A Proper Harmonic Map from the Unit Disk to the Complex Plane

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Reference

- A. A. and J.A. Gálvez, *Proper harmonic maps from hyperbolic Riemann surfaces into the Euclidean plane.* Preprint 2009 (arXiv:0906.2638).

Harmonic Maps

Definition

Consider $D \subset \mathbb{C} \equiv \mathbb{R}^2$ an open domain and $F = (F_1, F_2) : D \to \mathbb{R}^2$ a map.

- F is said to be harmonic iff $F_i : D \to \mathbb{R}$ is a harmonic function $\forall i = 1, 2$.
- F is said to be holomorphic iff it is harmonic and F_1 and F_2 satisfy the Cauchy-Riemann equations.

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Motivation

Question

Are $\mathbb{D}$ and $\mathbb{C}$ equivalent under harmonic/holomorphic diffeomorphisms?

Motivation

Theorem (Picard, 1879)

There is no nonconstant holomorphic function from the complex plane to the unit disk.

Theorem

There is no proper holomorphic function from the unit disk into the complex plane.

A map $F : \mathbb{D} \to \mathbb{C}$ is said to be proper iff the image of any divergent curve on $\mathbb{D}$ is a divergent curve on $\mathbb{C}$, i.e., the pre-image of any compact subset of $\mathbb{C}$ is a compact subset of $\mathbb{D}$.

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There is no nonconstant harmonic map from the Euclidean plane to the unit disk.

Theorem (Heinz, 1952)

There is no harmonic diffeomorphism from the unit disk onto the Euclidean plane.

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The Conjecture

Conjecture (Shoen and Yau, 1985)

There are no proper harmonic maps from the unit disk to the complex plane.

Proper Harmonic Maps

Main Theorem (A. and Gálvez, 2009)

There exists a proper harmonic map from $\mathbb{D}$ to $\mathbb{C}$.

Sketch of the Proof

There exist a sequence of closed disks, $\{D_n\}_{n\in\mathbb{N}}$, and a sequence of harmonic maps, $X_n : D_n \to \mathbb{R}^2$ such that

1. $D_{n-1} \subset \mathrm{Int}(D_n) \subset D_n \subset \mathbb{D}$.
2. $|X_n(z) - X_{n-1}(z)| < 1/n^2$, $\forall z \in D_{n-1}$.
3. $|X_n(z)| > n$, $\forall z \in \partial D_n$.
4. $|X_n(z)| > n-1$, $\forall z \in D_n - D_{n-1}$.

The set $D = \cup_{n\in\mathbb{N}} \mathrm{Int}(D_n)$ is an open bounded disk and the limit map $X = \lim_{n\to\infty} X_n : D \to \mathbb{R}^2$ is harmonic.

Given $K \subset \mathbb{R}^2$ a compact set there exists $n \in \mathbb{N}$ such that $|p| < n$ $\forall p \in K$, then $X^{-1}(K) \subset D_{n'}$ and so $X : D \to \mathbb{R}^2$ is proper.

Let $Y : \mathbb{D} \to D$ be a holomorphic diffeomorphism. Then

$$Z = X \circ Y : \mathbb{D} \to \mathbb{R}^2$$

is proper and harmonic.

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The sequence

Lemma

Let $D_1 \subset Int(\mathcal{D}) \subset \mathcal{D} \subset \mathbb{D}$ two closed disks, $r < R$ two real numbers and $F : \mathcal{D} \to \mathbb{R}^2$ a harmonic map such that

$$r < |F(z)| < R, \quad \forall z \in \mathcal{D} - D_1.$$

Given ϵ_1, ϵ_2, and ϵ_3 positive constants, there exists a closed disk D_2 and a harmonic map $G : D_2 \to \mathbb{R}^2$ such that

- $D_1 \subset Int(D_2) \subset D_2 \subset Int(\mathcal{D})$.
- $|F(z) - G(z)| < \epsilon_1, \forall z \in D_1$.
- $R - \epsilon_2 < |G(z)| < R, \forall z \in \partial D_2$.
- $r - \epsilon_3 < |G(z)|, \forall z \in D_2 - D_1$.

