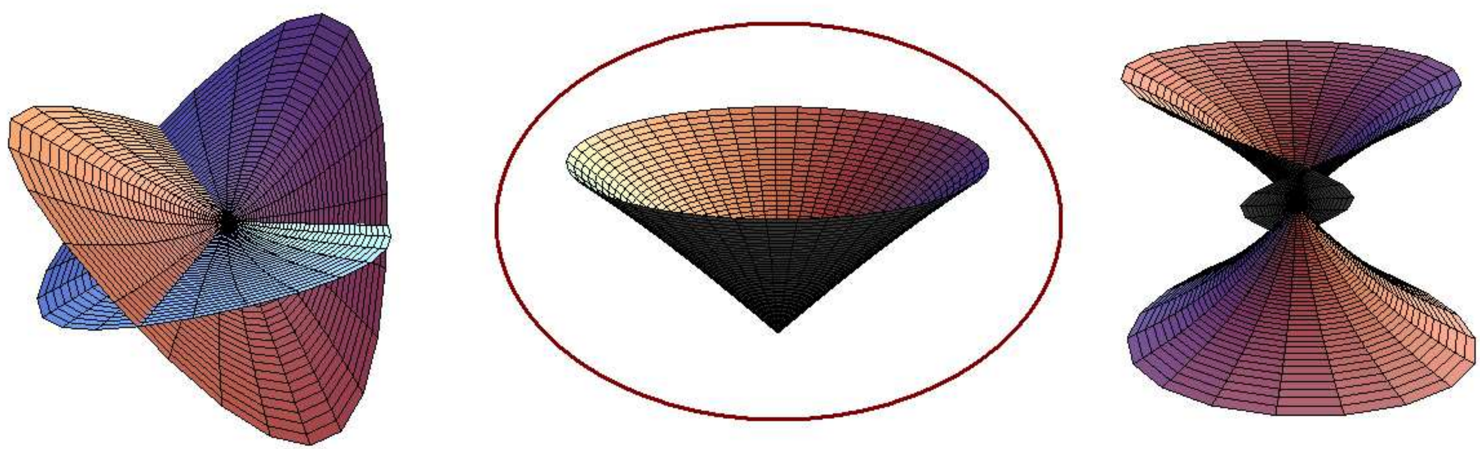


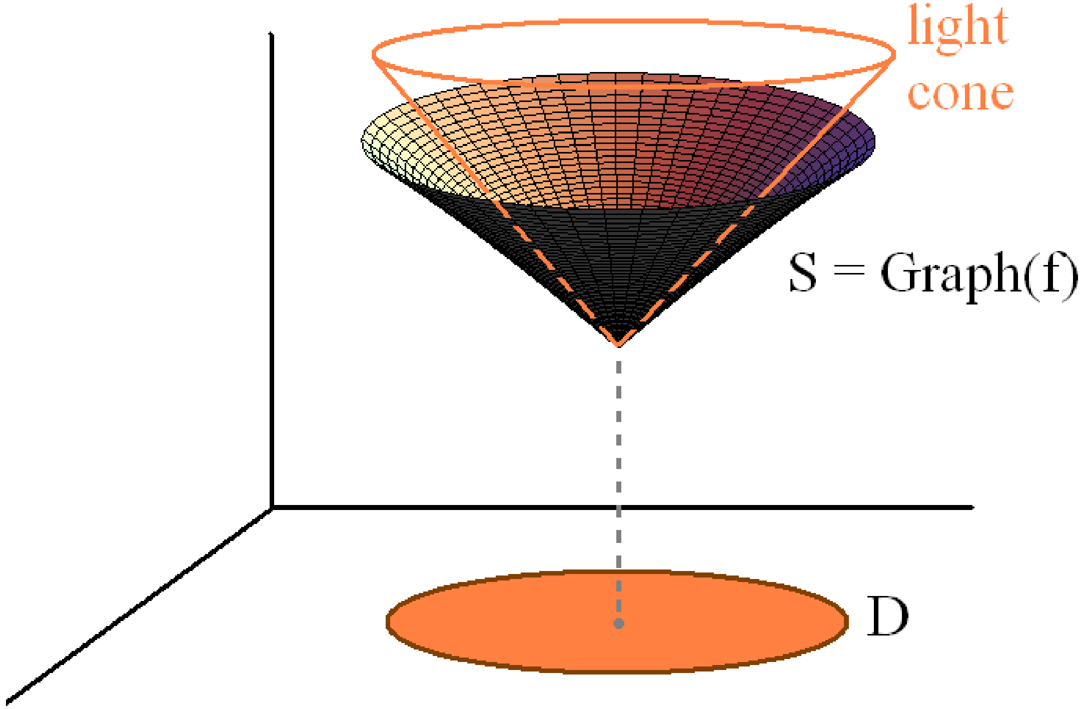
- Isolated embedded singularities $\equiv$ conelike singularities