“Deforming” Harmonic Maps

Let $D \subset \mathbb{C}$ be a closed disk and $F : D \to \mathbb{R}^2$ a map. Given an orthonormal basis $S = \{e_1, e_2\}$ of $\mathbb{R}^2$, we shall denote

$$F_{(1,S)} := \langle F, e_1 \rangle, \quad F_{(2,S)} := \langle F, e_2 \rangle.$$

Thus, F is a harmonic map into $\mathbb{R}^2$ if and only if $F_{(j,S)} : D \to \mathbb{R}$ are harmonic functions, $j = 1, 2$.

Given a harmonic map $F : D \to \mathbb{R}^2$, a harmonic function $h : D \to \mathbb{R}$ and an orthonormal basis $S = \{e_1, e_2\}$, then the new map $G : D \to \mathbb{R}^2$ defined in local coordinates as

$$G_{(1,S)} = F_{(1,S)} + h, \quad G_{(2,S)} = F_{(2,S)}$$

is also harmonic.

“We can modify a harmonic map $F : D \to \mathbb{R}^2$ in a direction of $\mathbb{R}^2$ preserving the perpendicular one”

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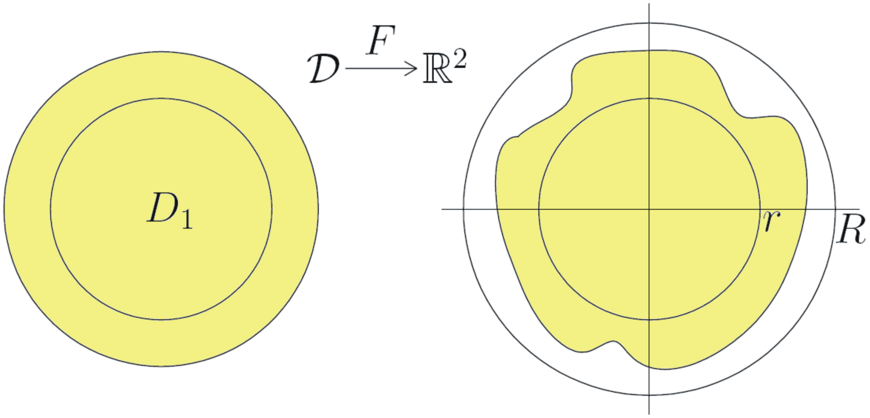

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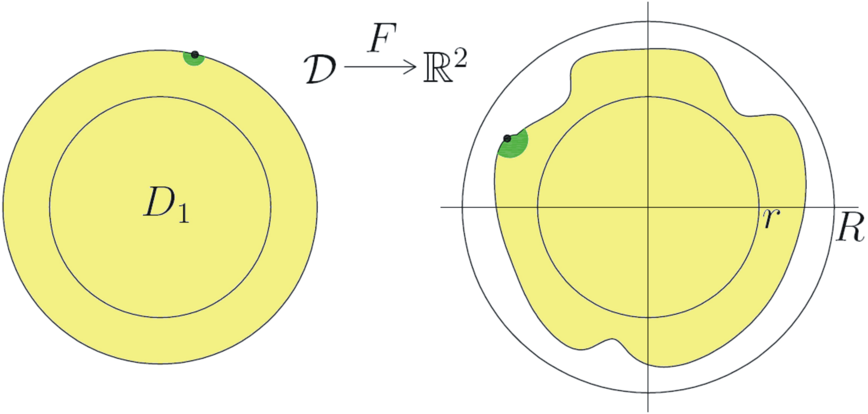