Conelike singularities

- Local behavior:

Conelike singularities

- Local behavior: $\mathcal{S}$ maximal graph of a function f with a singular point, then
  - $|\nabla f| \longrightarrow 1$ at the singularity,
  - the normal vector becomes lightlike,
  - the surface is asymptotic to half of the light cone at the singularity.



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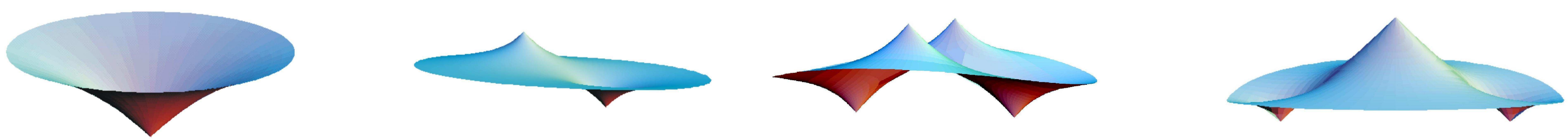
  - $|\nabla f| \to 1$ at the singularity,
  - the normal vector becomes lightlike,
  - the surface is asymptotic to half of the light cone at the singularity.
- Global behavior: Let $\mathcal{S} \subset \mathbb{L}^3$ be a maximal surface with a finite number of singular points. Are equivalent:
  - $\mathcal{S}$ is complete and embedded.
  - $\mathcal{S}$ is complete and its singularities are of conelike type.
  - $\mathcal{S}$ is an entire maximal graph.

  Moreover, $\mathcal{S}$ is asymptotic at infinity to either a half catenoid or a plane.

Entire graphs with a finite set of singularities



Conformal parametrizations

- A parametrization $X : \Omega \subset \mathbb{C} \to \mathcal{S} \subset \mathbb{L}^3$ is conformal if preserves the angles.
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- The conformal structure of entire maximal graphs with $n+1$ singularities is a n-connected circular domain

$$\Omega = \mathbb{C} \setminus \cup_{j=1}^{n+1} D_j, \qquad D_j = \text{disjoint open disks}$$

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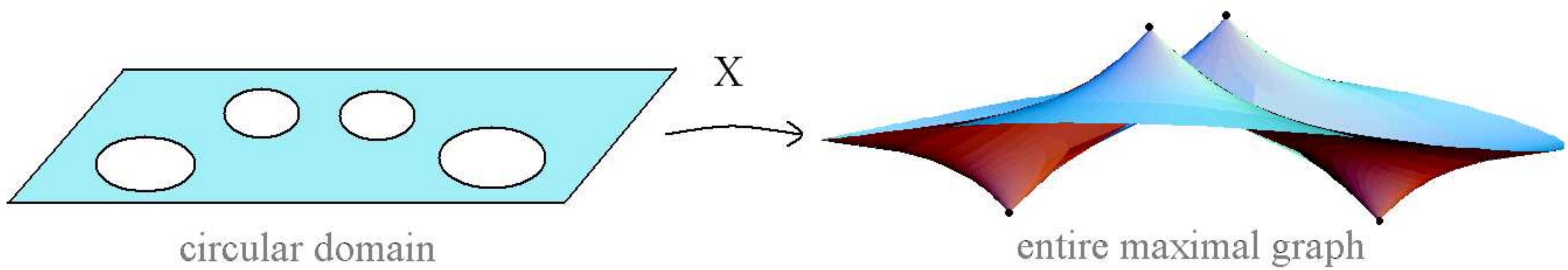

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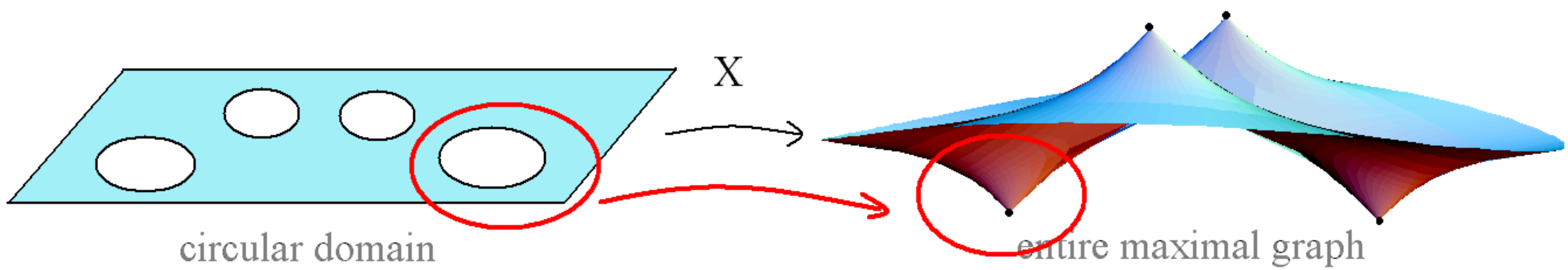

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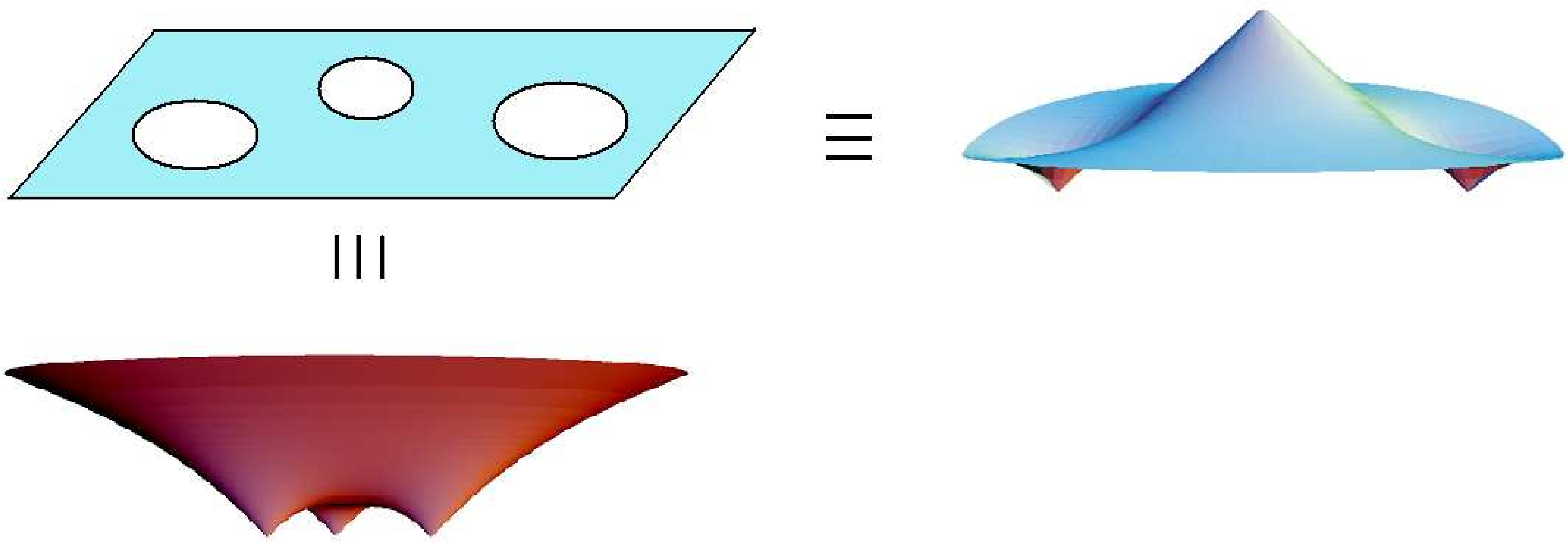
- Our (re-new) aim: Does the conformal structure determine the surface?