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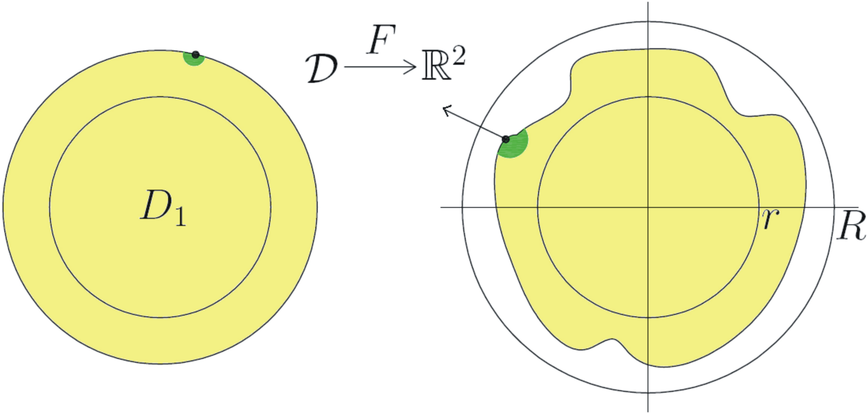

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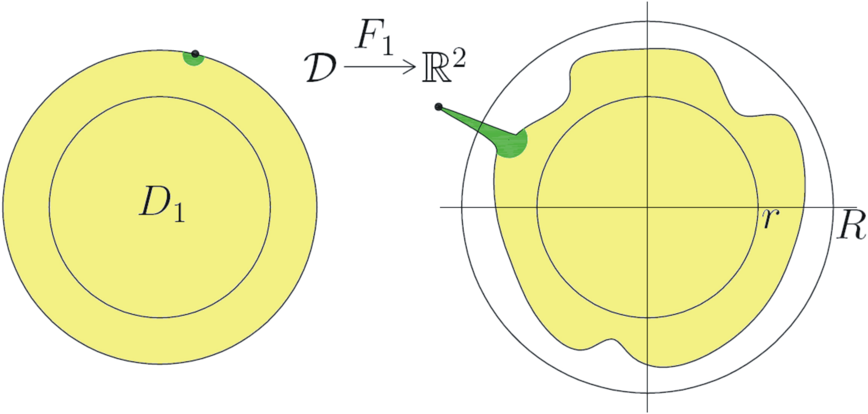

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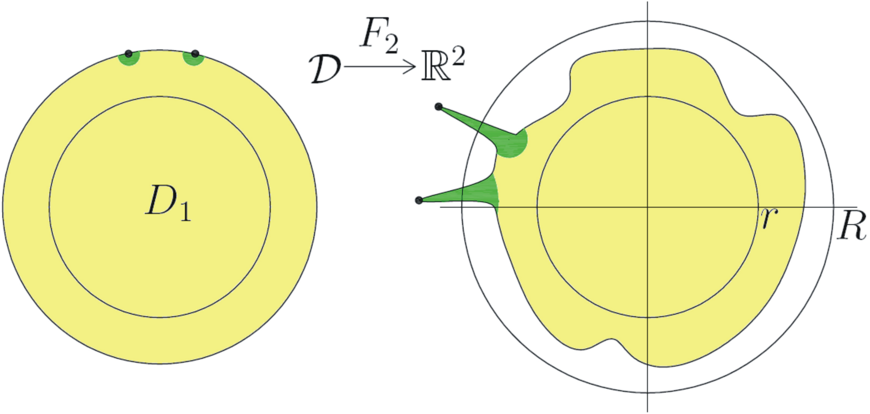

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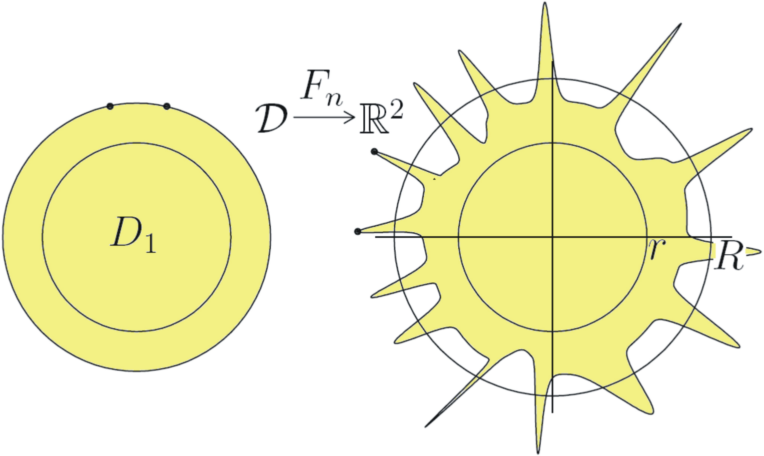


Runge's Theorem

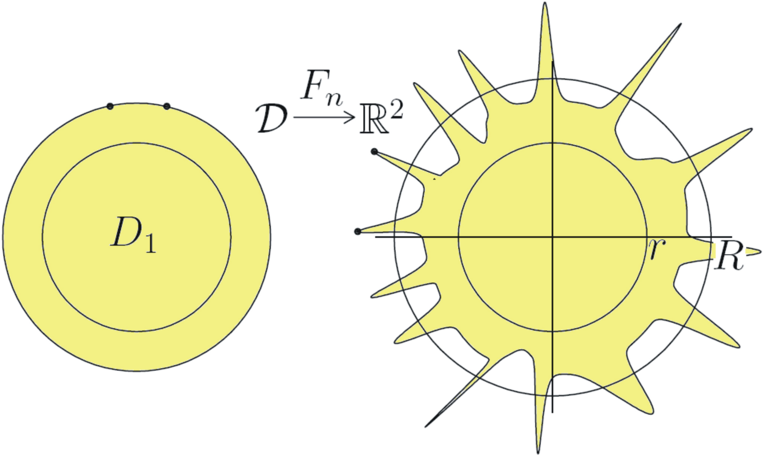
Theorem (Runge, 1885)

Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} - K$ is connected and $f : K \to \mathbb{C}$ a holomorphic function.
Then, for any $\epsilon > 0$, there exists a holomorphic function $h : \mathbb{C} \to \mathbb{C}$ such that

$$|h(z) - f(z)| < \epsilon, \quad \forall z \in K.$$

- $f : K \to \mathbb{C}$ is holomorphic if it is the restriction to K of a holomorphic function defined on an open domain $U \supset K$.

The Second Deformation



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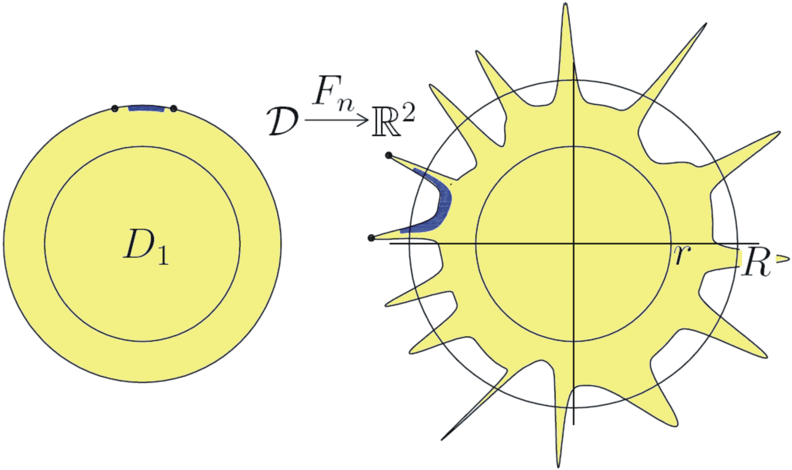