Main result

- Problem:
  Given a n-connected circular domain Ω, how many different entire maximal graphs are there with this conformal structure?



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  1. This number only depends on n.
  2. Compute this number for a specific circular domain.

Outline of the proof

Weierstrass data

Let $X : U \subset \mathbb{C} \to \mathcal{S} \subset \mathbb{L}^3$ a conformal parametrization of a regular maximal surface. Then X is of the form

$$X(z) = \text{Re} \int_{z_0}^{z} \left(\frac{i}{2}\left(\frac{1}{g} - g\right), \frac{-1}{2}\left(\frac{1}{g} + g\right), 1\right)\phi_3,$$

- ϕ_3 is a holomorphic 1-form ($\phi_3 = f(z)dz$), and
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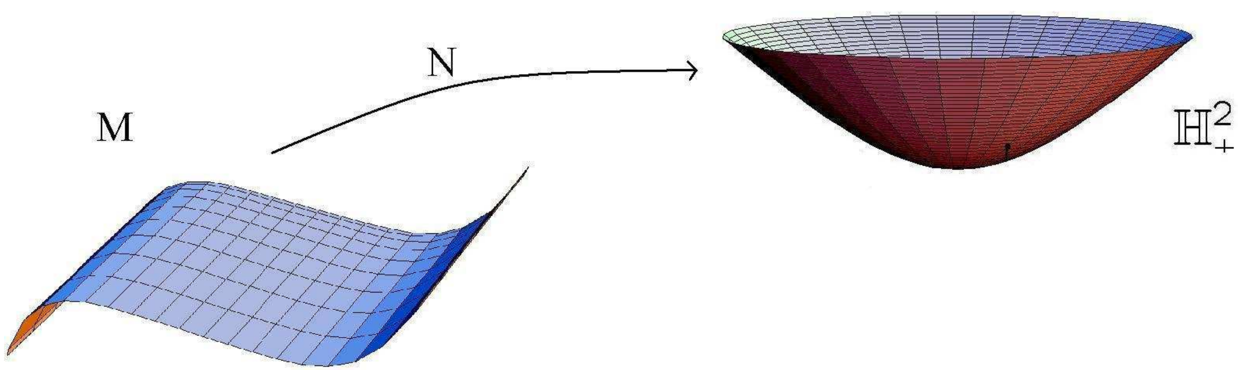

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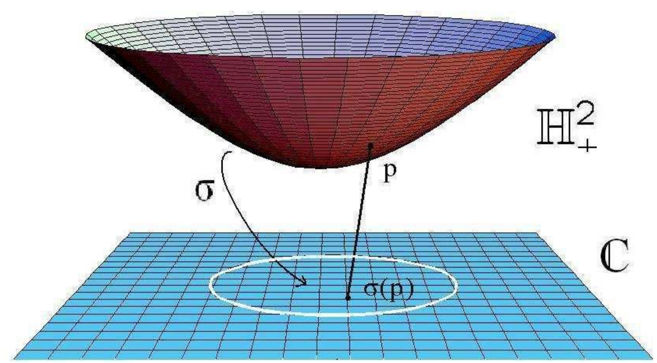