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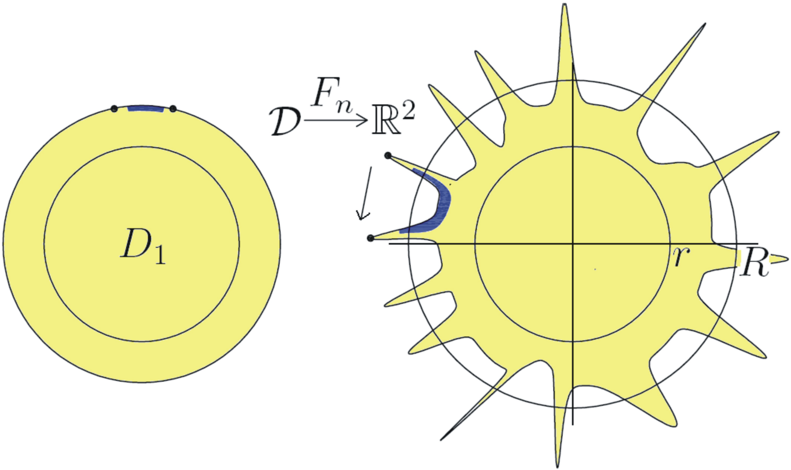

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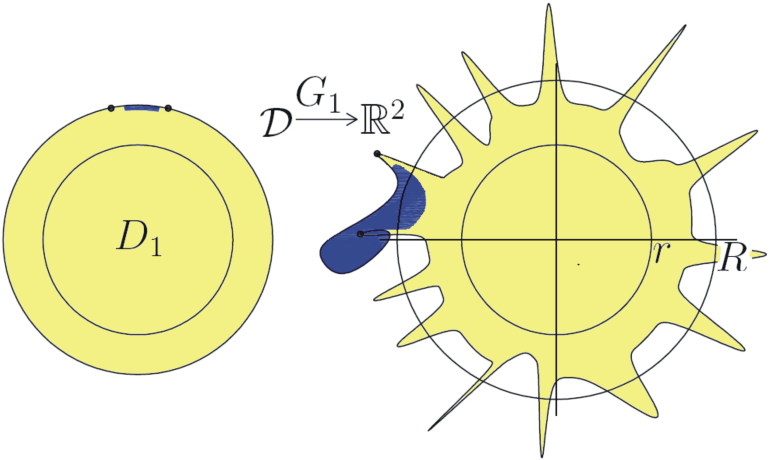

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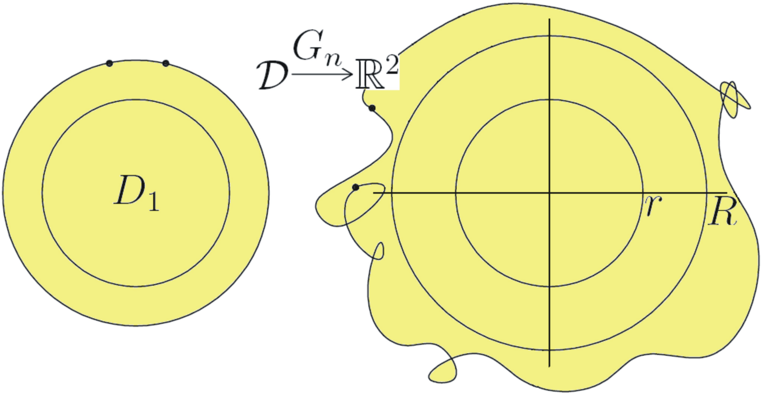
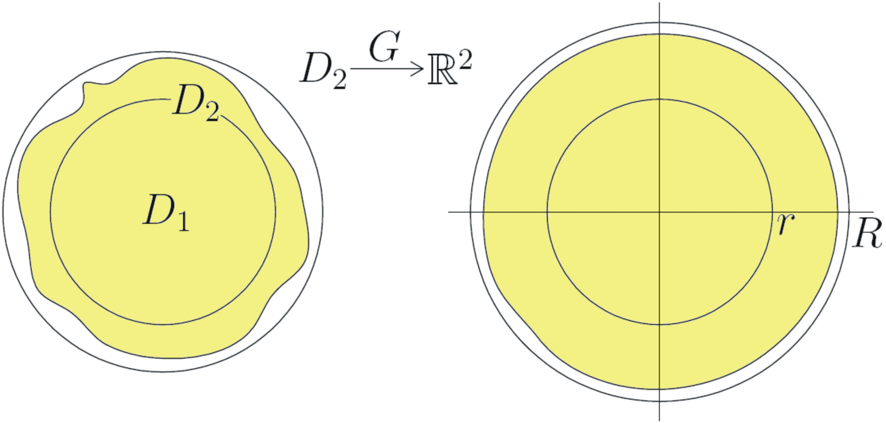