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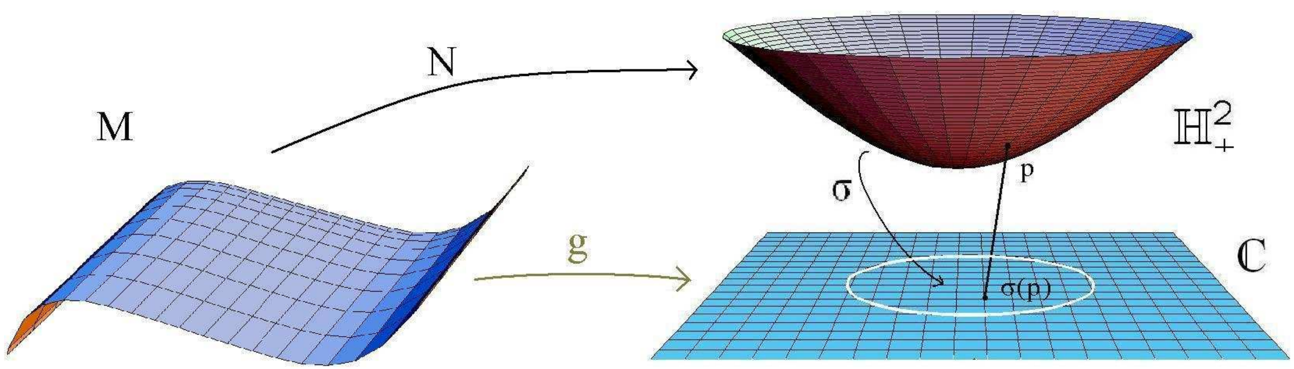
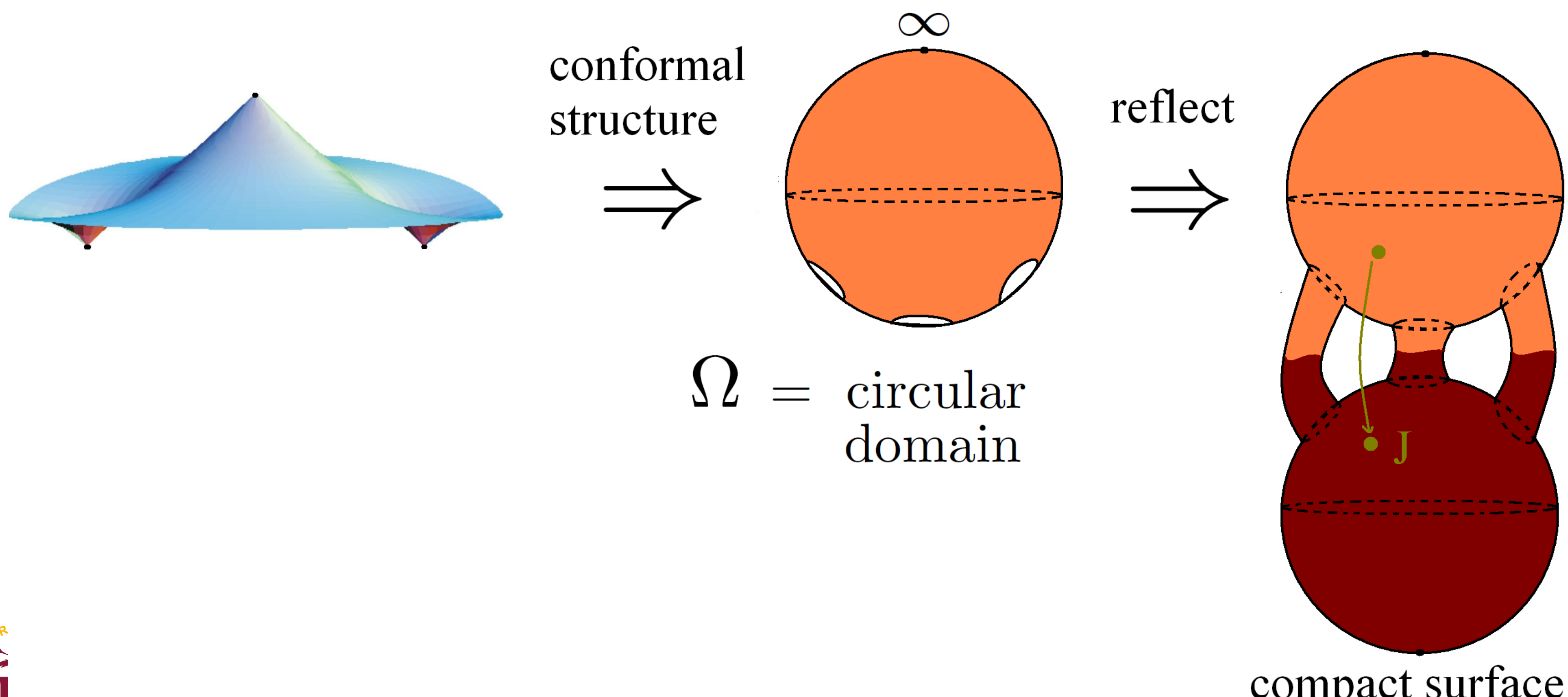


Weierstrass data II

For surfaces with (conelike) singularities...

The Weierstrass data of an entire maximal graph with $n+1$ singularities extend to a compact genus n Riemann surface (called the double surface)

satisfying $g \circ J = 1/\bar{g}$, and $J^*(\phi_3) = -\overline{\phi_3}$.