Shrinking the domain of definition



Riemann Surfaces

Definition

- A Riemann surface is a 1-dimensional complex submanifold, i.e., a smooth surface with a holomorphic atlas.
- Let $\mathcal{M}$ be a Riemann surface. A map $F : \mathcal{M} \to \mathbb{C}$ is said to be harmonic (resp. holomorphic, subharmonic,...) iff $F \circ z^{-1} : z(U) \subset \mathbb{C} \to \mathbb{C}$ is harmonic (resp. holomorphic, subharmonic,...) for any holomorphic chart (U, z) on $\mathcal{M}$.

Definition

An open Riemann surface is said to be hyperbolic iff it carries a negative non-constant subharmonic function. Otherwise, it is said to be parabolic. Compact Riemann surfaces are said to be elliptic.

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Proper Harmonic Maps

Theorem (A. and Gálvez, 2009)

There exist proper harmonic maps from (hyperbolic) Riemann surfaces of arbitrary finite topological type to $\mathbb{C}$.

The Main Theorem is “sharp”

- The Theorem fails if we change harmonic map for holomorphic function.

- The Theorem fails if we change $\mathbb{C}$ for any complete flat surface $\mathcal{S}$ non-isometric to the Euclidean plane:

  There is no proper harmonic map from a hyperbolic Riemann surface into such a surface $\mathcal{S}$.

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An Open Problem

Theorem (Shoen and Yau, 1997)

There is no harmonic diffeomorphism from the unit disk onto a complete surface with non-negative Gauss curvature.

Open Problem

Does a proper harmonic map from a hyperbolic Riemann surface into a complete surface with non-negative Gauss curvature exist?

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A Related Problem

- Schoen and Yau related their conjecture with the problem of non-existence of a hyperbolic minimal surface in $\mathbb{R}^3$ which properly projects into a plane.
  - A hyperbolic (resp. parabolic) minimal surface in $\mathbb{R}^3$ is the image of a hyperbolic (resp. parabolic) Riemann surface by a conformal harmonic immersion to $\mathbb{R}^3$.
  - The projection to $\mathbb{R}^2$ of a minimal immersion is a harmonic map.

Proper Hyperbolic Minimal Surfaces

Theorem (Morales, 2003)

There exists proper hyperbolic simply connected minimal surfaces in $\mathbb{R}^3$.

Theorem (A., Ferrer and Martín, 2008)

There exists proper hyperbolic minimal surfaces in $\mathbb{R}^3$ *with arbitrary finite topological type.*

Theorem (Ferrer, Martín and Meeks, 2009)

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Hyperbolic Minimal Surfaces in $\mathbb{R}^3$ properly projecting onto $\mathbb{R}^2$.

Theorem (A. and López, work in progress)

Any open Riemann surface admits a conformal minimal immersion in $\mathbb{R}^3$ properly projecting onto $\mathbb{R}^2$.

Theorem (Scheinberg, 1978)

Let K be a compact subset of an open Riemman surface $\mathcal{N}$ such that $\mathcal{N} - K$ is connected and $f : K \to \mathbb{C}$ a continuous map such that

$$f\Big|_{\overline{Int(K)}} \quad \text{is holomorphic.}$$

Then, for any $\epsilon > 0$, there exists a holomorphic function $h : \mathcal{N} \to \mathbb{C}$ such that

$$|h(z) - f(z)| < \epsilon, \quad \forall z \in K.$$

Thanks!