2 Steps

Theorem [–,09]: For each n-connected circular domain there are exactly 2^n entire maximal graphs (with $n+1$ singularities) with this conformal structure.

- Step 1: The number of conformally equivalent maximal graphs does not depend on the circular domain, but only on the number of singularities.
- Step 2: For each n, there exists a n-connected circular domain Ω_n, such that the number of maximal graphs conformally equivalent to Ω_n is exactly 2^n.

Proof: Step 1

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- Fix n. To each entire maximal graph $\mathcal{S}$ with $n+1$ singularities, we assign an upla (v, D), where
  - $v = (c_0, r_0, \ldots, c_n, r_n) \in \mathbb{R}^{3n+3}$ is the sequence of centers and radii of the circular domain Ω associated to $\mathcal{S}$,
  - D is the *set* of zeroes of g in Ω.

To avoid congruences we will assume $c_0 = 0, r_0 = 1, c_1 \in \mathbb{R}^+$.

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- Theorem [–, Lopez, Souam]:
  The above defined map $\mathcal{S} \mapsto (v, D)$ is a bijection.
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Proof: Step 2

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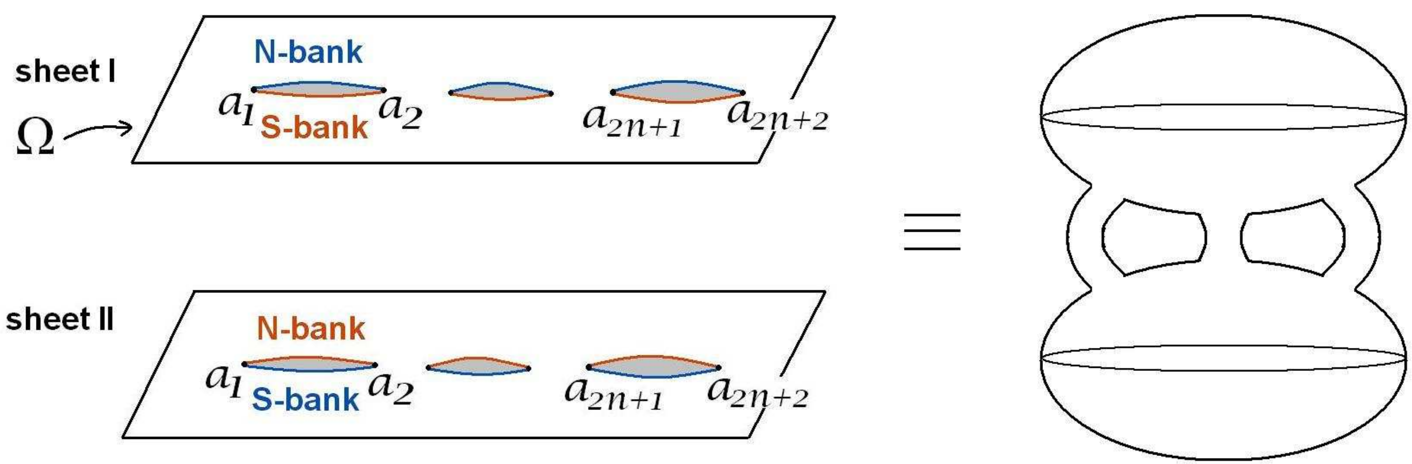
$$N = \{(z,w) \in \bar{\mathbb{C}} \times \bar{\mathbb{C}} \; : \; w^2 = \prod_{j=1}^{2n+2} (z - a_j)\}$$



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- We fix a circular domain Ω_n (depending on $a_1 < \ldots < a_{2n+2} \in \mathbb{R}$)
- We prove that any maximal surface conformally equivalent to Ω_n must have Weierstrass data of the form

$$g = \frac{w + P(z)}{w - P(z)} \quad \phi_3 = \left(\frac{w}{P(z)} - \frac{P(z)}{w}\right) dz^2,$$

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- The above data provides congruent surfaces if and only if the sets of $b_j's$ are complementary. Therefore we can assume $b_1 = a_1$.
- Thus, the number of possible choices of $b_2, \ldots, b_{n+1}$ is 2^n. $\square$

The end...

Thank you!